ARTU Spring 2015 Report *

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April 15, 2015

Over the past semester, my research work has mainly involved learning the scheme-theoretic viewpoint through which papers such as [1, 2] are rightly understood. Schemes offer the benefit of not requiring a fixed embedding of a curve, as the notion of a variety does. Moreover, scheme theory, in a sense, contains the entirety of Galois theory: If E is an extension field of F, then there is a bijective correspondence between the automorphisms of E fixing F and the automorphisms of $\operatorname{Spec} E$ commuting with the map $\operatorname{Spec} E \to \operatorname{Spec} F$ induced by the inclusion of fields. The interplay between the geometric notion of a morphism of curves and the Galois-theoretic behavior of induced maps of points is central to the mathematics I have been studying.

In working over $\mathbb{Q}$, we set $\mathbb{P}^1 = \operatorname{Proj} \mathbb{Q}[x_0, x_1]$ per the construction in [3]. We wish to examine curves over $\mathbb{P}^1$ whose specializations control the Galois theory of associated classes of polynomials. In [1] Bruin and Elkies consider trinomials in Y of the form $Y^n + TY + T$. Such a trinomial can be viewed as corresponding to the rational point T on $\mathbb{P}^1$ (more precisely the prime ideal of $\mathbb{Q}[x_0, x_1]$ generated by $x_1 - Tx_0$). Let $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ be the map sending to T precisely those Y satisfying $Y^n + TY + T = 0$. (Here f is actually defined as sending to the ideal generated by an irredicible polynomial $g(x_0, x_1)$ any prime ideal containing $g(x_0^{n-1}x_1 + x_0^n, -x_1^n)$. The map of sheaves is given by $x_0 \mapsto x_0^{n-1}x_1 + x_0^n$ and $x_1 \mapsto -x_1^n$.)

Given a scheme S and a point p, there is some neighborhood of p isomorphic to an affine scheme $\operatorname{Spec} A$. The residue field of S at p may be defined as $\kappa_p = A_p/m$, where m is the maximal ideal of the localized ring A_p (cf. exercise II.2.7 of [3]). It is easy to check that κ_p is independent of our choice of neighborhood. Since each element of the stalk at p of the sheaf on the space $\operatorname{sp} S$ uniquely determines an element of A_p as the image of p, we have a canonical map taking the single point of $\operatorname{Spec} \kappa_p$ to p in S. The fibered product of schemes is proved always to exist in [3].

$$\begin{array}{ccc} C \times_{\mathbb{P}^1} \operatorname{Spec} \kappa_p & \longrightarrow & \operatorname{Spec} \kappa_p \\ \downarrow & & \downarrow \\ C & \longrightarrow & \mathbb{P}^1 \end{array}$$

A first step in understanding the specialization behavior of a curve C over $\mathbb{P}^1$ is to describe the fibered product in the above diagram. If $C = \mathbb{P}^1$ and the bottom arrow is f, then for a rational point p corresponding to T, over which f is not ramified, the fibered product is given by $\coprod_i \operatorname{Spec} K_i$, where for each irreducible factor h_i of $x^n + Tx + T$ we have that K_i is the field extension of $\kappa_p = \mathbb{Q}$ obtained by adjoining a root of h_i. Elkies and Bruin go on to consider a curve over $\mathbb{P}^1$ corresponding to the Galois closure of the map $\mathbb{Q}(x) \to \mathbb{Q}(x)$ of function fields induced by f. The fibered product now obtained turns out to capture the entire Galois behavior of the trinomial $x^n + Tx + T$.

*This material is based upon work supported by the National Science Foundation under agreement No. DMS-1055897. Any opinions, findings and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

Future directions for study include analyzing the above fibered products by means of representation theory, seeing how the genus of a curve influences the situation, and looking at some questions concerning specialization addressed in [2].

References

[1] Nils Bruin and Noam D. Elkies. Trinomials $ax^7 + bx + c$ and $ax^8 + bx + c$ with Galois groups of order 168 and $8 \cdot 168$. *Algorithmic Number Theory*, 2369:172–188, 2002.

[2] Pierre Dèbes and François Legrand. Specialization results in Galois theory. *Transactions of the American Mathematical Society*, 365(10):5259–5275, October 2013.

[3] Robin Hartshorne. *Algebraic Geometry*, volume 52 of *Graduate Texts in Mathematics*. Springer-Verlag, 1977.