QUANTIZATION OF DYNAMICAL BILLIARDS

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Consider two point-like particles constrained to move in one dimension and bounded by two fixed walls (figure 1). The particles collide elastically with each other and experience specular reflection at the walls, as governed by Newtonian mechanics. This is an example of a dynamical billiards system. Our objective is to understand the quantization of this system. In other words, we would like a description of this system that both comports with quantum mechanics and when passed through an appropriate limit, reduces to the classical picture.

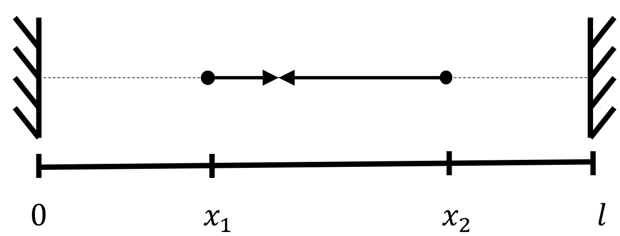

Figure 1: Schematic of a dynamical system of two billiard-like particles stuck between two walls

We can describe the two billiard-like particles as a single particle in coordinate space, where each point in the space corresponds to a particular configuration of both particles, as seen in figure 2. The problem has now been reduced to understanding the motion of a single particle in two dimensions.
We can further simplify the problem by rescaling the triangular boundary such that the composite particle behaves like a billiard. The path of a billiard constrained by a polygonal boundary is of particular mathematical interest. The motion naturally corresponds to a geodesic on a Riemann surface [3]. Therefore, our task is to understand quantization on flat Riemann surfaces with possibly a nontrivial topological structure.

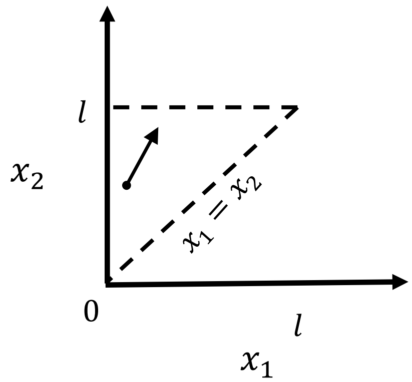

Figure 2: Transformation of two particles in one dimension to one particle in two dimensions. The particle is bounded by $x_1 = 0, x_2 = l,$ and $x_1 = x_2$.

There are a number of ways one might go about quantizing this system. One approach is through the path integral formulation of quantum mechanics. Extensive research has been done on the quantization of chaotic billiard dynamical systems and a technique dependent upon Feynman path integrals has been employed by Gutzwiller [2]. We may be able to apply similar techniques to the polygonal boundaries we consider here. An alternative approach is to make use of Markov chain Monte Carlo (MCMC) methods. MCMC methods have proven useful in quantizing certain problems in quantum field theory, and presumably, may be applied to our much more elementary system [4].

So far, my efforts have been devoted to understanding quantum mechanics at a mathematically rigorous level. This entailed studying some functional analysis, for which, I have been following Reed and Simon's *Functional analysis*. I have also investigated the path integral formulation of quantum mechanics and MCMC methods as they pertain to quantization of problems in field theory. Further preliminary work will include studying Riemann surfaces.

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