SUMMARY OF EVAN MILLER'S ARTU RESEARCH

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Ever since Joseph Fourier's seminal study of heat conduction in the early 19th century, the classical *heat equation*,

$$u_t = \Delta u,$$

has been one of the cornerstones of partial differential equations and mathematical physics. Yet, this equation allows for faster-than-light conduction of heat, which is inconsistent with special relativity. Various authors have therefore proposed replacing the (parabolic) heat equation by variants of the (hyperbolic) *telegraph equation*,

$$c^{-2} u_{tt} + u_t = \Delta u,$$

which reduces to the heat equation in the limit $c \to \infty$. While propagation is correctly limited to the speed of light c, this solution is rather unsatisfying, being more "wave-like" than "heat-like," with the attendant problems of shocks and regularity that arise in moving from parabolic to hyperbolic equations.

Recently, though, Yann Brenier proposed a new approach to relativistic heat conduction, based on the theory of *optimal transportation*, which concerns moving mass from one distribution to another while minimizing some cost function. Brenier observed that the classical heat equation solves an optimal transportation problem involving Boltzmann entropy and a non-relativistic, kinetic-energy-like cost function. Furthermore, if one replaces this classical cost function by a relativistic cost function—one for which faster-than-light transport is infinitely expensive—one arrives at the *relativistic heat equation*,

$$u_t = \nabla \cdot \frac{u \nabla u}{\sqrt{u^2 + c^{-2} |\nabla u|^2}},$$

whose $c \to \infty$ limit is again the classical heat equation (provided $u > 0$).

Understanding the properties of this nonlinear partial differential equation and its solutions has been the focus of Evan Miller's ARTU research project. First, he shows that, although the classical heat equation exhibits the same behavior for positive, negative, and zero temperatures, the relativistic heat equation is qualitatively different for positive and negative temperatures, and correctly "breaks down" at absolute zero. Second, Evan studies *relativistically harmonic functions*, which are stationary solutions to the relativistic heat equation (by analogy with harmonic functions, which are stationary solutions to the classical heat equation). He proves that these functions exhibit many features in common with their classical counterparts, including maximum/minimum principles and regularity. Finally, he studies time-dependent solutions to the relativistic heat equation, proving that—in contrast with solutions to the classical heat equation—they satisfy a weak maximum/minimum principle but *not* a strong maximum/minimum principle. In fact, a concrete counterexample shows that the strong principle *must* be violated whenever propagation speed is finite.