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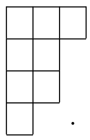
My work has focused around the representation theory of the symmetric groups, with much of my reading being from [1]. In general, a (finite-dimensional) representation of a group G is a homomorphism $\varphi : G \to GL_n(\mathbb{F})$ for some field $\mathbb{F}$, or equivalently, a module over the group algebra $\mathbb{F}[G]$. If we restrict ourselves to $\mathbb{F} = \mathbb{C}$, for a fixed group, every representation can be decomposed into a direct sum of certain irreducible representations for that group, and the total number of such irreducible representations (up to isomorphism) is the same as the number of conjugacy classes in the group. Also, if we consider the character of a representation $\varphi : G \to GL_n(\mathbb{C})$, defined as $\chi : G \to \mathbb{C}$ by $\chi(g) = \mathrm{tr}(\varphi(g))$, it is a function that is constant on conjugacy classes of G, i.e. it is a class function. It turns out that up to isomorphism, a character is unique to the representation that generates it. Thus, considering characters is equivalent to considering representations.

Moving to the symmetric groups specifically, There is a standard construction of the irreducible $\mathbb{C}[S_n]$-modules, known as the Specht modules S^λ for partitions $\lambda \vdash n$. This opens up the representation theory of S_n to combinatorial techniques, as the conjugacy classes in S_n are naturally determined by cycle type of the permutations, which are given by integer partitions $\lambda \vdash n$. The standard way that this analysis is brought about is through Young diagrams and Young Tableaux. The Young diagram of a partition $\lambda = (\lambda_1, \ldots, \lambda_l) \vdash n$ is the set of integer lattice points $\{(i,j) | 1 \leq j \leq l, 1 \leq i \leq \lambda_j\}$, and is often viewed as a diagram of 1×1 boxes, with l rows and λ_j boxes in the jth row. For example, the Young diagram of $(3,2,2,1) \vdash 8$ is



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A standard Young tableau of shape λ is defined to be a filling of the Young diagram of $\lambda \vdash n$ with the numbers 1 through n in such a way that every row and every column are increasing. For example

1	2	3
4	5	
6	7	
8		

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is a Young tableau of shape $(3, 2, 2, 1)$, whereas

1	5	8
4	2	
6	3	
7		

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is a nonstandard tableau. Letting f^λ denote the number of standard Young tableau of shape λ, we have that the dimensions of the Specht modules are given by $\dim(S^\lambda) = f^\lambda$.

Inspired by a problem in Stanley's *Enumerative Combinatorics* [2], we investigated when certain polynomials in the number of fixed points of a permutation define character of S_n for all $n \in \mathbb{N}$. One result that we have been able to show bounds the size of the dimension of S^λ based on n and the number of rows of λ.

References

[1] Bruce E. Sagan. *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions.* Graduate Texts in Mathematics. Springer, 2001.

[2] Richard P. Stanley. *Enumerative Combinatorics.* Number v. 2 in Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1999.