THE RELATIVISTIC HEAT EQUATION: ARTU FINAL REPORT

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This semester, and the past summer, I worked on a generalization of the heat equation designed to better comport with relativity. The heat equation is one of the oldest equations in PDE, going all the way back to Fourier. But while the heat equation is an effective and simple model for diffusion processes, it has serious theoretical problems related to relativity. The heat kernel is a Gaussian with standard deviation $\frac{1}{\sqrt{t}}$. This means that even with finitely supported initial conditions, for all positive times, the solutions to the heat equation on the real line will have the whole real line as support. This infinite propagation of signals is an obvious violation of relativity.

This shortcoming of the heat equation was addressed from an optimal transportation point of view in [1]. In this paper Brenier replaces the standard heat equation with:

$$u_t = \nu \nabla \cdot \frac{u \nabla u}{\sqrt{u^2 + \frac{\nu^2 |\nabla u|^2}{c^2}}} \tag{1}$$

For the most part we will work with this equation in dimensionless terms with $\nu, c = 1$, in which case the equation becomes:

$$u_t = \nabla \cdot \frac{u \nabla u}{\sqrt{u^2 + |\nabla u|^2}} \tag{2}$$

Note that for positive values of u, the ordinary heat equation is recovered in the limit as c goes to infinity. The standard heat equation can be derived using optimal transportation methods, as a gradient descent problem of the appropriate entropy functional [4]. If one replaces the classical Lagrangian with the relativistic Lagrangian as a cost function, then (2) becomes the solution to the optimal transportation problem. We are also interested in the elliptic case, where we set $u_t = 0$, and look at the equilibrium temperature distribution. This equation corresponds to Laplace's equation in the classical case. The relativistic analog of Laplace's equation is given by:

$$\nabla \cdot \frac{u \nabla u}{\sqrt{u^2 + |\nabla u|^2}} = 0 \tag{3}$$

One interesting difference between the standard heat equation and the relativistic heat equation is that the nonlinear character of the latter leads it to respect absolute zero. The standard heat equation is linear and any constant function is a solution, so no special behavior is observed around absolute zero. If u is a solution, than so is $u - C$, for any $C \in \mathbb{R}$. Along similar lines, if u is a solution to the standard heat equation that so is $-u$. On the other hand, for the relativistic heat equation $u(x,t)$ a solution implies that $-u(x,-t)$ is a solution, so negative temperatures result in a time reversal symmetry. Both the standard and relativistic heat equation are invariant under positive scalar multiples: $u(x,t)$ is a solution implies $u(\lambda x, \lambda t)$ is a solution for every $\lambda > 0$. Furthermore, even the elliptic equation (3) behaves interestingly around absolute zero. The simplest problem involving stationary states of the equation is to take the unit interval in one dimension, and let $u(0) = 0$ $u(1) = C$. This corresponds to finding the equilibrium of a thin rod with fixed temperatures at the ends for the relativist equation. Shockingly, this problem is ill-posed. However, when you think about it, this is not so surprising after all. Because we cannot simply add or subtract a constant and still have a solution to (2), our choice of units matters. We must use Kelvin (or any other unit where the temperature of absolute zero in those units is zero). The linearity of the heat equation means that from it's point of view their is no absolute zero, however, for the (2), $u(0) = 0$ corresponds to setting the temperature at one end of the bar to absolute zero. This is completely unphysical, so it is a good sign that our equation comes back ill posed. A strong model should not only be able to model reasonable situations correctly; it should also break down in situations where it is no longer reasonable. The relativistic heat equation, unlike the standard heat equation, respects absolute zero.

The bulk of my work during the semester has been focused on extending the maximum principles for the standard heat equation and Laplace's equation to the the relativistic heat equation (2) and the stationary states of the relativistic heat equation (3). Because we do not have an explicit average value theorem for solutions to (3) like we do for harmonic functions, it is much more difficult to prove a strong maximum principle for the stationary states in the relativistic case. In the end, we had to use the fact that solutions to (3) must be analytic on the interior by a previous general result about nonlinear elliptic equations proven in [2]. We then picked an integral estimate that allowed us to exploit both the Gauss divergence theorem and the mean value theorem for integrals to obtain the desired result. In the case of the relativistic heat equation, we proved the weak maximum principle by using the substitution $v_\epsilon(x,t) = u(x,t) - \epsilon t$. We can prove a weak maximum principle for v_ϵ for any $\epsilon > 0$ by extending our result from the elliptic case, and then take the limit $\epsilon \to 0$ to complete the proof. This approach is entirely analogous to the way the weak maximum principle for the standard heat equation is proved in any undergraduate course in PDE, such as in [3]. We actually cannot do any better than a weak maximum principle for the standard heat equation, because finite propagation speeds means we can exploit the light cone to find counterexamples. If $u(x,t)$ solves (2) for $t > 0$, and $u(\cdot, 0)$ is constant on an interval $[a, b]$, then for each $t \in (0, \frac{b-a}{2})$, $u(\cdot, t)$ will also be constant on $[a+t, b-t]$. Basically, if u is constant on an interval at $t = 0$, it will stay constant on an interval that shrinks inward at the speed of light. This proves that we cannot have a strong maximum principle for (2) and that solutions to (2) are not necessarily analytic.

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