The Relativistic Heat Equation

Maximum Principles and Behavior near Absolute Zero

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The Heat Equation

- The heat equation is the standard model for diffusion and heat flow, going back to Fourier's seminal study of heat conduction in the early 1800s.
- The heat equation is:

$$u_t = \Delta u \tag{1}$$

Theorem

Suppose $u(\cdot, 0) = f$ and $f \in S'$, the class of tempered distributions. Then for each $t > 0$ the solution to the heat equation is given by:

$$u(x,t) = \big(K(\cdot,t) * f\big)(x,t) = \int_{\mathbb{R}^n} f(y) \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} dy \tag{2}$$

- $K(x,t)$ is the heat kernel, and is the solution to the heat equation when the initial condition is the Dirac distribution, δ.

$$K(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} \tag{3}$$

.

- For initial conditions $f \geq 0$, $f \not\equiv 0$ with compact support, we have for each $t > 0$, and each $x \in \mathbb{R}^n$, $u(x,t) > 0$. This poses a major theoretical problem for the heat equation, because Einstein's theory of special relativity does not allow the infinite propagation of signals.

Problems with the Heat Equation

- The heat equation can also be written as a transport equation with transport velocity $-\frac{\nabla u}{u}$:

$$u_t = \nabla \cdot \left(\frac{\nabla u}{u} u \right) \tag{4}$$

- This is clearly the same equation as (1) because $\nabla \cdot \nabla u = \Delta u$.
- The transport velocity $-\frac{\nabla u}{u}$ is clearly unbounded, further evidence that the heat equation does not respect relativity. The transport velocities ought to be bounded by the speed of light.

The Optimal Transportation Viewpoint

- The framework of optimal transportation induces an alternative differential structure (rather than L^2), and we can view these diffusive processes as the gradient descent of a convex entropy functional with respect to this gradient.
- The general evolution equation for this gradient descent problem is given by:

$$u_t = \nabla \cdot \left(u \nabla k^* \left(\nabla \frac{\delta F}{\delta u} \right) \right), \tag{5}$$

  where k is a convex cost function and F is the entropy of the system.
- In both the relativistic and classical case we will take F to be the Boltzmann Entropy:

$$F(u) = \int_U u \log(u), \tag{6}$$

  where $u \geq 0$ and $U \subset \mathbb{R}^n$ is open In this case we find that:

$$\frac{\delta F}{\delta u} = \log(u) + 1 \tag{7}$$

  Taking the gradient we find that:

$$\nabla \frac{\delta F}{\delta u} = \nabla \log(u) = \frac{\nabla u}{u} \tag{8}$$

Optimal transport and the standard heat equation

- The general evolution equation for this gradient descent problem is given by:

$$u_t = \nabla \cdot \left(u \nabla k^* \left(\nabla \frac{\delta F}{\delta u} \right) \right), \tag{9}$$

- In both the relativistic and classical case we will take F to be the Boltzmann Entropy:

$$F(u) = \int_U u \log(u), \tag{10}$$

- For the ordinary heat equation we take our cost function k to be the classical Lagrangian,

$$k(v) = \frac{1}{2}|v|^2 \tag{11}$$

- The Legedre transform of K is the Hamiltonian:

$$k^*(p) = \frac{1}{2}|p|^2 \tag{12}$$

- Plugging into our general PDE, we find,

$$u_t = \nabla \cdot \left(u \nabla k^* \left(\nabla \frac{\delta F}{\delta u} \right) \right) = \nabla \cdot \left(\frac{\nabla u}{u} u \right) = \Delta u \tag{13}$$

Optimal transportation and the relativistic heat equation

- In the relativistic case we take the cost function to be the relativistic Lagrangian:

$$k(v) = \begin{cases} 1 - \sqrt{1 - |v|^2}, & \text{if} \quad |v| \leq 1, \\ +\infty, & \text{otherwise} \end{cases} \tag{14}$$

- As in the classical case, the Legendre dual of the relativistic Lagrangian is the relativistic Hamiltonian:

$$k^*(p) = \sqrt{1 + |p|^2} - 1 \tag{15}$$

- Taking the gradient of this cost function, we find:

$$\nabla k^*(p) = \frac{p}{\sqrt{1 + |p|^2}} \tag{16}$$

- Finally we plug into our general PDE and find:

$$u_t = \nabla \cdot \left(u \nabla k^* \left(\nabla \frac{\delta F}{\delta u} \right) \right) = \nabla \cdot \frac{u \nabla u}{\sqrt{u^2 + |\nabla u|^2}} \tag{17}$$

Behavior near absolute zero

- $$u_t = \nabla \cdot \frac{u \nabla u}{\sqrt{u^2 + |\nabla u|^2}} \tag{18}$$
- $$u_t = \nabla \cdot \frac{u \nabla u}{\sqrt{u^2 + \frac{|\nabla u|^2}{c^2}}} \tag{19}$$
- The standard heat equation is linear so $u(x,t)$ is a solution implies $-u(x,t)$ is a solution.
- For the relativistic heat equation however, $u(x,t)$ a solution implies $-u(x,-t)$ is a solution.
- In the limit as $c \to \infty$ (19) interpolates the the standard heat equation for $u > 0$, but interpolates the reverse heat equation $u_t = -\Delta u$ for $u < 0$.
- What we have observed is that (18) exhibits time reversal symmetry for negative temperatures.

An implicit solution in 1 dimension

- The simplest problem we can look at is the stationary states in 1 dimension. In dimension our elliptic equation is:

$$\frac{\partial}{\partial x}\left(\frac{uu_x}{\sqrt{u^2+u_x^2}}\right)=0 \tag{20}$$

- Because this is an ODE, we can solve implicitly for an exact solution:

$$\frac{uu_x}{\sqrt{u^2+u_x^2}}=C \tag{21}$$

- Squaring both sides we find:

$$\frac{u^2u_x^2}{u^2+u_x^2}=C^2 \tag{22}$$

- Solving for u_x we find:

$$u_x=\frac{Cu}{\sqrt{u^2-C^2}} \tag{23}$$

- Taking the reciprocal we find:

$$\frac{dx}{du}=\frac{\sqrt{u^2-C^2}}{Cu} \tag{24}$$

- Integrating both sides we find:

$$x + b = \frac{\sqrt{u^2 - C^2}}{2C} - \frac{C\log(u + \sqrt{u^2 - C^2})}{2u} \tag{25}$$

- Recall that by definition: $\frac{uu_x}{\sqrt{u^2+u_x^2}} = C$
- Therefore, $C = 0 \implies u$ is constant. So if u is not constant, then $C \neq 0$, and the implicit solution we found above will hold for some choice of constants.
- The boundary value problem $u(0) = 0, u(L) = A > 0$ does not hold for any choice of constants. The equation is ill posed!
- The boundary value problem described above corresponds to setting one end of a thin rod to absolute zero.

Analyticity and Ellipticity

Theorem (Morrey)

Suppose $\phi(x, u, Du, D^2u) = 0$ *is a uniformly elliptic PDE and* ϕ *is analytic in* x, u, Du, D^2u*. If* u *is a solution to* $\phi(x, u, Du, D^2u) = 0$*, then* u *is real analytic.*

Theorem

Let $U \subset \mathbb{R}^n$ *be open and bounded. Suppose* $u \in C^2(U) \cap C(\overline{U})$*,* $u > 0$*, and* $\nabla \cdot \frac{u\nabla u}{\sqrt{u^2+|\nabla u|^2}} = 0$*. Then u is real analytic in U.*

Proof.

Our PDE is analytic in it's arguments and uniformily elliptic, so we just apply Morrey's theorem.

$$\nabla \cdot \frac{u\nabla u}{\sqrt{u^2 + |\nabla u|^2}} = \frac{u^3\Delta u + u|\nabla u|^2\Delta u - u(\nabla u)^T \mathrm{Hess}(u)\nabla u + |\nabla u|^4}{(u^2 + |\nabla u|^2)^{3/2}}$$

$$\nabla \cdot \frac{u\nabla u}{\sqrt{u^2 + |\nabla u|^2}} = |\nabla u|^4 + \sum_{i=1}^{n}\sum_{j=1}^{n}\Big((u^3 + u|\nabla u|^2)\delta_{ij} - uu_{x_i}u_{x_j}\Big)u_{x_ix_j},$$

□

Strong Maximum Principle

Lemma

Suppose u *is real analytic in* $U \subset \mathbb{R}^n$*, and u has a local maximum at* $x \in U$*. Then there exists* $r > 0$ *such that for all* $y \in B(x, r), \nabla u(y) \cdot n \leq 0$*, where* n *is the unit normal,* $n = \frac{y-x}{|y-x|}$.

Proof.

This is just the statement that non-constant real analytic functions can have only isolated local maxima. It is stated in this somewhat inelegant form, because this is the precise statement we will need to prove the strong maximum principle. □

Theorem (Strong maximum principle for the relativistic Laplace equation)

Let $U \subset \mathbb{R}^n$ *be a connected, bounded open set. Suppose* $u \in C^2(U) \cap C(\overline{U})$, $\nabla \cdot \frac{u \nabla u}{\sqrt{u^2 + |\nabla u|^2}} = 0$*, and* $u > 0$ *on U. If u attains its maximum on the interior of U, then u is constant on U. In other words, if there exists* $x \in U$ *such that* $u(x) = \sup_{\overline{U}} u$*, then u is constant on U.*

Strong Maximum Principle (cont.)

Proof.

$$\int_{B(0,r)} \nabla \cdot \frac{u\nabla u}{\sqrt{u^2+|\nabla u|^2}} dy = 0 \tag{26}$$

Next we apply the Stokes Divergence Theorem to (26) and find:

$$\int_{\partial B(0,r)} \frac{u}{\sqrt{u^2+|\nabla u|^2}} \nabla u \cdot n \, dS(y) = 0. \tag{27}$$

For all $y \in \partial B(0,r)$,

$$\nabla u(y) \cdot n = 0 \tag{28}$$

For every $y \in B(x,\delta)$:

$$u(y) - u(x) = \int_0^{|y-x|} \frac{\partial}{\partial r} u(x+rn) dr = \int_0^{|y-x|} \nabla u(x+rn) \cdot n \, dr = 0$$

Hence we can conclude $u = M$ in $B(x,\delta)$. This establishes that $u^{-1}\left[\{M\}\right] = \{x \in U : u(x) = M\}$ is open and closed, by continuity, because U is a connected open set, the onlyclosed and open, nonempty set in the supspace topology is U, so $u^{-1}\left[\{M\}\right]$=U. $u = M$ on U. □

Ultrastrong Maximum Principle for strict subsolutions

Theorem (Ultrastrong maximum principle for strict subsolutions)

Let $U \subset \mathbb{R}^n$ be a connected, bounded open set. Suppose $u \in C^2(U) \cap C(\overline{U})$, $\nabla \cdot \frac{u \nabla u}{\sqrt{u^2+|\nabla u|^2}} > 0$, and $u > 0$ on U. Then for all $x \in U$, $u(x) < \sup_{\overline{U}} u$.

Proof.

Let $M = \sup_{\overline{U}} u$. Suppose towards contradiction $\exists x \in U$ s.t $u(x) = M$. Then clearly u has a local max at x. This implies that $\nabla u(x) = 0$. But we know that:

$$\nabla \cdot \frac{u \nabla u}{\sqrt{u^2+|\nabla u|^2}} = \frac{u^3 \Delta u + u|\nabla u|^2 \Delta u - u(\nabla u)^T \text{Hess}(u) \nabla u + |\nabla u|^4}{(u^2+|\nabla u|^2)^{3/2}} > 0.$$

Therefore we can conclude that $\Delta u(x) > 0$. This means that u cannot have a local max at x, contradicting our assumption that $\exists x \in U$ such that $u(x) = M$. This completes the proof. □

Weak Maximum Principle, time dependent case

Theorem (Weak maximum principle for the relativistic heat equation)

Let $U \subset \mathbb{R}^n$ be a connected, bounded open set, $U_T = U \times (0, T)$. Suppose $u \in C^2(U_\infty) \cap C(\overline{U_\infty})$ and $u_t = \nabla \cdot \frac{u \nabla u}{\sqrt{u^2 + |\nabla u|^2}}$ with $u > 0$ in U_∞. Then:

$$\sup_{\overline{U_T}} u = \max \left\{ \sup_{\partial U \times [0,T]} u, \sup_{U \times \{t=0\}} u \right\}$$

More straightforwardly, the maximum must be attained on the boundary of U_T, and not at the top.

Proof.

Let $u^\epsilon(x,t) = u(x,t) - \epsilon t$. Clearly we have: $u_t^\epsilon = u_t - \epsilon$.

$$u_t^\epsilon(x,t) \geq 0 \implies \Delta u_t^\epsilon(x,t) > 0 \tag{29}$$

So u^ϵ cannot attain it's max on the interior or the top. □

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