THE RELATIVISTIC HEAT EQUATION: MAXIMUM/MINIMUM PRINCIPLES AND OTHER PROPERTIES

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This summer I worked on a generalization of the heat equation designed to better comport with relativity. The heat equation is one of the oldest equations in PDE, going all the way back to Fourier. But while the heat equation is an effective and simple model for diffusion processes, it has serious theoretical problems related to relativity. The heat kernel is a Gaussian with standard deviation $\frac{1}{\sqrt{t}}$. This means that even with finitely supported initial conditions, for all positive times, the solutions to the heat equation on the real line will have the whole real line as support. This infinite propagation of signals is an obvious violation of relativity.

This shortcoming of the heat equation was addressed from an optimal transportation point of view in [1]. In this paper Brenier replaces the standard heat equation with:

$$\frac{\partial}{\partial t} u = \nu \nabla \cdot \frac{u \nabla u}{\sqrt{u^2 + \frac{\nu^2 |\nabla u|^2}{c^2}}} \tag{1}$$

Note that for positive values of u, the ordinary heat equation is recovered in the limit as c goes to infinity. The standard heat equation can be derived using optimal transportation methods, as a gradient descent problem [2]. If one replaces the classical Lagrangian with the relativistic Lagrangian as a cost function, then (1) becomes the solution to the optimal transportation problem.

One interesting difference between the standard heat equation and the relativistic heat equation is that the nonlinear character of the latter leads it to respect absolute zero. The standard heat equation is linear and any constant function is a solution, so no special behavior is observed around absolute zero. If u is a solution, than so is $u - C$, for any Real C. Along similar lines, if u is a solution to the standard heat equation that so is $-u$. On the other hand, for the relativistic heat equation $u(x,t)$ a solution implies that $-u(x,-t)$ is a solution, so negative temperatures result in a time reversal symmetry. Both the standard and relativistic heat equation are invariant under positive scalar multiples. Furthermore, even the elliptic operator behaves interestingly around absolute zero. The simplest problem involving stationary states of the equation is to take the unit interval in one dimension, and let $u(0) = 0$ $u(1) = C$. This corresponds to finding the equilibrium of a thin rod with fixed temperatures at the ends for the relativist equation. Shockingly, this problem is ill-posed. However, when you think about it, this is not so surprising after all. Because we cannot simply add or subtract a constant and still have a solution to RHC, our choice of units matters. We must use Kelvin (or any other unit where the temperature of absolute zero in those units is zero). The linearity of the heat equation means that from it's point of view their is no absolute zero, however, for the RHC, $u(0) = 0$ corresponds to setting the temperature at one end of the bar to absolute zero. This is completely unphysical, so it is a good sign that our equation comes back ill posed. A strong model should not only be able to model reasonable situations correctly; it should also break down in situations where it is no longer reasonable.

The bulk of my work this summer was concerned with extending the classical maximum and minimum principles, well known for the stationary states of the heat equation, to the stationary states of the relativistic heat equation. Setting the time rate of change to zero, we have a nonlinear elliptic problem. If (1) is to be a physically reasonable model of the relativistic diffusion, then there ought to be strong maximum and minimum principles, stating that the maximum and minimum values for the solutions to the elliptic equation on a open set will be obtained on the interior if and only if the solution is constant. Thanks to a concavity term in the equations, the strong minimum principle is relatively straightforward. Extending the results we already have in one dimension for the strong maximum principle and the well posedness of solutions that are zero on the boundary is the main focus of our current efforts. In the case of the maximum principle we are trying to use variational methods to get the proof.

References

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