Geometric Theory of Boundary Value Problems for Schrödinger Operators

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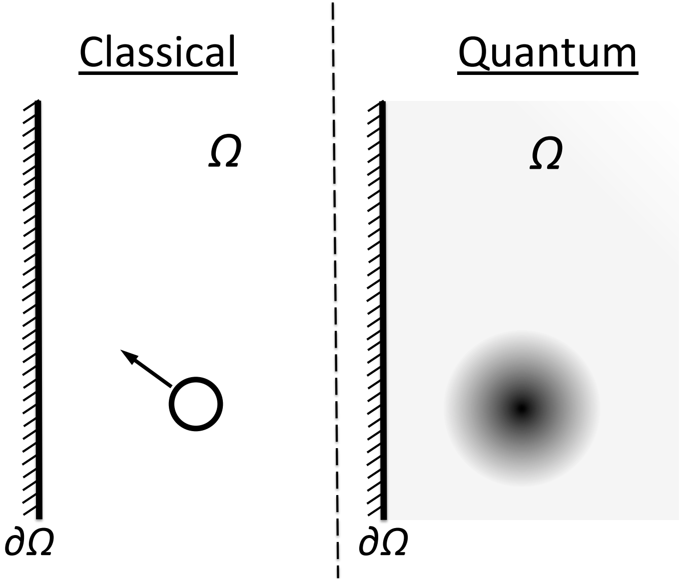
Outline

Motivation



Motivation

- We can describe the "state of the system" with a unit vector ψ in an $L_2(\Omega)$ Hilbert space .
- That is, $\psi \in L_2(\Omega) = \{f : \Omega \to \mathbb{C} : \|f\|^2 := \int_\Omega |f(\mathbf{x})|^2 d\mathbf{x} < +\infty\}$.
- ψ also satisfies the Schrödinger equation $\boxed{\imath\hbar \frac{\partial}{\partial t}\psi = \frac{-\hbar^2}{2m}\Delta\psi}$ with $\boxed{\text{appropriate boundary conditions}}$.

Self-adjoint Extensions of the Hamiltonian Operator

$$\boxed{\imath\hbar\frac{\partial}{\partial t}\psi = \frac{-\hbar^2}{2m}\Delta\psi \rightarrow \imath\hbar\frac{\partial}{\partial t}\psi = \hat{H}\psi}$$

- For a free particle, the differential operation $\frac{-\hbar^2}{2m}\Delta$ generates the hamiltonian operator $\hat{H}$
- $\hat{H}_o$ with domain $D(\hat{H}_o) = C_o^\infty$, is called the initial symmetric operator, where C_o^∞ is the set of all compactly supported smooth functions in $L_2(\Omega)$.
- For simple one dimensional regions, self-adjoint extensions of $\hat{H}_o$ correspond to different boundary conditions.

Complex Symplectic Vector Spaces

Definition (complex symplectic space)

A complex symplectic space is a complex linear space V with a two form $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ satisfying the following properties for $u, v \in V$

- $\langle \cdot, \cdot \rangle$ is linear in its first entry.
- $\langle \cdot, \cdot \rangle$ is skew-hermitian. That is, $\langle u, v \rangle = -\overline{\langle v, u \rangle}$.
- $\langle \cdot, \cdot \rangle$ is nondegenerate. Meaning, if $\langle u, v \rangle = 0$, $\forall v \in V$, then $u = 0$.

Definition (skew-orthogonality)

We call two vectors $u, v \in V$ skew-orthogonal if $\langle u, v \rangle = 0$. Furthermore, we call two subspaces $W, U \subset V$ skew-orthogonal, written as $\langle W, U \rangle = 0$, if $\forall w \in W$ and $\forall u \in U$, $\langle w, u \rangle = 0$.

Lagrangian Subspaces

Definition (Lagrange plane)

For a finite dimensional complex symplectic space V with complex symplectic form $\langle \cdot, \cdot \rangle$, a Lagrange plane L is a subspace of V which is skew-orthogonal to itself $\langle L, L \rangle = 0$ and complete in the following sense. If $\langle v, u \rangle = 0 \ \forall u \in L$, then $v \in L$.

Definition (Lagrange Grassmannian)

A Lagrangian Grassmannian is the set of all Lagrange planes for a given complex symplectic space.

Boundary Value Space

- The minimal symmetric extension $\hat{H}_1$ of $\hat{H}_o$ is the closure of $\hat{H}_o$ in $L_2(\Omega)$.
- The maximal extension $\hat{H}_2$ of $\hat{H}_o$ is the adjoint of the minimal symmetric extension of $\hat{H}_o$.

Definition (boundary value space for $\hat{H}_o$)

The boundary value space for $\hat{H}_o$ is the complex symplectic vector space $S = D(\hat{H}_2)/D(\hat{H}_1)$ with complex symplectic product

$$\langle \cdot, \cdot \rangle_S = (\tfrac{-\hbar^2}{2m}\Delta \cdot, \cdot)_{L_2(\Omega)} - (\cdot, \tfrac{-\hbar^2}{2m}\Delta \cdot)_{L_2(\Omega)}.$$

Bringing it All Together

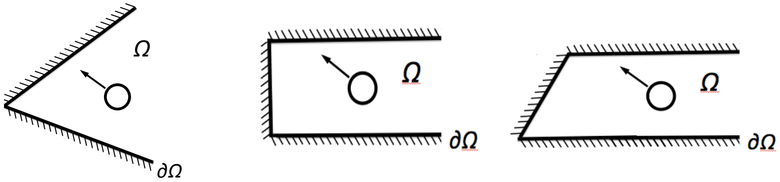
Theorem (Glazman-Krein-Naimark-Everitt-Markus)

There is a bijective correspondence between the Lagrangian Grassmannian for the boundary value space of $\hat{H}_o$ and the set of all self-adjoint extensions of $\hat{H}_o$. Furthermore, each complete Lagrangian gives the domain of the corresponding self-adjoint extension.

- For example, the aptly named Dirichlet-Lagrangian is given by $L_{Dr} := \{[f] \in S : f \in W^2(\Omega) \text{ and } f|_{\partial\Omega} = 0\}$, where $W^2(\Omega)$ is the Sobolev Hilbert space.

Current Investigations

- Geometric classification of all possible self-adjoint extensions for $\hat{H}_o$.
- Does conservation of probability imply self-adjointness?



- Different billiard tables
- Quantization
- Connection to Scattering Theory

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