Categories and Groupoids

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Definition - Category

A category $\mathcal{C}$ consists of:

1. A collection of objects: $\mathrm{Ob}(\mathcal{C}) = V(\mathcal{C})$
2. A collection of morphisms $\mathrm{Mor}_{\mathcal{C}}(A, B) = E_{AB}$ for each $A, B \in \mathrm{Ob}(\mathcal{C})$
3. A law of composition: if $\alpha \in \mathrm{Mor}_{\mathcal{C}}(A, B)$ and $\beta \in \mathrm{Mor}_{\mathcal{C}}(B, C)$ then $\alpha\beta \in \mathrm{Mor}_{\mathcal{C}}(A, C)$

Additionally, we require that:

1. if $\alpha \in E_{AB}$, $\beta \in E_{BC}$, and $\gamma \in E_{CD}$ then $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ (associativity)
2. There exists an identity element $\epsilon_A \in E_{AA}$ for each $A \in V(A)$ such that if $\alpha \in E_{AB}$ then $\epsilon_A\alpha = \alpha = \alpha\epsilon_B$ (*identity*)

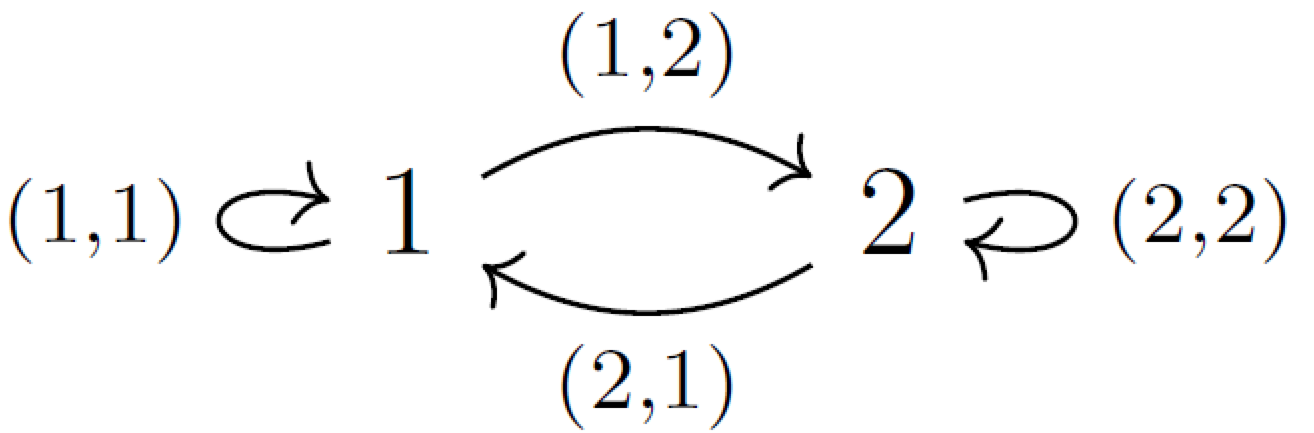
Definition - Groupoid

A groupoid is a small category in which every morphism is invertible:

- $\forall \alpha \in E_{AB},\ \exists \alpha^{-1} \in E_{BA}$ such that $\alpha\alpha^{-1} = \epsilon_A$ and $\alpha^{-1}\alpha = \epsilon_B$

Example (Δ^1):

- $\mathrm{V}(\Delta^1) = \{1, 2\}$
- $\mathrm{E}_{ij} = (i, j)$



Definition – Functor

If $\mathcal{A}$ and $\mathcal{B}$ are categories then a functor $F : \mathcal{A} \to \mathcal{B}$

1. Assigns to each $A \in \mathrm{Ob}(\mathcal{A})$ an object $F(A) \in \mathrm{Ob}(\mathcal{B})$
2. Assigns to each $\alpha \in \mathrm{Mor}_{\mathcal{A}}(A_1, A_2)$ a morphism $F(\alpha) \in \mathrm{Mor}_{\mathcal{B}}(F(A_1), F(A_2))$ such that:
   - Identities are preserved: $F(\epsilon_A) = \epsilon_{F(A)}$
   - For $\alpha \in \mathrm{Mor}_{\mathcal{A}}(A_1, A_2)$ and $\beta \in \mathrm{Mor}_{\mathcal{A}}(A_2, A_3)$, $F(\alpha\beta) = F(\alpha)F(\beta)$

Examples of Categories

1. The category of sets, $\mathcal{S}$
   - $V(\mathcal{S}) = \text{Sets}$
   - $E_{AB} = B^A$

2. The category of directed graphs, $\mathcal{D}$
   - $V(\mathcal{D}) = \text{Directed Graphs}$
   - $E_{AB} = \text{All graph maps } \theta : A \to B$

3. The category of (small) categories, $\mathcal{C}$
   - $V(\mathcal{C}) = \text{All (small) categories}$
   - $E_{AB} = \text{All functors } F : A \to B$

4. The category of groupoids, $\mathcal{G}$
   - $V(\mathcal{G}) = \text{Groupoids}$
   - $E_{AB} = \text{All functors } F : A \to B$

Limits

• We can define a diagram $\boldsymbol{A}$ in a category $\mathcal{C}$ to be a graph with vertices some subset of $\mathrm{Ob}(\mathcal{C})$ and arrows some subset of the morphisms in $\mathcal{C}$.

• We then define right and left limits of $\boldsymbol{A}$ to be objects, $L, L' \in \mathrm{Ob}(\mathcal{C})$, that satisfy some desired universal property.

Consider the following diagram, $\boldsymbol{A}$, in $\mathcal{S}$:

$$\boldsymbol{A} = \{1\} \hookrightarrow \{1,2\} \hookrightarrow \{1,2,3\} \hookrightarrow \{1,2,3,4\} \hookrightarrow \cdots$$

It turns out that

$$\varinjlim \boldsymbol{A} = \mathbb{N}$$

$$\varprojlim \boldsymbol{A} = \{1\}$$

Theorem 1

Theorem 1 *Any diagram in* $\mathcal{S}, \mathcal{D}, \mathcal{C}$ *or* $\mathcal{G}$ *has both a right and a left limit.*
We say that $\mathcal{S}, \mathcal{D}, \mathcal{C}$ *and* $\mathcal{G}$ *are complete.*

Universal Morphisms

Universal morphisms are a type of right limit.

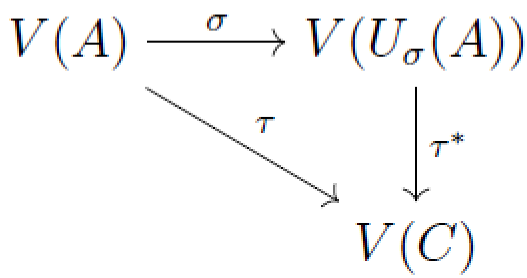
Suppose A is a category and σ is any function defined on $V(A)$.

If we can find a category $U_\sigma(A)$ such that $V(U_\sigma(A)) = \sigma[V(A)]$ and a functor $\theta : A \to U_\sigma(A)$ that satisfies the following universal property, then we say that θ is a universal morphism:

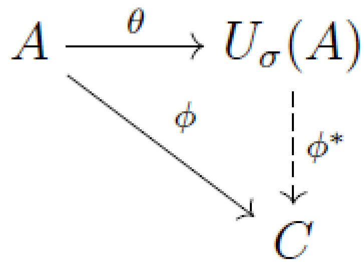
For every functor $\phi : A \to C$ ($C \in \mathrm{Ob}(\mathcal{C})$) whose vertex map is of the form $\tau = \sigma\tau^*$, there exists a unique functor $\phi^* : U_\sigma(A) \to C$ such that
$\phi = \theta\phi^*$.

Universal Morphisms

That is, given:



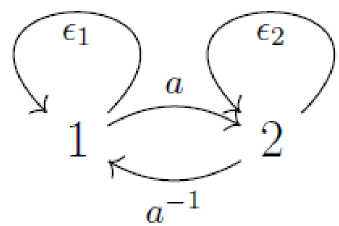
we also have:



Theorem - *For any vertex map*, σ, a universal morphism exists.

Example

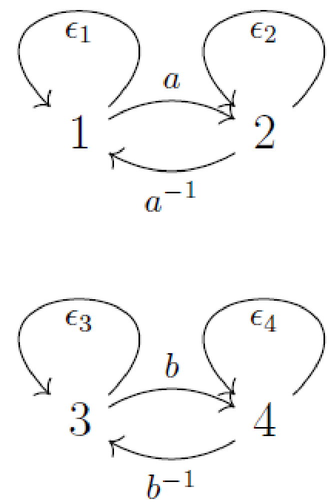
Let $\mathcal{A}$ be the following category:



Let $\sigma(1) = \sigma(2) = x$
Then $U_\sigma(A)$ is isomorphic to the free group on one letter, $U_\sigma(A) = < a >$

Example

If instead we let $\mathcal{A}$ be:



Again, if $\sigma(1) = \sigma(2) = \sigma(3) = \sigma(4) = x$
Then $U_\sigma(A)$ is the free group on two letters, $< a, b >$

Theorem 2

In general, the image of a universal morphism, $U_\sigma(A)$, will consist of "words" on the alphabet of morphisms in A modulo some equivalence relation

Theorem 2 *Every edge (morphism) of* $U_\sigma(A)$ *is represented by exactly one* σ *- reduced path.*

Corollary *- Every word in a free group has a unique reduced word.*

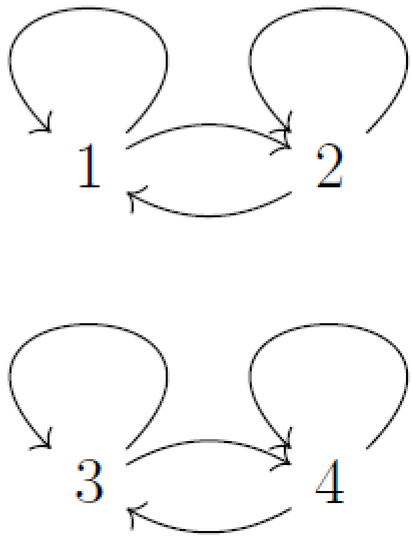
Normal and Quotient Groupoids

We say that N is a normal subgroupoid of G if:

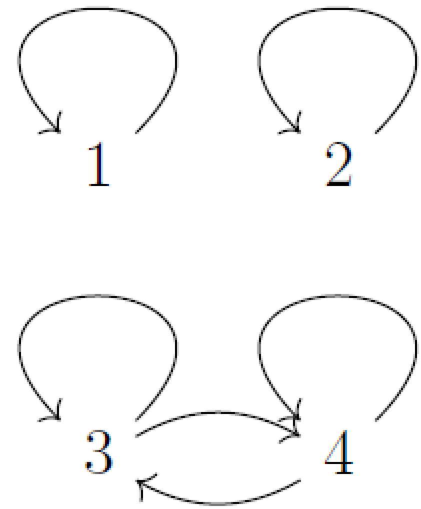
- N contains all the identity elements of G
- if $x \in N_{ii}$ and $a \in G_{ij}$ then $a^{-1}xa \in N_{jj}$

Given a normal subgroupoid, we can define a quotient groupoid, G/N

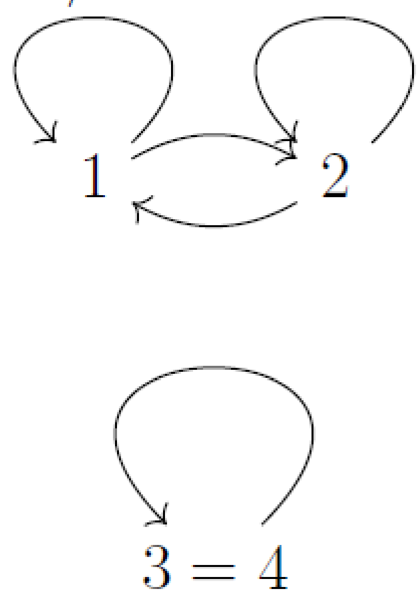
Let G be the following groupoid:



Let N be:

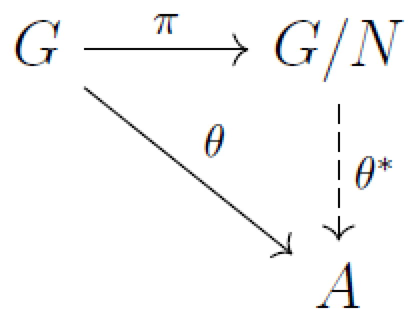


Then G/N will look like:



Theorem 3

Theorem 3 *Let G be a groupoid and N a normal subgroupoid. Let $\pi : G \to G/N$ be the canonical projection. Let $\theta : G \to A$ be any groupoid-map with kernel $M \supseteq N$. Then there exists a unique groupoid-map $\theta^* : G/N \to A$ such that $\theta = \pi\theta^*$ Furthermore,* $\operatorname{Ker}\theta^* = M/N$



Future goals

- Covering morphisms
- Homology groups
- Defining cardinality on categories

References

[1] Higgins, Philip J. (1971). *Notes on Categories and Groupoids*, Van Nostrand Reinhold Company, London.

[2] Baez, John C. and Dolan, James (2000). *From Finite Sets to Feynman Diagrams*, arXiv:math/0004133.