Categorification

Benjamin Pollak

During the Fall of 2014, Professor Tang and I began to apply the tools we obtained from studying category theory over the Summer. In particular, we focused on categorification. Categorification is the process of replacing sets and equations by categories and isomorphisms. Much of the work presented in this report is from the research of Baez, and we refer you to [3] for a more thorough treatment.

Perhaps the simplest object to categorify is the set of natural numbers, $\mathbb{N}$.

Definition 1. *Let FinSet denote the category of finite sets.*
• $\mathrm{Ob}(FinSet)$ = *All finite sets and*
• *for* $X, Y \in \mathrm{Ob}(FinSet)$, $\mathrm{Mor}_{FinSet}(X, Y)$ = *All functions mapping* X *to* Y*.*

It turns out the *FinSet* is actually a categorification of $\mathbb{N}$.

Definition 2. *Let* $\mathcal{A}$ *be a category. For* $A_1, A_2 \in \mathcal{A}$*, we say that* A_1 *is isomorphic to* A_2 *if there exists an isomorphism* $\alpha \in \mathrm{Mor}_{\mathcal{A}}(A_1, A_2)$ *(i.e. there exists* $\alpha^{-1} \in \mathrm{Mor}_{\mathcal{A}}(A_2, A_1)$*). In this case we write* $A_1 \cong A_2$*.*

Definition 3. *Being isomorphic to an object is an equivalence relation. So, for* $A \in \mathrm{Ob}(\mathcal{A})$*, let* $[A]$ *denote the equivalence class containing* A*. We call the equivalence classes isomorphism classes.*

The first thing to note is that in FinSet, two objects are isomorphic if and only if they have the same cardinality as sets. This is because in FinSet an isomorphism is simply a bijective a function, and a bijection exists between two objects in FinSet precisely when those objects have the same cardinality as sets. Thus there is one isomorphism class for each finite cardinal number (i.e. all n-element sets belong to the same isomorphism class). There is now a natural mapping linking $\mathbb{N}$ to FinSet where we let $n \in \mathbb{N}$ correspond to the isomorphism class of n-element sets in FinSet.

With this mapping it is not too difficult to show how *FinSet* is a categorification of $\mathbb{N}$: coproducts play the role of addition and products the role of multiplication. In fact, we even have a categorification of the additive and multiplicative identities in $\mathbb{N}$ - they are the empty set and the isomorphism class of one element sets respectively.

After studying the relatively simple categorification of the natural numbers, we spent much of Fall studying [3] which demonstrates how using groupoids we can obtain a categorification of L^2 - functions. Crucial to this construction is the notion of cardinality.

Definition 4. *Let X be a category. Let $\underline{X}$ denote the set of isomorphism classes of objects of X. For $x \in \mathrm{Ob}(X)$ let aut(x) denote the automorphism group of x. We let $|X|$ denote the cardinality of X and define it as follows (whenever the sum converges):*

$$|X| = \sum_{[x]\in\underline{X}} \frac{1}{|\mathrm{aut}(x)|}$$

If the above sum converges we call the category tame.

Using cardinalities of groupoids, it is possible to obtain a categorification of the vector spaces of real-valued function defined on any discrete set. The main idea is to fix a groupoids, X. Once we have our fixed groupoid, X, we consider all possible combinations (Ψ, v) where Ψ is any other groupoid and v is a functor $v : \Psi \to X$. This naturally gives rise to a real-valued function, $\utilde{\Psi}$, as follows:

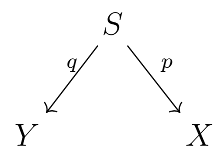
$$\utilde{\Psi}([x]) = |v^{-1}(x)|.$$

If we let $\mathbb{R}^{\underline{X}}$ denote the $\mathbb{R}$ - vector space of functions defined on $\underline{X}$ then each pair (Ψ, v) gives rise to a vector.

Theorem 1. *The pairs (Ψ, v) are a categorification of $\mathbb{R}^{\underline{X}}$. Every such function in $\mathbb{R}^{\underline{X}}$ can be realized by a pair (Ψ, v).*

In addition to categorifying vector spaces, we can also obtain an analogue of linear transformations using groupoids. Linear transformations correspond to spans.

Definition 5. *Let X and Y be groupoids. A span from X to Y is pair of functors $p : S \to X$ and $q : S \to Y$.*

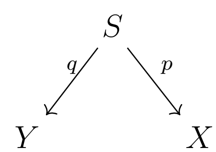


We say a span is tame if whenever $v : \Psi \to X$ is tame then $S\Psi$ over Y is tame.

The reason a span corresponds to al linear transformation is that using pullbacks, a span along with a groupoid over X naturally gives rise to a groupoid over Y.

The following theorem, proved in [3], gives an explicit formula demonstrating how a span corresponds to a linear transformation.

Theorem 2. *Let*



be a tame span of groupoids from X to Y. Then there exists a unique linear transformation $\underset{\sim}{S} : \mathbb{R}^{\underline{X}} \to \mathbb{R}^{\underline{Y}}$ such that for any groupoid Ψ over X, $\underset{\sim}{S}\underset{\sim}{\Psi} = \underset{\sim}{S\Psi}$. Furthermore we have an explicit formula for $\underset{\sim}{S}$. For any function $\psi \in \mathbb{R}^{\underline{X}}$,

$$(\underset{\sim}{S}\psi)([y]) = \sum_{[x]\in\underline{X}} \sum_{[s]\in\underline{p^{-1}(x)}\cap\underline{q^{-1}(y)}} \frac{|\mathrm{Aut}(x)|}{|\mathrm{Aut}(s)|}\psi([x]).$$

Even better than just categorifying linear transformations and vector spaces, we can also obtain a categorification of inner products.

Definition 6. *For two groupoids Ψ and Φ over X, define the inner product of Ψ and Φ to be the pullback. Denote this groupoid by $\langle \Psi, \Phi \rangle$.*

Theorem 3. *Consider the set of groupoids Ψ over X where $\langle \Psi, \Psi \rangle$ is tame and the subspace of $\mathbb{R}^{\underline{X}}$ generated by such groupoids. For any two such groupoids, Ψ and Φ, there exists a unique inner product such that*

$$\langle \underset{\sim}{\Psi}, \underset{\sim}{\Phi} \rangle = |\langle \Psi, \Phi \rangle|.$$

The work in [3] only considers $\underline{X}$ as a discrete set. After studying the work in [3], we began to develop a method to categorify continuous functions between topological spaces using well-behaved groupoids.

Definition 7. *A topological groupoid is a groupoid whose objects and morphisms form a topological space, and whose source, target, identity, and composition maps are all continuous.*

Definition 8. *A Hausdorff groupoid is a groupoid in which the object and morphism topologies are both Hausdorff.*

Definition 9. *An étale groupoid is a topological groupoid whose source map is a local homeomorphism.*

Our end goal is to determine if we can obtain a categorification of functions between arbitrary manifolds. Thus far, the main result we have developed is the following:

Theorem 4. *Suppose X is an open set in $\mathbb{R}^n$. Then every real-valued continuous function has a categorification by a Hausdorff, étale groupoid.*

Over the coming weeks we hope to generalize the results of theorem 4 to a larger class of spaces. In particular, we first want to find a way to categorify quotient spaces of $\mathbb{R}^n$. After that, we wish to extend the results to orbit spaces defined by a topological group acting on another space as well as to cones in $\mathbb{R}^n$.

References

[1] Higgins, Philip J. (1971). *Notes on Categories and Groupoids*, Van Nostrand Reinhold Company, London.

[2] Baez, John C. and Dolan, James (2000). *From Finite Sets to Feynman Diagrams*, arXiv:math/0004133.

[3] Baez, John C., Hoffnung, Alexander E., and Walker, Christopher D. (2008). *Groupoidification Made Easy*, arXiv:0812.4864.