ARTU Research Project: Third Report

Anthony Grebe under the direction of Professor Victor Wickerhauser

For the better part of this semester, my ARTU project has been a detour from my previous study of high-dimensional data sets. This semester, I have focused on learning about wavelets on $\mathbb{R}^1$, but the ultimate goal of my project will be to generalize methods of wavelet analysis so that they can be used to study data sets on arbitrary-dimensional manifolds.

To construct a wavelet, one first defines the father function, which satisfies the two-scale equation

$$\phi\left(\frac{x}{2}\right) = \sum_n \sqrt{2}h(n)\phi(x-n)$$

for some given set of coefficients $h(n)$. In addition, ϕ has the property that $\phi(x)$ is orthogonal to any integer translate of itself (that is, $\langle \phi(x), \phi(x-n) \rangle = \delta_{0n}$). The simplest example of such a function is the Haar father function with $\phi(x) = \chi_{[0,1)}(x)$ and $h(0) = h(1) = 1/\sqrt{2}, h(n) = 0 \; \forall \, n \neq 0, 1$. From the father function, one defines the wavelet function

$$\psi\left(\frac{x}{2}\right) = \sum_n \sqrt{2}g(n)\phi(x-n)$$

with $g(n) = h(1-n)(-1)^n$. (For the Haar wavelet, $\psi(x) = \chi_{[0,1/2)}(x) - \chi_{[1/2,1)}(x)$.) The significance of wavelets is that any function that can be written $f(x) = \sum_n a_n^0 \phi(x-n)$ can also be written as $\sum_n a_n^1 \phi\left(\frac{x}{2}-n\right) + d_n^1 \psi\left(\frac{x}{2}-n\right)$. If f is smooth and not rapidly varying, the coefficients d_n (which store information about the differences between nearby values of f) will be small, which makes wavelets useful in data compression and analysis. This process of finding a_n^1 and d_n^1 from the original coefficients a_n^0 is called the wavelet transform, and in practice, it is iterated multiple times by using a_n^1 to construct a_n^2 and d_n^2 and so on. Both the wavelet transform and the reconstruction algorithm (to obtain the original coefficients a_n^0 from a_n^1 and d_n^1) are computationally simple, making them practical computational tools.

In her paper "Orthonormal Bases of Compactly Supported Wavelets," Daubechies explains how a large class of wavelets and father functions can be constructed. In particular, her focus on compactly supported wavelets is useful in signal processing, where all signals are necessarily of finite length. I have read the paper and have implemented the wavelet transform and reconstruction algorithms for one of the wavelets she describes.

Currently, I am investigating methods of calculating the original set of coefficients a_n from the function f. In typical applications, $f(x)$ is defined on a discrete set of points and is not an exact linear combination of the functions $\phi(x-n)$, so it is necessary to estimate coefficients a_n that best describe the function. In practice, the crude estimation $a_n \approx f(n)$ is often used, but I am investigating the error in this approximation to see if better estimates are possible.