Truncated Toeplitz Operators

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I spent this winter learning about truncated Toeplitz operators under the supervision of Professor Rochberg. Classical Toeplitz operators are defined on the Hardy space $H^2(\mathbb{T})$. If $f \in L^\infty(\mathbb{T})$ and M_f is the corresponding multiplication operator, the Toeplitz operator with symbol f is $T_f = P_{H^2} M_f$, where P_{H^2} is the projection onto H^2. Much is known about these operators, and they have interesting applications in many areas of mathematics. For example, they are used in certain index theorems in noncommutative geometry.

Recently, Sarason introduced the notion of a truncated Toeplitz operator and established many of their basic properties [2]. If u is an inner function, uH^2 is a shift-invariant subspace, and we define the model space $K_u = H^2 \ominus uH^2$. Such subspaces are useful for modeling a large class of contraction operators [1]. If P_u is the projection onto K_u and $f \in L^\infty(\mathbb{T})$, we define the truncated Toeplitz operator A_f on K_u as $A_f = P_u M_f$.

I am currently trying to find necessary and sufficient conditions on f for A_f to be in the Schatten classes S_p, where p ranges from 1 to infinity. The class S_p consists of all operators whose singular values are l^p-summable. (Recall that the singular values of an operator T are the eigenvalues of $|T| = \sqrt{T^*T}$.) The analogous problem for classical Toeplitz operators has an elegant answer: T_f is in S_p if and only if the symbol function f is in the analytic Besov space B_p [3]. However, this question remains open for truncated Toeplitz operators. I am currently exploring past results concerning Schatten class Toeplitz and Hankel operators on other function spaces and hope to build on them to resolve this problem.

References

[1] N.K. Nikolskii. *Treatise on the Shift Operator: Spectral Function Theory.* Springer, 2011.

[2] D Sarason. Algebraic properties of truncated Toeplitz operators. *Oper. Matrices*, 2007.

[3] K. Zhu. *Operator Theory in Function Spaces.* American Mathematical Society, 2nd edition, 2007.