The Relativistic Heat Equation

The Strong Minimum Principle and Behavior near Absolute Zero

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- The heat equation is the standard model for diffusion and heat flow. It is a parabolic equation, and also be seen, from the point of view of stochastic processes, as the equation for the governing the time evolution of a particle undergoing Brownian motion.
- The heat equation is:

$$u_t = \Delta u \tag{1}$$

- this equation can also be written as a transport equation with transport velocity $\frac{\nabla u}{u}$:

$$u_t = \nabla \cdot (\frac{\nabla u}{u} u) \tag{2}$$

- This is clearly the same equation as (1) because $\nabla \cdot \nabla u = \Delta u$.

Problems with the Heat Equation

- The heat equation is a transport equation by velocity $\frac{\nabla u}{u}$. This is problematic, however, because this transport velocity is unbounded, and allows particles to move, or heat to flow, faster than the speed of light.
- Furthermore, in one dimension the solution to the heat equation with initial condition $u(x,0) = \phi \in L^1(\Re)$ is:

$$u(x,t) = \int_{\Re} \phi(y) \frac{1}{\sqrt{4\pi t}} e^{-(x-y)^2/4t} \mathrm{dy} \tag{3}$$

- Because the heat kernel has $\Re$ as support $\forall t > 0$, solutions to the heat equation have $\Re$ for support, with respect the spatial variable, $\forall t > 0$ as well.
- This does not comport well with relativity. Relativistically we would expect that $\mathrm{supp}(u(x,0)) \subset (a,b) \implies \mathrm{supp}(u(x,t)) \subset (a-ct, b+ct)$. That is, we expect to stay inside a light cone.

The Relativistic heat equation

- The standard heat equation can be derived as a solution to an optimal transportation problem corresponding to the convex cost function

$$k(z) = \frac{1}{2}z^2 \tag{4}$$

- For the relativistic heat equation we replace the cost function, which corresponds to the classical Lagrangian, with:

$$k(z) = \left\{ {}^{1-\sqrt{1-z^2}, |z| \leq 1}_{\infty, |z| > 1} \right\} \tag{5}$$

- Our relativistic heat equation is:

$$u_t = \nabla \cdot \frac{u \nabla u}{\sqrt{u^2 + |\nabla u|^2}} \tag{6}$$

Relativistic Equation in the asymptotic limit

- the version above is dimensionless with the speed of light and diffusivity of the medium set to 1. We do this for simplicity and because it does not change our solutions up to a scalar multiple change in units. It is however, useful to see the equation with constants, because then it is clear that the heat equation is recovered in the limit as $c \to \infty$, so long as we have the additional stipulation that $u > 0$

$$u_t = \nu \nabla \cdot \frac{u \nabla u}{\sqrt{u^2 + \frac{\nu^2 |\nabla u|^2}{c^2}}}. \tag{7}$$

Behavior at absolute zero

- The standard heat equation is linear so $u(x,t)$ is a solution implies $-u(x,t)$ is a solution.
- For the relativistic heat equation however, $u(x,t)$ a solution implies $-u(x,-t)$ is a solution.
- In the limit as $c \to \infty$ (6) interpolates the the standard heat equation for $u > 0$, but interpolates the reverse heat equation $u_t = -\Delta u$ for $u < 0$. This limit equation, when $u < 0$, compresses solutions back to one point and will eventually take a smooth function to a discontinuous one, so this is our first hint that (6) is ill posed for $u < 0$.
- What we have observed is that (6) exhibits time reversal symmetry for negative temperatures.

An implicit solution in 1 dimension

- The simplest problem we can look at is the stationary states in 1 dimension. In dimension our elliptic equation is:

$$\left(\frac{uu_x}{\sqrt{u^2+u_x^2}}\right)_x = 0 \tag{8}$$

- Because this is an ODE, we can solve implicitly for an exact solution:

$$\frac{uu_x}{\sqrt{u^2+u_x^2}} = C \tag{9}$$

- Squaring both sides we find:

$$\frac{u^2u_x^2}{u^2+u_x^2} = C^2 \tag{10}$$

- Solving for u_x we find:

$$u_x = \frac{Cu}{\sqrt{u^2-C^2}} \tag{11}$$

- Taking the reciprocal we find:

$$\frac{dx}{du} = \frac{\sqrt{u^2-C^2}}{Cu} \tag{12}$$

- Integrating both sides we find:

$$x + b = \frac{\sqrt{u^2 - C^2}}{2C} - \frac{C \log(u + \sqrt{u^2 - C^2})}{2u} \tag{13}$$

- Recall that by definition: $\frac{uu_x}{\sqrt{u^2+u_x^2}} = C$
- Therefore, $C = 0 \implies u$ is constant. So if u is not constant, then $C \neq 0$, and the implicit solution we found above will hold for some choice of constants.
- The boundary value problem $u(0) = 0, u(L) = A > 0$ does not hold for any choice of constants. The equation is ill posed! While the fact that this seemingly straightforward boundary value problem is not well posed, it is actually not surprising. The relativistic heat equation (6) is nonlinear, so if we add a constant a solution will cease to be a solution. This means choice of units matters, and, as in all cases where choice of Temperature units matter, we must use Kelvin. The boundary value problem described above corresponds to setting one end of a thin rod to absolute zero. This is completely unphysical, so it is good that it comes back ill posed.

Strong Minimum Principle

Theorem (Minimum Principle)

Let $\Omega \subset \Re^n$ be a simply connected, bounded open set with a smooth boundary. Suppose $\forall x \in \Omega$, $\nabla \cdot \frac{u\nabla u}{\sqrt{u^2+|\nabla u|^2}} = 0$ and u is continuous on $\bar{\Omega}$. Then u attains its minimum in $\Omega \iff$ u is constant on Ω.

Proof.

We begin the proof by rewriting the equation.

$$\nabla \cdot \frac{u\nabla u}{\sqrt{u^2+|\nabla u|^2}} = \frac{u}{\sqrt{u^2+|\nabla u|^2}}\Delta u + \nabla \left(\frac{u}{\sqrt{u^2+|\nabla u|^2}} \right) \cdot \nabla u \qquad (14)$$

With some basic calculus we find:

$$\nabla \left(\frac{u}{\sqrt{u^2+|\nabla u|^2}} \right) = \frac{|\nabla u|^2 \nabla u - u\mathrm{Hess}(u)\nabla u}{(u^2+|\nabla u|^2)^{3/2}} \qquad (15)$$

□

Strong Minimum Principle (cont.)

Proof.

Therefore, taking the dot product in (14), and combining terms we find:

$$\nabla \cdot \frac{u\nabla u}{\sqrt{u^2+|\nabla u|^2}} = \frac{u^3\Delta u + u|\nabla u|^2\Delta u - u(\nabla u)^T \mathrm{Hess}(u)\nabla u + |\nabla u|^4}{(u^2+|\nabla u|^2)^{3/2}} = 0 \tag{16}$$

$$u^3\Delta u + u(\nabla u)^T(\Delta u I_n - \mathrm{Hess}(u))\nabla u + |\nabla u|^4 = 0 \tag{17}$$

Now we define the operation $T : \Re^{n\times n} \to \Re^{n\times n}$, by $T(A) = \mathrm{tr}(A)I_n - A$. Clearly T preserves positive semi-definiteness. We have the same eigenvectors, and the eigenvalues are the sum of all the other eigenvalues. Observe that $\Delta u = \mathrm{tr}(\mathrm{Hess}(u))$, and conclude:

$$u^3\Delta u + u(\nabla u)^T T(\mathrm{Hess}(u))\nabla u + |\nabla u|^4 = 0 \tag{18}$$

If u has a minimum at $x \in \Omega$, then $\mathrm{Hess}(u)$ is positive semi-definite on a neighborhood of x, so on some neighborhood of x each of the terms in (18) is ≥ 0. Because they sum to zero, this implies that all of the terms in (18) are zero on some neighborhood, N of x.

□

Proof.

$\nabla u = 0$ on N$\Longrightarrow$ u is constant on N. But each point on the boundary of N, an open neighborhood, is also then a minimum, so we can apply the same argument again, until we have covered Ω. Leaving aside some technical topological issues, this shows that u must be constant on Ω.

□

References

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