Categorification

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Categorification

Definition **:** A *category*, C, consists of:

1. A collection of objects, $\mathrm{Ob}(C)$.
2. A collection of morphisms, $\hom(C)$.
3. A law of composition between morphisms.

Along with 2 axioms:

1. Composition of morphisms is associative.
2. Every object has an identity morphism.

Definition **:** *Categorification* is the process of replacing sets, functions, and equations by categories, functors, and isomorphisms.

- Knot theory
- Quantum Mechanics

Finite Sets and Natural Numbers

$\boldsymbol{Definition}$ **:** $FinSet$ is the category of finite sets.

- $\text{Ob}(FinSet) =$ All finite sets and
- for $X, Y \in \text{Ob}(FinSet)$,

$\text{Mor}_{FinSet}(X, Y) =$ All functions from X to Y.

Theorem 1. *$FinSet$ is a categorification of $\mathbb{N}$.*

Isomorphism Classes

$\boldsymbol{Definition}$ **:** Let $\mathcal{A}$ be a category. For $A_1, A_2 \in \mathcal{A}$, we say that A_1 is isomorphic to A_2 if there exists an isomorphism $\alpha \in \mathrm{Mor}_{\mathcal{A}}(A_1, A_2)$ (i.e. there exists $\alpha^{-1} \in \mathrm{Mor}_{\mathcal{A}}(A_2, A_1)$). In this case we write $A_1 \cong A_2$.

Isomorphism defines an equivalence relation, and for $A \in \mathrm{Ob}(\mathcal{A})$, let $[A]$ denote the equivalence class of A.

Isomorphisms in $FinSet$ are bijections.
So, for $X, Y \in \mathrm{Ob}(FinSet)$, $X \cong Y \iff |X| = |Y|$.

We now have a natural map between $\mathbb{N}$ and $FinSet$ where each $n \in \mathbb{N}$ corresponds to the isomorphism class of n-element sets.

Arithmetic

Using products and coproducts we can also categorify arithmetic in $\mathbb{N}$.

In $FinSet$, the coproduct is the disjoint union. Therefore,

$$|X \coprod Y| = |X| + |Y|.$$

Thus the coproduct is the categorification of addition in $\mathbb{N}$.
This is well - defined: $X \cong X'$ and $Y \cong Y' \implies X \coprod Y \cong X' \coprod Y'$.

Similarly, the product is a categorification of multiplication as

$$|X \prod Y| = |X||Y|.$$

Identities

We can also obtain categorified equivalents of the additive and multiplicative identities in $\mathbb{N}$.

Additive identity: The empty set corresponds to the additive identity since

$$X \coprod \emptyset \cong X.$$

Multiplicative identity: The isomorphism class of one element sets corresponds to the multiplicative identity since

$$X \prod (\text{any one element set}) \cong X$$

Thus, $FinSet$ is a categorification of $\mathbb{N}$.

Cardinality

Introducing the notion of cardinality we can obtain a categorification of linear algebra.

$\boldsymbol{Definition:}$ Let X be a category. Let $\underline{X}$ denote the set of isomorphism classes of objects of X. For $x \in \mathrm{Ob}(X)$ let aut(x) denote the automorphism group of x. We let $|X|$ denote the cardinality of X and define it as follows (whenever the sum converges):

$$|X| = \sum_{[x] \in \underline{X}} \frac{1}{|\mathrm{aut}(x)|}$$

If the above sum converges we call the category tame.

<u>Example:</u> The cardinality of a group, G, is $\frac{1}{|G|}$

Example – Finite Sets

In *FinSet* each isomorphism class is simply the collection of sets with a given cardinality $n \in \mathbb{N}$ or the empty set. For an n-element set, X, note that $\mathrm{aut}(X) = S_n$. So, $|\mathrm{aut}(X)| = n!$. Thus we conclude that

$$|FinSet| = \sum_{n=0}^{\infty} \frac{1}{n!} = e$$

Vector Spaces

Now suppose X is any fixed groupoid. We will produce a categorification of $\mathbb{R}^{\underline{X}}$, the $\mathbb{R}$ - vector space of real valued functions defined on $\underline{X}$.

With X fixed, consider all pairs (Ψ, v) where Ψ is a groupoid and $v : \Psi \to X$ is a functor. This naturally gives rise to a real-valued function, $\underset{\sim}{\Psi} \in \mathbb{R}^{\underline{X}}$, as follows:

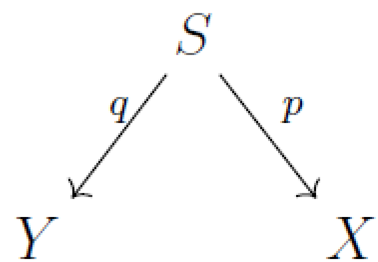
$$\underset{\sim}{\Psi}([x]) = |v^{-1}(x)|.$$

Theorem 1. *The pairs (Ψ, v) are a categorification of $\mathbb{R}^{\underline{X}}$. Every such function in $\mathbb{R}^{\underline{X}}$ can be realized by a pair (Ψ, v).*

Example: The identity functor $i : FinSet \to FinSet$ has categorification $\underset{\sim}{i}([n]) = \frac{1}{n!}$.

Spans

***Definition* :** Let X and Y be groupoids. A *span* from X to Y is pair of functors $p : S \to X$ and $q : S \to Y$.

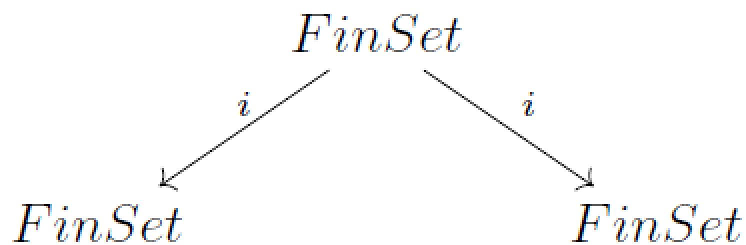


We say a *span is tame* if whenever $v : \Psi \to X$ is tame then $S\Psi$ over Y is tame.

Spans are categorifications of linear transformations from $\mathbb{R}^{\underline{X}}$ to $\mathbb{R}^{\underline{Y}}$ since using pullbacks, a groupoid Ψ over X naturally gives rise to a groupoid $S\Psi$ over Y.

Example

The following is an example of a span from $FinSet$ to $FinSet$.



This span is a categorified version of the identity transformation from $\mathbb{R}^{\underline{FinSet}}$ to itself.

Explicit Formula

Theorem 2. *Let*

$$Y \xleftarrow{\;q\;} S \xrightarrow{\;p\;} X$$

be a tame span of groupoids from X *to* Y. *Then there exists a unique linear transformation* $\underset{\sim}{S} : \mathbb{R}^{\underline{X}} \to \mathbb{R}^{\underline{Y}}$ *such that for any groupoid* Ψ *over* X, $\underset{\sim}{S}\underset{\sim}{\Psi} = \underset{\sim}{S\Psi}$. *Furthermore we have an explicit formula for* $\underset{\sim}{S}$. *For any function* $\psi \in \mathbb{R}^{\underline{X}}$,

$$(\underset{\sim}{S}\psi)([y]) = \sum_{[x]\in\underline{X}} \; \sum_{[s]\in\underline{p^{-1}(x)}\cap\underline{q^{-1}(y)}} \frac{|\mathrm{Aut}(x)|}{|\mathrm{Aut}(s)|}\psi([x]).$$

Continuous Functions

Definition: A topological groupoid is a groupoid whose objects and morphisms form a topological space, and whose source, target, identity, and composition maps are all continuous.

Definition: A Hausdorff groupoid is a topological groupoids whose object and morphism topologies are Hausdorff.

Definition: An étale groupoid is a topological groupoid whose source map is a local homeomorphism.

Example: The groupoid with object set $\mathbb{R}$ and only the identity morphisms with the usual topology on $\mathbb{R}$ is a Hausdorff, étale groupoid.

Continuous Functions

The following theorem is an original result:

Theorem 6. *Suppose X is an open set in $\mathbb{R}^n$. Then every real-valued continuous function has a categorification by a Hausdorff, étale groupoid.*

Generalization:

- Measure
- Lie groupoids

References

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[3] Baez, John C., Hoffnung, Alexander E., and Walker, Christopher D. (2008). *Groupoidification Made Easy*, arXiv:0812.4864.

[4] Pollak, Benjamin (2014). *Senior Thesis - Categories, Groupoids, and Groupoidifications.*