Homology Representations of the Symmetric Groups[1]

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- a multiplication $G \times A \to A$, $(g, x) \mapsto gx$ such that $\forall x \in A$ and $\forall g, h \in G$
  - $1_G x = x$ and
  - $(gh)x = g(hx)$.

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- Equivalently, G-module: A vector space V, with a multiplication $G \times V \to V$, $(g, \mathbf{v}) \mapsto \mathbf{v}$, such that $(\forall \mathbf{v}, \mathbf{w} \in V)(\forall c, d \in \mathbb{C})(\forall g, h \in G)$,
  - $1_G \mathbf{v} = \mathbf{v}$.
  - $(gh)\mathbf{v} = g(h\mathbf{v})$.
  - $g(c\mathbf{v} + d\mathbf{w}) = c(g\mathbf{v}) + d(g\mathbf{w})$.

An Example: Permutation Representations

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- Multiplication: $g(r_1 a_1 + \cdots + r_m a_m) = r_1(ga_1) + \cdots + r_m(ga_m)$ ($r_i \in \mathbb{C}, a_i \in A, g \in G$).

Homology

- Chain complex: $(C_., \partial_.)$ a sequence

$$\cdots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \xrightarrow{\partial_{-1}} \cdots$$

  of vector spaces C_k ("chain spaces") and linear transformations $\partial_k : C_k \to C_{k-1}$ ("boundary maps") such that $\partial_k \circ \partial_{k+1} = 0$ $(\forall k \in \mathbb{Z})$.

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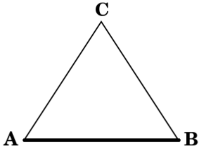
- $\partial_k \circ \partial_{k+1} = 0 \iff \mathrm{Im}(\partial_{k+1}) \subset \mathrm{Ker}(\partial_k)$.
- Homology spaces: $H_k := \dfrac{\mathrm{Ker}(\partial_k)}{\mathrm{Im}(\partial_{k+1})}$

Example: Homology of a triangle

We assign a chain complex to the boundary of triangle ΔABC:



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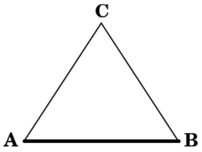

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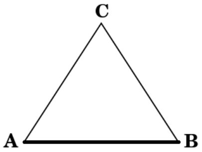

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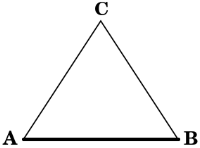

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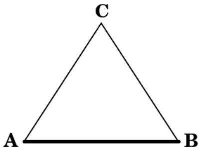

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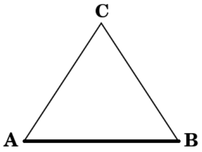

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- It is easy to check that in this case we have
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- Further generalization (via category theory): Simplicial sets.

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- Suppose further: multiplication by group elements commutes with boundary maps.
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- Then the homology spaces become G-modules in a natural way.

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- It turns out that $g(AB + BC + CA) = \text{sign}(g)(AB + BC + CA)$, where $\text{sign}(g) = \begin{cases} 1 & \text{if } g \text{ is an even permutation} \\ -1 & \text{if } g \text{ is an odd permutation} \end{cases}$.

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  of vector spaces C_k and linear transformations $\partial_k : C_k \to C_{k-1}$ such that $\partial^N = 0$.
  - So a 2-complex is a chain complex.
- Choose $k \in \mathbb{Z}$, $1 \leq i < N$. Then

$$\cdots \xrightarrow{\partial^i} C_{k+N-i} \xrightarrow{\partial^{N-i}} C_k \xrightarrow{\partial^i} C_{k-i} \xrightarrow{\partial^{N-i}} \cdots$$

  is a chain complex. The homology at C_k is

$$H_{k,i} = \frac{\ker(\partial^i : C_k \to C_{k-i})}{\operatorname{Im}(\partial^{N-i} : C_{k+N-i} \to C_k)}$$

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- In fact, the usual chain complex associated to the simplicial set is just the case $N = 2$, $\omega = -1$.

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- Boundary map:

  $\partial_k((a_1, \ldots, a_k)) = \sum_{i=1}^{k} \omega^{i-1}(a_1, \ldots, \hat{a}_i, \ldots, a_k)$ ($\hat{a}_i$ indicates that a_i has been removed from the list).

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- $(W_., \partial_.)$ is an N-complex, the (total) word complex.
- I_k, subspace of W_k spanned by k-tuples with no repeats. Then $(I_., \partial_.)$ is also an N-complex, the injective word complex.

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- S_n acts on the collection of (injective) k-tuples of the numbers $1, \ldots, n$: W_k and I_k are permutation modules for S_n.

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- This action of S_n commutes with the boundary maps ∂_k, so $H_{k,i}(W)$ and $H_{k,i}(I)$ are S_n-modules as well.

Total Word Complex

Theorem:

Total Word Complex

Theorem: The complex $(W_., \partial_.)$ has trivial homology. That is, for all k and all $0 < i < N$, $H_{k,i}(W) = 0$.

Injective Word Complex

Conjecture: For the injective word complex, all of the homology is "concentrated towards the top".

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$$\cdots \xrightarrow{\partial^i} I_{k+N-i} \xrightarrow{\partial^{N-i}} I_k \xrightarrow{\partial^i} I_{k-i} \xrightarrow{\partial^{N-i}} \cdots ,$$

all of the homology spaces except the top one are the zero space, unless the sequence starts

$$I_n \xrightarrow{\partial} I_{n-1} \xrightarrow{\partial^{N-1}} \cdots ,$$

in which case the top two homology spaces may be nontrivial.

Injective Word Complex

Theorem: If the homology is concentrated as conjectured, then the collection of these homology modules for all n breaks up naturally into sequences of representations $(X_n)_{n=1}^{\infty}$ (where each X_n is a representation of S_n) that are what Stanley, in his *Enumerative Combinatorics*, calls elementary sequences. Moreover, the case $N = 2$, $\omega = -1$ gives exactly the sequence of representations that he gives as an example there.

Current Work

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- We are currently trying to use methods of algebraic discrete Morse theory as well as methods involving spectral sequences to prove our conjecture about the concentration of the homology of the injective word complex.

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- We are currently trying to use methods of algebraic discrete Morse theory as well as methods involving spectral sequences to prove our conjecture about the concentration of the homology of the injective word complex.
- We are also using the zig-zag lemma, applied to the triple I, W, and the quotient complex W/I, to convert the conjecture to an equivalent conjecture about W/I, the "non-injective word complex", which seems to behave better with respect to the manipulations necessary for the algebraic discrete Morse theory, and opens up more possibilities for the use of spectral sequences.

Future work

We can also assign N-complexes analogous to W and I to any simplicial complex. We are particularly interested in whether there are connections between these homology spaces and the usual simplicial homology for a given simplicial complex.