Categories and Groupoids

Benjamin Pollak

Along with Professor Tang, I spent the summer of 2013 reading necessary background material for my upcoming senior thesis in mathematics. In particular, we spent much of our time learning the fundamentals of category theory, with a particular emphasis on groupoids.

A category $\mathcal{C}$ consists of:

1. A collection of objects: $\mathrm{Ob}(\mathcal{C})$
2. A collection of morphisms $\mathrm{Mor}_{\mathcal{C}}(A, B)$ for each $A, B \in \mathrm{Ob}(\mathcal{C})$
3. A law of composition: if $\alpha \in \mathrm{Mor}_{\mathcal{C}}(A, B)$ and $\beta \in \mathrm{Mor}_{\mathcal{C}}(B, C)$ then $\alpha\beta \in \mathrm{Mor}_{\mathcal{C}}(A, C)$

We also require the following two axioms:

1. if $\alpha \in \mathrm{Mor}_{\mathcal{C}}(A, B),\ \beta \in \mathrm{Mor}_{\mathcal{C}}(B, C),$ and $\gamma \in \mathrm{Mor}_{\mathcal{C}}(C, D)$ then $(\alpha\beta)\,\gamma = \alpha\,(\beta\gamma)$ (*associativity*)
2. There exists an identity element $\epsilon_A \in \mathrm{Mor}_{\mathcal{C}}(A, A)$ for each $A \in \mathrm{Ob}(\mathcal{C})$ such that if $\alpha \in \mathrm{Mor}_{\mathcal{C}}(A, B)$ then $\epsilon_A\alpha = \alpha = \alpha\epsilon_B$ (*identity*)

If $\mathcal{G}$ is a category such that given $A, B \in \mathrm{Ob}(\mathcal{C})$ and $\alpha \in \mathrm{Mor}_{\mathcal{C}}(A, B)$ $\exists\alpha^{-1} \in \mathrm{Mor}_{\mathcal{C}}(B, A)$ satisfying $\alpha\alpha^{-1} = \epsilon_A$ and $\alpha^{-1}\alpha = \epsilon_B$ we say that $\mathcal{G}$ is a groupoid.

The fundamental structure preserving maps between categories are functors: If $\mathcal{A}$ and $\mathcal{B}$ are categories then a functor $F : \mathcal{A} \to \mathcal{B}$

1. Assigns to each $A \in \mathrm{Ob}(\mathcal{A})$ an object $F(B) \in \mathrm{Ob}(\mathcal{B})$
2. Assigns to each $\alpha \in \mathrm{Mor}_{\mathcal{A}}(A_1, A_2)$ a morphism $F(\alpha) \in \mathrm{Mor}_{\mathcal{B}}(F(A_1), F(A_2))$ such that:
   - Identities are preserved: $F(\epsilon_A) = \epsilon_{F(A)}$

- For $\alpha \in \mathrm{Mor}_{\mathcal{A}}(A_1, A_2)$ and $\beta \in \mathrm{Mor}_{\mathcal{A}}(A_2, A_3)$, $F(\alpha\beta) = F(\alpha)F(\beta)$

If there exist functors $F : (\mathcal{A}) \to \mathcal{B}$ and $G : (\mathcal{B}) \to \mathcal{A}$ such that $FG = 1_{\mathcal{A}}$ and $GB = 1_{\mathcal{B}}$ we say that $\mathcal{A}$ and $\mathcal{B}$ are *isomorphic categories*, $\mathcal{A} \cong \mathcal{B}$.

We can also define a way to "transform" one functor into another. If F and G are functors $F, G : \mathcal{A} \to \mathcal{B}$ then a *natural transformation* $\tau : F \to G$ associates to $X \in \mathrm{Ob}(\mathcal{A})$ a $\mathcal{B}$-morphism $\tau(X) \in \mathrm{Mor}_{\mathcal{B}}(F(X), G(X))$ such that for any $\alpha \in \mathrm{Mor}_{\mathcal{A}}(X, Y)$ the following diagram commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{\tau(X)} & G(X) \\ {\scriptstyle F(\alpha)}\downarrow & & \downarrow{\scriptstyle G(\alpha)} \\ F(Y) & \xrightarrow{\tau(Y)} & G(Y) \end{array}$$

If, additionally, for all $X \in \mathrm{Ob}(\mathcal{A})$, $\tau(X)$ is an isomorphism then we say that τ is a natural equivalence and that $F \simeq G$. If there exist functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ such that $FG \simeq 1_{\mathcal{A}}$ and $GB \simeq 1_{\mathcal{B}}$ we say that $\mathcal{A}$ and $\mathcal{B}$ are *equivalent categories*, $\mathcal{A} \simeq \mathcal{B}$.

After exploring examples of categories and functors between them, Professor Tang and I moved on to more interesting topics. In particular, we studied the construction of free categories and free groupoids as well as how a topological space naturally gives rise to a fundamental groupoid on the space.

After becoming comfortable with the basics of category theory, we studied limits in categories. Limits characterize the construction of all objects satisfying some universal property (for example: products, coproducts, pullbacks, and pushouts) as well as distinguish certain functors (universal morphisms) and characterize the construction of free products of categories. One of the main theorems we proved over the summer is:

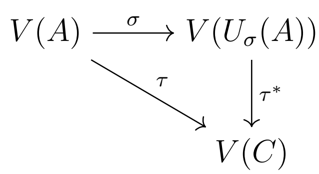
Theorem 1 *The categories $\mathcal{D}$ (the category of directed graphs), $\mathcal{C}$ (the category of categories), and $\mathcal{G}$ (the category of groupoids) are complete. That is, $\mathcal{D}, \mathcal{C},$ and $\mathcal{G}$ admit both right and left limits.*

In particular, we devoted a fair amount of time studying universal morphism. In $\mathcal{C}$, the category of categories, a universal morphism is a distinguished functor intimately related to push-outs, a type of right limit. More precisely, suppose A and $U_\sigma(A)$ are categories (A,$U_\sigma(A) \in \mathrm{Ob}(\mathcal{C})$)

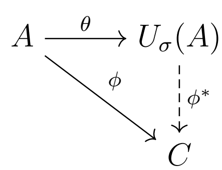
and $\theta \in \mathrm{Mor}_{\mathcal{C}}(A, U_\sigma(A))$. As a functor, θ consists of a vertex map $V(\theta) = \sigma : \mathrm{Ob}(A) \to \mathrm{Ob}(U_\sigma(A))$. If θ satisfies the following universal property, then we call θ a universal morphism:

For every functor $\phi : A \to C$ $(C \in \mathrm{Ob}(\mathcal{C}))$ whose vertex map is of the form $\tau = \sigma\tau^*$, there exists a unique functor $\phi^* : U_\sigma(A) \to C$ such that $\phi = \theta\phi^*$.

That is, given:



we also have:



During the summer, we proved:

Theorem 2 *Universal morphisms exist. That is, given a category A and a function, σ, defined on $V(A)$, there exists a category $U_\sigma(A)$ and a morphism $\theta : A \to U_\sigma(A)$ such that θ is a universal morphism.*

The universal property for universal morphisms immediately implies that $U_\sigma(A)$ is unique up to isomorphism. Over the summer, we studied the structure $U_\sigma(A)$, and learned that it is a category with object set $\mathrm{Ob}(U_\sigma(A)) = \sigma[V(A)]$ and morphisms which are words on the alphabet of morphisms of A modulo some equivalence relation and satisfying certain restrictions on whether two words may be composed. Just as in the case of free groups, the equivalence relation on these words says that two words are identical if they represent the same "σ − reduced word." In particular, we proved:

Theorem 3 *Every edge (morphism) of $U_\sigma(A)$ is represented by exactly one σ-reduced path.*

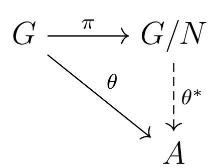
This theorem is particularly useful as it completely defines the structure of $U_\sigma(A)$. Additionally, an immediate corollary to Theorem 2 is that every word in a free group has a unique reduced word.

After studying the properties of limits in categories, we focused more intensely on groupoids. In particular, we studied how to define quotient maps in $\mathcal{G}$, the category of groupoids. Let G be a groupoid and N a subgroupoid of G. $\forall i, j \in V(N)$, let $N_{ij} = \mathrm{Mor}_N(i, j)$ We say that N is *Normal* in G if

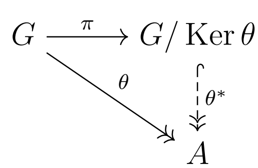
- N contains all the identity elements of G
- if $x \in N_{ii},\ a \in G_{ij}$ then $a^{-1}xa \in N_{jj}$

Just as in the case of groups, if θ is a functor $\theta : A \to B$ then $\operatorname{Ker}\theta$ is a normal subgroupoid of G. Using N, we can define an equivalence relation on G and obtain a well defined quotient groupoid, G/N. These quotient groupoids satisfy a nice universal property:

Theorem 4 *Let G be a groupoid and N a normal subgroupoid. Let $\pi : G \to G/N$ be the canonical projection. Let $\theta : G \to A$ be any functor with kernel $M \supseteq N$. Then there exists a unique groupoid-map $\theta^* : G/N \to A$ such that $\theta = \pi\theta^*$ Furthermore,* $\operatorname{Ker}\theta^* = M/N$

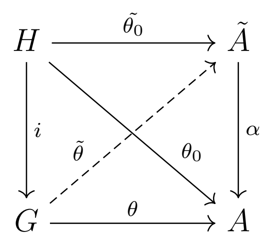


Theorem 3 implies an analogous result to the first isomorphism theorem for groups. Since π is always surjective, an immediate corollary to Theorem 3 is that for any surjective functor $\theta : G \to A$, we have $G/\operatorname{Ker}\theta$ is isomorphic to A



Towards the end of the summer we studied covering morphisms in $\mathcal{G}$, the category of groupoids. For a groupoid $\tilde{A}$, we define for $i \in I = V(\tilde{A})$, $\tilde{A}_{i*} = \bigcup_{j \in I} \tilde{A}_{ij}$. We say that $\alpha : \tilde{A} \to A$ is a covering morphism if α is functor such that $\alpha_{|\tilde{A}_{i*}} = \alpha_{i*} : \tilde{A}_{i*} \to A_{\alpha(i)*}$ is a bijection $\forall i \in I$. We proved that the category $\mathrm{Cov}(A)$ consisting of all covering morphisms of A is equivalent to the category $\mathcal{S}^A$, the category of representations of A (a representation of A is simply a functor mapping A to $\mathcal{S}$, the category of sets). The main result we proved about covering morphisms is that they satisfy a lifting property analogous to the homotopy lifting property of topological spaces. If $\theta : G \to A$ is a functor and $\alpha : \tilde{A} \to A$ is a covering map, we say that θ can be lifted to $\tilde{A}$ if there exists a unique functor $\tilde{\theta} : G \to \tilde{A}$ such that $\tilde{\theta}\alpha = \theta$. In this case, $\tilde{\theta}$ is called a lifting of θ.

Theorem 5 *Suppose $\alpha : \tilde{A} \to A$ is a covering of groupoids. Let $\theta : G \to A$ be a functor. Let H be a retract of G and $\theta_0 = \theta_{|H}$. If θ_0 lifts to $\tilde{\theta}_0 : H \to \tilde{A}$, then θ has a unique lifting $\tilde{\theta} : G \to \tilde{A}$ extending $\tilde{\theta}_0$.*



In the coming weeks we plan to examine how one may define homology and cohomology groups on a groupoid. Once we accomplish this task, we plan to apply the tools gained this past summer to explore how we may extend the notion of cardinality to category theory. Through our readings, we learned that one may define the cardinality of a category by looking at automorpihsm classes of its objects. Now that we have covered the fundamentals of category theory, we plan to more deeply dissect this definition and determine what results can be derived from it.

References

[1] Higgins, Philip J. (1971). *Notes on Categories and Groupoids*, Van Nostrand Reinhold Company, London.

[2] Baez, John C. and Dolan, James (2000). *From Finite Sets to Feynman Diagrams*, arXiv:math/0004133.