Kapranov N-Complexes and Elementary Sequences of S_n-Characters*

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Abstract

We examine a class of N-complexes, as defined by Kapranov in [5]. These turn out to be related to interesting S_n-representations. Specifically, the Lefschetz characters of these N-complexes give elementary sequences, as defined by Stanley in problem 7.65 in [13], and address an open problem he poses there.

1 Introduction

N-complexes were first introduced by Kapranov in the 1996 paper [5]. They are a generalization of chain complexes, relaxing the requirement that $\partial^2 = 0$ to just $\partial^N = 0$ for some N. Kapranov develops a nice theory of the homology of N-complexes. Some results include that the homology of an N-complex can be put together to form an $N-1$ complex (and in the case $N = 3$ this chain complex is exact), development of the hom and tensor product functors of N-complexes, and defining the homotopy category of N-complexes.

Considering N-complexes of $\mathbb{F}[G]$-modules for G a finite group and $\mathbb{F}$ a field, the homology modules of such N-complexes will also be $\mathbb{F}[G]$-modules. Thus, sequences of representations $\mathfrak{X}_k$ of G over F that intertwine with the boundary maps of an N-complex as $\mathfrak{X}_{k-1}(g)\partial_k = \partial_k\mathfrak{X}_k(g)$ give us new $\mathbb{F}$-representations of G from the homology modules. Once the concept of N-complexes has been introduced, this is a straightforward generalization of the concept of homology representations of chain complexes.

In problem 7.65 in [13], Richard Stanely defines a sequence of characters of symmetric groups $\psi_n : S_n \to C$ to be elementary if for every character value in the sequence is plus or minus the degree of some (weakly) earlier character, or 0. He gives an example there, after which he poses the open questions: “What other ‘interesting’ elementary sequences are there? Can all elementary sequences be completely classified?”([13], page 470)

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The example that Stanley gives in the problem is the sequence of characters afforded by the representations of S_n on the homology of the injective word complex on n letters. These were studied, for example, by Reiner and Webb in [10]. Earlier work in topological combinatorics had shown that the injective word complex (on n letters) could be realized as an $(n-1)$-dimensional regular CW complex, and that this CW complex was homotopy equivalent to a wedge of $(n-1)$-spheres. As such, its (reduced) homology is concentrated in the top dimension, and so the action of S_n on the regular CW complex yields a single representation of S_n on the homology group. Reiner and Webb then used this to derive representation theoretic analogs of several well-known combinatorial formulas related to the derangement numbers, which are the degrees of the corresponding S_n representations.

We generalize the sequence from Stanley's example by constructing analogous N-complexes, which will still be called the injective word complexes. While in the context of chain complexes the injective word complex on n letters gives a single S_n-representation to use, when we convert to N-complexes each of the N-complexes (for $N > 2$) give multiple such representations. After a little bit of work, it turns out that these, too, can be arranged into elementary sequences, and so the N-complexes actually give rise to an infinite family of elementary sequences. We also investigate a related class of N-complexes, called the (total) word complexes, which contain the injective word complexes as subcomplexes.

The rest of the paper is arranged as follows. In section 2 we briefly go through some definitions of combinatorial objects that will be needed later in connection with the symmetric groups and their representations. All relevant definitions and results from the representation theory and character theory of finite groups are presented in section 3, and we explain those constructions and results that are specific to the symmetric groups and their representation theory in section 4. Section 5 gives one of the definitions of the Grothendieck group of a ring from the end of chapter III, §4 of [6], which provides a broader context for results that would otherwise just be about group characters. This is especially the case for the results we discuss in section 6 about certain aspects of chain complexes. Section 7 defines and establishes only the most elementary properties of the q-analog numbers, factorials, and binomial coefficients, which will be needed for "the q-analog of homological algebra", as Kapranov calls the study of N-complexes.

We start section 8 by re-introducing N-complexes and defining several connected notions. The rest of the section is then dedicated to defining, and proving the correctness of, a generalization of Kapranov's construction of an N-complex of $\mathbb{C}$-vector spaces for a simplicial set. Having set up a general framework for construction of N-complexes, in section 9 we move to specifics, and are concerned with the construction and basic properties of the total word complexes and injective word complexes. Section 10 is then dedicated to showing that the total word complexes have trivial homology. Our investigation of the injective word complexes, including the elementary sequences, is in section 11.

2 Partitions and Young Tableaux

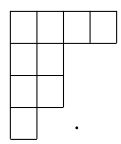
Recall that for $n \in \mathbb{N}$, a *partition of* n is a weakly decreasing sequence $\lambda = (\lambda_1 \lambda_2, \dots, \lambda_l)$ of positive integers such that $\sum_{i=1}^{l} \lambda_i = n$. We write $\lambda \vdash n$ to mean that λ is a partition of n. We may also write λ in the form $(1^{m_1}, 2^{m_2}, \dots, n^{m_n})$ where for $1 \leq i \leq n$, m_i is the number of times i appears in the list $\lambda_1, \dots, \lambda_l$. If $m_i = 0$ for some i, we often leave out the corresponding term. Further, for any i with $m_i = 1$, we usually leave off the superscript.

Example 2.1. The partition $(4, 2, 2, 1) \vdash 9$ can also be written $(1^1, 2^2, 3^0, 4^1, 5^0, 6^0, 7^0, 8^0, 9^0)$, $(1^1, 2^2, 4^1)$, or $(1, 2^2, 4)$.

A *composition of* n is a sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ of non-negative integers such that $\sum_{i=1}^{l} \lambda_i = n$. Clearly every partition of n is a composition of n.

Definition 2.2. The *Ferrers diagram* of a partition $\lambda = (\lambda_1, \dots, \lambda_l) \vdash n$ is the set of integer lattice points $\{(i, j) \in \mathbb{Z}^2 | 1 \leq j \leq l,\ 1 \leq i \leq \lambda_j\}$, and is often viewed as a diagram of 1×1 boxes, with l rows and λ_j boxes in the jth row.

Example 2.3. The Ferrers diagram of $(4, 2, 2, 1) \vdash 9$ is



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Definition 2.4. Let $\lambda \vdash n$. A *Young tableau of shape* λ or a λ*-tableau* is an array t obtained by filling in the boxes of the Ferrers diagram of λ with the numbers $1, \dots, n$ such that each number appears exactly once. A *generalized Young tableau of shape* λ is an array T obtained by filling in the boxes of the Ferrers diagram of λ with positive integers, allowing repetitions. The *content of* T is the composition $\mu = (\mu_1, \mu_2, \dots, \mu_m)$ where μ_i is the number of times i appears in T and m is the largest number appearing in T.

Example 2.5.

5	1	4	7
9	6		
2	3		
8			

is a $(4, 2, 2, 1)$-tableau. On the other hand,

1	2	1	5
1	3		
3	2		
6			

is not a Young tableau, but it is a generalized Young tableau of shape $(4, 2, 2, 1)$ and content $(3, 2, 2, 0, 1, 1)$.

Note that a Young tableau is exactly a generalized Young tableau of content (1^n).

Definition 2.6. A *semistandard Young tableau of shape* λ or a *semistandard* λ*-tableau* is a generalized Young tableau with rows that are weakly increasing and columns that are strictly increasing. The *Kostka number* $K_{\lambda\mu}$ is the number of semistandard λ-tableau of content μ.

Example 2.7.

$$\begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 5 \\ \hline 2 & 3 \\ \cline{1-2} 3 & 6 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array}$$

is a semistandard $(4,2,2,1)$-tableau of content $(3,1,2,1,1,1)$. Neither of the arrays from Example 2.5 are semistandard.

Example 2.8. Let $\lambda = (3,1)$. We compute all nonzero $K_{\lambda\mu}$ with $\mu \vdash 4$. For $\mu = (3,1)$ we get the single tableau

$$\begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 \\ \cline{1-1} \end{array}$$

so $K_{(3,1),(3,1)} = 1$. For $\mu = (2,1,1)$ we have

$$\begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \text{ and } \begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}$$

so $K_{(3,1),(2,1,1)}$. For $\mu = (2,2)$ we have

$$\begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 \\ \cline{1-1} \end{array}$$

so $K_{(3,1),(2,2)} = 1$. Finally, for $\mu = (1,1,1,1)$ we have

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} \end{array}\ , \quad \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1} \end{array}\ , \text{ and } \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1} \end{array}$$

so $K_{(3,1),(1,1,1,1)} = 3$.

Definition 2.9. Say $\lambda \vdash n$. Two λ-tableaux are said to be *row equivalent* if the corresponding rows of the two tableaux contain the same elements. A λ*-tabloid* is an equivalence class of λ-tableaux under row equivalence. The λ-tabloid given as the equivalence class of a tableau t is drawn the same as t but without the vertical lines.

Example 2.10. The (2,1)-tabloid containing the (2,1)-tableau

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}$$

is

$$\begin{array}{cc} \hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}\ .$$

Fix $\lambda = (\lambda_1, \ldots, \lambda_l) \vdash n$. It is easy to see that the action of S_n on the set of λ-tableaux by permuting their entries gives us an action of S_n on the set of λ-tabloids. Moreover, it is clear that this action is isomorphic to the action of S_n on P_λ, the set of ordered tuples $(S_1, S_2, \ldots, S_l)$ of subsets of $\{1, \ldots, n\}$ that are pairwise disjoint, whose union is all of $\{1, \ldots, n\}$, and which satisfy $|S_i| = \lambda_i$ for all $1 \leq i \leq l$.

Note that in a given such l-tuple $(S_1, \ldots, S_l)$, any one S_i is determined by all of the others. For any $1 \leq i \leq l$, let $P_{\lambda,i}$ denote the collection of $(l-1)$tuples obtained by removing the i^{th} set from every l-tuple in P_λ. Then it is obvious that the action of S_n on $P_{\lambda,i}$ given by the natural bijection from P_λ to $P_{\lambda,i}$ is the same as the action of S_n on $P_{\lambda,i}$ given by permuting the elements of the sets in a given $(l-1)$-tuple. So we have that the action of S_n on the set of λ-tabloids is isomorphic to the action of S_n on $P_{\lambda,i}$ for any i.

In particular, letting $\lambda = (1^k, n-k)$ and chosing $i = 1$, this says that the action of S_n on $(1^k, n-k)$-tabloids is isomorphic to the S_n action on sequences of length k that contain no repeats and whose elements are from $\{1, \ldots, n\}$, i.e. words of length k from the alphabet $\{1, \ldots, n\}$ with no repeated letters.

3 Representation Theory of Finite Groups

All results given without proof in this section are proved in [3]. For the case when the coefficient field is $\mathbb{C}$, a more elementary presentation can be found in Chapter 1 of [12].

Throughout this paper, G is a finite group. Let R be a ring. The *group ring of G over R*, denoted $R[G]$, is the collection of formal R-linear combinations of elements of G. That is, $R[G] = \left\{ \sum_{g \in G} a_g g \middle| a_g \in R \right\}$. $R[G]$ is an R-module in the obvious way, and we also give it a multiplication by $\left(\sum_{g \in G} a_g g\right)\left(\sum_{g \in G} b_g g\right) = \sum_{g,h \in G} a_g b_h (gh)$, which makes $R[G]$ a ring. When $R = \mathbb{F}$ is a field, $\mathbb{F}[G]$ is called the *group algebra of G over $\mathbb{F}$*.

Definition 3.1. An *$\mathbb{F}$-representation of G* (or a *representation of G over $\mathbb{F}$*) is a group homomorphism $\mathfrak{X} : G \to GL_n(\mathbb{F})$. The number n is called the *dimension* or the *degree* of the representation. When the field $\mathbb{F}$ is clear, we may also call $\mathfrak{X}$ a representation of G or a G-representation.

Example 3.2. For any group, the *trivial representation* of G, $1_G : G \to GL_1(\mathbb{F})$, is given by $1_G(g) = [1]$.

In order to avoid complications, we require $\mathbb{F}$ to be algebraically closed. Note that this doesn't restrict our notion of a representation because every field has an algebraic closure, and because if $\mathbb{E} \hookrightarrow \mathbb{F}$ is a field extension, then matrices over $\mathbb{E}$ can be viewed as matrices over $\mathbb{F}$. For a discussion of $\mathbb{F}$-representations of G when $\mathbb{F}$ is not necessarily algebraically closed, see chapters 9 and 10 of [3].

As will be explained shortly, $\mathbb{F}$-representations of G are equivalent to $\mathbb{F}[G]$-modules. Here, and throughout the rest of this paper, by a module over a ring we mean a finitely generated left module over that ring. Also, if V and W are $\mathbb{F}[G]$-modules, we will write $V \cong_{\mathbb{F}[G]} W$ to indicate that they are isomorphic as $\mathbb{F}[G]$-modules.

Towards seeing the equivalence of $\mathbb{F}$-representations of G and $\mathbb{F}[G]$-modules, first note that a representation $\mathfrak{X} : G \to GL_n(\mathbb{F})$ can be uniquely extended to an $\mathbb{F}$-algebra homomorphism $\overline{\mathfrak{X}} : \mathbb{F}[G] \to M_n(\mathbb{F})$, given by $\overline{\mathfrak{X}}(\sum_{g\in G} a_g g) = \sum_{g\in G} a_g \mathfrak{X}(g)$. Now $\mathbb{F}^n$ becomes an $\mathbb{F}[G]$-module with the multiplication given by $rv = \overline{\mathfrak{X}}(r)v$ for all $r \in \mathbb{F}[G]$ and $v \in \mathbb{F}^n$.

On the other hand, say V is an $\mathbb{F}[G]$-module. The inclusion $\mathbb{F} \to \mathbb{F}[G]$, $a \mapsto a \cdot 1_G$, shows that V is an $\mathbb{F}$-vector space. Note that for every $g \in G$, the map $\tilde{\mathfrak{X}}(g) : V \to V$ given by $v \mapsto gv$ is $\mathbb{F}$-linear. Moreover, $\tilde{\mathfrak{X}}(g^{-1}) = \tilde{\mathfrak{X}}(g)^{-1}$, so $\tilde{\mathfrak{X}}(g) \in GL(V)$. It is clear that $\tilde{\mathfrak{X}} : G \to GL(V)$ is a homomorphism, and so taking an $\mathbb{F}$-basis for V, we get a representation $\mathfrak{X} : G \to GL_{\dim_{\mathbb{F}}(V)}(\mathbb{F})$.

Now if we start with a representation $\mathfrak{X}$, let $V = \mathbb{F}^n$ be the corresponding $\mathbb{F}[G]$-module, and let $\mathfrak{Y}$ be the representation obtained by using the standard basis for $\mathbb{F}^n$, then $\mathfrak{Y} = \mathfrak{X}$. Similarly, if we start with an $\mathbb{F}[G]$-module V and basis B, let $\mathfrak{X}$ be the corresponding representation, and then let $W = \mathbb{F}^n$ be the corresponding $\mathbb{F}[G]$ module, then the $\mathbb{F}$-linear map taking B to the standard basis for $\mathbb{F}^n$ is an $\mathbb{F}[G]$-module isomorphism. In light of this equivalence between $\mathbb{F}$-representations of G and $\mathbb{F}[G]$-modules, we will often define a concept for one of the two kinds of objects and then use it freely for both.

Example 3.3 (**Permutation representations**)**.** Say G acts on a finite set X. Then for any field $\mathbb{F}$ the function $\mathfrak{X} : G \to GL_{|X|}(\mathbb{F})$ which assigns to $g \in G$ the corresponding permutation matrix is a representation of G called the *permutation representation* corresponding to X. The associated *permutation module* is defined as follows. Let $\mathbb{F}[X]$ denote the $\mathbb{F}$-vector space with basis given by X. We have the natural action $g(a_1x_1 + \cdots + a_mx_m) = a_1(gx_1) + \cdots + a_m(gx_m)$. Extending this by linearity to all of $\mathbb{F}[G]$ makes $\mathbb{F}[X]$ an $\mathbb{F}[G]$-module called the *permutation module* corresponding to X.

We recall several definitions for modules that will be applied to representations as well. These definitions work equally well for modules over an arbitrary ring, so we fix a ring R. An R-module is called *irreducible* if it contains no nontrivial proper submodule. An R-module is called *completely reducible* if it is a direct sum of irreducible submodules. A ring R is called *semisimple* if every R-module is completely reducible.

Theorem 3.4 (**Maschke's Theorem**)**.** *Say G is a group and $\mathbb{F}$ is a field of characteristic not dividing the order of G. Then every $\mathbb{F}[G]$-module is completely reducible. That is, $\mathbb{F}[G]$ is semisimple.*

Proof. This is Theorem 1.9 in [3], and the case $\mathbb{F} = \mathbb{C}$ is Theorem 1.5.3 in [12]. □

In particular, if $\mathbb{F}$ has characteristic 0 then $\mathbb{F}[G]$ is semisimple for any group G.

3.1 Characters and Brauer Characters

The idea of characters of a group is that not all of the information given by a specific representation is required to determine the isomorphism class of the representation.

In particular, if $\mathbb{F}$ has characteristic 0, then rather than assigning n^2 numbers to an element of G for a degree n representation, it suffices to assign to each $g \in G$ a single number.

Definition 3.5. Say $\mathbb{F}$ has characteristic 0. If $\mathfrak{X}$ is an $\mathbb{F}$-representation of G, then the *character of G corresponding to $\mathfrak{X}$* is the function $\chi : G \to \mathbb{F}$ given by $\chi(g) = \mathrm{tr}(\mathfrak{X}(g))$. We may also say that χ is the character of G *arising from*, *given by*, or *afforded by* $\mathfrak{X}$.

Example 3.6. Say χ is the character given by the permutation representation corresponding to the action of G on X. Then $\chi(g)$ is the number of fixed points of g, i.e. $|\{x \in X | gx = x\}|$. In this case χ is called the *permutation character* corresponding to this action.

Example 3.7. The character afforded by the trivial representation 1_G, called the *trivial character* of G, is just the constant function with value 1 on all of G. As such, we also use 1_G to denote the trivial character of G.

Example 3.8. If $\mathfrak{X}$ is a representation of G and χ is the corresponding character, then $\chi(1) = \deg(\mathfrak{X})$. As such, we may also write $\deg(\chi)$ to denote this value.

We will say that a character has a certain property of representations (or of $\mathbb{F}[G]$-modules) if some representation that affords it has that property. In particular, if $\mathfrak{X}$ is an irreducible representation of G, and χ is the corresponding character, then we say that χ is an irreducible character of G.

It is easy to see that isomorphic representations give the same character, because an isomorphism from $\mathfrak{X}$ to $\mathfrak{Y}$, both G-representations of degree n, is given by a $P \in GL_n(\mathbb{F})$ such that for all $g \in G$, $P\mathfrak{X}(g)P^{-1} = \mathfrak{Y}(g)$, and conjugate matrices have the same trace. This same fact immediately shows us that characters of G are *class functions*, i.e. functions on G (with values in $\mathbb{F}$) that are constant on conjugacy classes, because $\chi(hgh^{-1}) = \mathrm{tr}(\mathfrak{X}(h)\mathfrak{X}(g)\mathfrak{X}(h)^{-1}) = \mathrm{tr}(\mathfrak{X}(g)) = \chi(g)$. Although much less obvious, the converse to the first of these claims is also true.

Theorem 3.9. *If $\mathfrak{X}$ and $\mathfrak{Y}$ are two G-representations with corresponding characters χ and ψ, then $\mathfrak{X} \cong \mathfrak{Y}$ if and only if $\chi = \psi$.*

Proof. This is Corollary 2.9 in [3] and the $\mathbb{F} = \mathbb{C}$ case is part 5 of Corollary 1.9.4 in [12]. □

Theorem 3.9 holds because the irreducible characters of G form an orthonormal basis for the space of class functions on G with the inner product being $\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\psi(g)}$. This implies the theorem because every representation of G is a direct sum of irreducible representations, and if χ and ψ are the characters afforded by representations $\mathfrak{X}$ and $\mathfrak{Y}$ then $\mathfrak{X} \oplus \mathfrak{Y}$ affords the character $\chi + \psi$.

When $\mathbb{F}$ has positive characteristic p, one cannot hope for things to go as well. At a surface level, taking the traces can fail horribly, because if $\mathfrak{X}$ is any $\mathbb{F}$-representation of G, then the trace of the representation $\underbrace{\mathfrak{X} \oplus \cdots \oplus \mathfrak{X}}_{p}$ is just 0. This difficulty can be circumvented, however, if one cleverly assigns complex numbers to matrices over a field of characteristic p. Indeed, this can be done, and doing so was a major contribution of Richard Brauer to representation theory, which we will discuss shortly. But there is a deeper problem. Because $\mathbb{F}[G]$ is not semisimple when p divides the order of G, there are many $\mathbb{F}[G]$-modules that are not direct sums of irreducible modules. Thus, no "trace-like" function can completely determine the isomorphism type of representations of G over arbitrary fields of positive characteristic. However, as we will see, there exist such trace-like functions that determine the composition factors of a given $\mathbb{F}[G]$-module.

Towards understanding Brauer's construction, fix a prime p and let $\mathbb{F} = \overline{\mathbb{F}}_p$ be the algebraic closure of the finite field with p elements. It turns out (Lemma 15.1 in [3]) that the group of units $\mathbb{F}^\times$ is isomorphic to the group $U_p = \{\varepsilon \in \mathbb{C} | \varepsilon^m = 1 \text{ for some } m \in \mathbb{Z} \text{ with } p \not| m\}$. So fix an isomorphism $\Phi : \mathbb{F}^\times \to U_p$.

Given a group G, $g \in G$ is said to be *p-regular* if p does not divide the order of g. Let $\mathcal{R}_p = \mathcal{R}_p(G)$ be the set of p-regular elements of G. If $\mathfrak{X}$ is an $\mathbb{F}$-representation of G, the *Brauer character* of G afforded by $\mathfrak{X}$ is the function $\varphi : \mathcal{R}_p \to \mathbb{C}$ defined as follows. For $g \in \mathcal{R}_p$, let $\varepsilon_1, \ldots, \varepsilon_k \in \mathbb{F}^\times$ be the eigenvalues of $\mathfrak{X}(g)$, counting multiplicities (so $k = \deg(\mathfrak{X})$). Then the value of the Brauer character φ at g is defined as $\varphi(g) = \sum_i \Phi(\varepsilon_i)$.

As with characters, it is easy to see that isomorphic representations give the same Brauer character and that Brauer characters are constant on conjugacy classes. Both of these claims are true because $\mathfrak{X}(g)$ and $P\mathfrak{X}(g)P^{-1}$ have the same eigenvalues for any $P \in GL_{\deg(\mathfrak{X})}(\mathbb{F})$.

It is not hard to see that the Brauer character of a representation is determined just by its composition factors. It clearly suffices to show that if V is an $\mathbb{F}[G]$-module and W is a proper submodule, then the Brauer character afforded by V is the sum of the Brauer characters afforded by W and V/W. To see this, pick an $\mathbb{F}$-basis $\mathcal{B}_W$ for W and extend this to an $\mathbb{F}$-basis $\mathcal{B}$ for V. In this basis the representation $\mathfrak{X}$ corresponding to V has the form

$$\mathfrak{X}(g) = \begin{bmatrix} \mathfrak{Y}(g) & \mathfrak{U}(g) \\ 0 & \mathfrak{Z}(g) \end{bmatrix}$$

where $\mathfrak{Y}$ is the representation corresponding to W in the basis $\mathcal{B}_W$ and $\mathfrak{Z}$ is the representation corresponding to V/W in the basis given by the image of $\mathcal{B} \backslash \mathcal{B}_W$ under the quotient. So now what we want to show is that the Brauer character afforded by $\mathfrak{X}$ is the sum of the Brauer characters afforded by $\mathfrak{Y}$ and $\mathfrak{Z}$. But this is obvious, because the eigenvalues of $\mathfrak{X}(g)$ are exactly the eigenvalues of $\mathfrak{Y}(g)$ together with the eigenvalues of $\mathfrak{Z}(g)$.

So the Brauer character corresponding to an $\mathbb{F}$-module is the sum of the Brauer characters corresponding to its composition factors. This immediately gives us that the Brauer character afforded by a representation is determined only by its composition factors. The converse, that a Brauer character determines the composition factors of any representation that affords it, follows from this and the fact (Theorem 15.5 in [3]) that the Brauer characters corresponding to the different isomorphism classes of irreducible $\mathbb{F}$-representations of G are distinct and linearly independent over $\mathbb{C}$. We say that a Brauer character of G is irreducible if it is afforded by an irreducible $\mathbb{F}[G]$-module.

Brauer characters are a very powerful tool in the representation theory of finite groups, both in that they give tools for understanding representations of G in positive characteristic, and in that they give a tool for moving back and forth between characteristic 0 and characteristic p representations of G. One such result is that (Theorems 15.6 and 15.19 in [3]) the restrictions of the irreducible characters of G to $\mathcal{R}_p$ and the irreducible Brauer characters of G are $\mathbb{Z}$-linear combinations of each other.

This result is especially surprising given that the question of whether a given function $f : \mathcal{R}_p \to \mathbb{C}$ is a Brauer character of G is not even well defined until we choose a homomorphism $\Phi : \mathbb{F}^\times \to U_p$. The following known result is an example where the Brauer character afforded by a representation is independent of this choice.

Proposition 3.10. *Let G be a group and let $\overline{\mathbb{F}}_p$ be an algebraically closed field of characteristic p. Say G acts on a set X. Then the Brauer character of G afforded by $\overline{\mathbb{F}}_p[X]$ is the restriction of the character of G corresponding to $\mathbb{C}[X]$ to the p-regular elements of G.*

Proof. Let $\mathfrak{X}$ be the $\overline{\mathbb{F}}_p$-representation of G corresponding to $\overline{\mathbb{F}}_p[X]$ with its usual basis, and let φ be the corresponding Brauer character. Fix a p-regular element $g \in G$. Let $X_1, \ldots, X_m$ be the $\langle g \rangle$-orbits of X. Re-ordering the elements of X into these orbits block-diagonalizes $\mathfrak{X}(g)$. Clearly, if X_l consists of a single element, i.e. a fixed point of g in X, the corresponding block is the 1×1 matrix $[1]$, which therefore contributes 1 to the value of $\varphi(g)$. So, in light of Example 3.6, it suffices to show that for g acting on the $\overline{\mathbb{F}}_p$-vector space generated by an orbit of size > 1, the value of the "Brauer trace" is 0. Let $\{x_1, \ldots, x_k\}$ be such an orbit, labeled such that $gx_i = x_{i+1}$ where the subscripts are taken mod k. Because p does not divide the order of g and k must divide the order of g, $p \nmid k$, so there is some primitive k^{th} root of unity $\omega \in \overline{\mathbb{F}}_p^\times$. So for $0 \le i \le k-1$, $v_i = \displaystyle\sum_{j=1}^{k} \omega^{i(j-1)} x_j$ is an eigenvector of g with eigenvalue ω^i. Now because ω is a k^{th} root of unity in $\overline{\mathbb{F}}_p$, for any choice of the isomorphism $\Phi : \overline{\mathbb{F}}_p^\times \to U_p \subset \mathbb{C}$, $\Phi(\omega)$ is a k^{th} root of unity in $\mathbb{C}$. So the value of the "Brauer trace" of g acting on the space spanned by this orbit is $\displaystyle\sum_{i=0}^{k-1} \Phi(\omega^i) = \sum_{i=0}^{k-1} \Phi(\omega)^i = 0$. □

3.2 Virtual Modules, Representations, and Characters

We know that adding (Brauer) characters of G yields a (Brauer) character of G, and taking the direct sum of two $\mathbb{F}[G]$-modules or of two $\mathbb{F}$-representations of G yields an $\mathbb{F}[G]$-module or $\mathbb{F}$-representation of G, respectively. In all of these cases we would like to be able to have some notion of subtraction as well. We start with the case of $\mathbb{F}[G]$-modules, but our construction will work for modules over any ring R.

Definition 3.11. Let R be a ring. A *virtual R-module* is a formal difference $V - W$ where V and W are R-modules.

Clearly, two virtual modules $V_1 - W_1$ and $V_2 - W_2$ should be considered equal if $V_1 \cong_R V_2$ and $W_1 \cong_R W_2$. But we also want our subtraction to be an inverse for our addition, so we also require that $(V \oplus U) - (W \oplus U) = V - W$. In particular, $(V \oplus W) - W = V$ where we also write V for the formal difference $V - 0$. We would like to make this into a group.

Because we are only considering R-modules that are finitely generated, it is not hard to see that there is a set of R-modules $\mathcal{M} = \mathcal{M}(R)$ such that every R-module V is isomorphic to exactly one module in $\mathcal{M}$, denoted $[V]$ and called, in a useful abuse of notation, the isomorphism class of V. Let $\widetilde{\mathcal{VM}} = \widetilde{\mathcal{VM}}(R)$ be the set of formal differences of elements of $\mathcal{M}$. Letting $\sim$ be the equivalence relation on $\widetilde{\mathcal{VM}}$ generated by $([V \oplus U]) - ([W \oplus U]) \sim [V] - [W]$ for any R-modules V, W, U, define $\mathcal{VM} = \mathcal{VM}(R)$ as the quotient $\mathcal{VM} = \widetilde{\mathcal{VM}}/\sim$. Abusing notation and writing elements of $\widetilde{\mathcal{VM}}$ for their equivalence classes in $\mathcal{VM}$, it is now clear that with the addition $([V_1] - [W_1]) + ([V_2] - [W_2]) = [V_1 \oplus V_2] - [W_1 \oplus W_2]$, $\mathcal{VM}$ is a group, which we will call the group of virtual R-modules.

$\mathcal{VM}$ is what is sometimes known as the Grothendieck group of the commutative monoid $(\mathcal{M}, \oplus)$. We, however, avoid using this notation because we will be using the term "Grothendieck group" to denote a slightly different construction later on (see Section 5).

Moving to the case of $\mathbb{F}$-representations of G, we define a *virtual $\mathbb{F}$-representation* of G to be a formal difference of representations $\mathfrak{X} - \mathfrak{Y}$. It is clear that the same constructions work as in the case of R-modules, with the only difference being that it is somewhat easier to see that there are no set theoretic issues.

For (Brauer) characters, the construction is much easier. Because, for a fixed $\mathbb{F}$, (Brauer) characters of G are complex-valued functions defined on the same domain, we can simply subtract their values pointwise. A *virtual (Brauer) character* of G is just a difference $\chi - \psi$ of (Brauer) characters of G. It is clear that this forms a free abelian subgroup of the space of all class functions, or of the space of all $\mathbb{C}$-valued functions on $\mathcal{R}_p(G)$ that are constant on conjugacy classes, depending on whether $\mathbb{F}$ has characteristic 0 or $p > 0$, respectively. This group will be called the group of virtual (Brauer) characters of G and denoted $\mathcal{VC}(G, \mathbb{F})$.

Because isomorphic $\mathbb{F}[G]$-modules have equal (Brauer) characters, and because the (Brauer) character afforded by a direct sum is the sum of the (Brauer) characters corresponding to the summands, it is clear that the map assigning to an $\mathbb{F}[G]$-module its (Brauer) character descends to a well-defined function from virtual $\mathbb{F}[G]$-modules to virtual (Brauer) characters. For the same reasons, we also immediately get that this map is a group homomorphism $\mathcal{VM}(\mathbb{F}[G]) \to \mathcal{VC}(G, \mathbb{F})$. If $[V] - [W]$ maps to $\chi - \psi$ under this homomorphism, we say that $\chi - \psi$ is the virtual (Brauer) character of G *afforded by* $[V] - [W]$.

3.3 Induced representations

Fix a field $\mathbb{F}$ and a group G. It is clear that for any subgroup $H \leq G$ the inclusion $\mathbb{F}[H] \hookrightarrow \mathbb{F}[G]$ lets us consider any $\mathbb{F}[G]$-module as an $\mathbb{F}[H]$-module. We would like to be able to have a corresponding way to go from $\mathbb{F}[H]$-modules to $\mathbb{F}[G]$-modules.

Definition 3.12. Say $H \leq G$ and V is an $\mathbb{F}[H]$-module. The corresponding *induced module* is the $\mathbb{F}[G]$-module $V \uparrow_H^G := \mathbb{F}[G] \otimes_{\mathbb{F}[H]} V$, where for the tensor product we consider $\mathbb{F}[G]$ an $\mathbb{F}[G]$-$\mathbb{F}[H]$-bimodule in the obvious way.

This definition corresponds to the definition of induced $\mathbb{Z}[G]$-modules given in Chapter III, Section 5 of [1]. We wish to show that this agrees with the usual definitions of induced representations and induced characters. Note that both for induced modules and for the associated notions for representations and characters (once they are defined) we may write $V \uparrow^G$ for $V \uparrow_H^G$ when it will not cause confusion.

Let V be an $\mathbb{F}[H]$-module and pick an $\mathbb{F}$-basis $B = \{b_1, \ldots, b_m\}$ for V. Pick a set $T = \{t_1, \ldots, t_l\}$ of left coset representatives for H in G. Then T is a basis for $\mathbb{F}[G]$ as a right $\mathbb{F}[H]$-module, and so $T \otimes B = \{t_i \otimes b_k | 1 \leq i \leq l, 1 \leq k \leq m\}$ is an $\mathbb{F}$-basis for $V \uparrow_H^G = \mathbb{F}[G] \otimes_{\mathbb{F}[H]} V$.

Given $g \in G$, we want to know how g acts on the basis elements $t_j \otimes b_k$. For each t_j there is a unique i such that $gt_j \in t_i H$, and we write $gt_j = t_i h$, so $t_i^{-1} g t_j = h \in H$. Now $g(t_j \otimes b_k) = gt_j \otimes b_k = t_i h \otimes b_k = t_i \otimes h b_k$. So we see that the following definition corresponds to Definition 3.12.

Definition 3.13. Say $H \leq G$ and fix a set $\{t_1, \ldots, t_l\}$ of left coset representatives for H. If $\mathfrak{X}$ is a representation of H, then the corresponding induced representation $\mathfrak{X} \uparrow_H^G$ is defined by associating to each $g \in G$ the block matrix

$$\mathfrak{X} \uparrow_H^G (g) = (\mathfrak{X}(t_i^{-1} g t_j))_{1 \leq i,j \leq l} = \begin{bmatrix} \mathfrak{X}(t_1^{-1} g t_1) & \mathfrak{X}(t_1^{-1} g t_2) & \cdots & \mathfrak{X}(t_1^{-1} g t_l) \\ \mathfrak{X}(t_2^{-1} g t_1) & \mathfrak{X}(t_2^{-1} g t_2) & \cdots & \mathfrak{X}(t_2^{-1} g t_l) \\ \vdots & \vdots & \ddots & \vdots \\ \mathfrak{X}(t_l^{-1} g t_1) & \mathfrak{X}(t_l^{-1} g t_2) & \cdots & \mathfrak{X}(t_l^{-1} g t_l) \end{bmatrix}$$

where for $g \notin H$ we define $\mathfrak{X}(g) = 0 \in M_{\deg(\mathfrak{X})}(\mathbb{F})$.

This is just a rephrasing of Definition 1.12.2 in [12]. Note that the preceding discussion shows that different choices of the coset representatives give isomorphic representations.

Induced representations give a broad class of representations. The following proposition, which is equivalent to Proposition 1.12.3 in [12] and Lemma 5.14 in [3], gives us examples.

Proposition 3.14. *Say G acts transitively on a set X. Let $x \in X$ and let $H = G_\alpha$, the stabilizer of x in G. Then $(1_H) \uparrow^G$ is isomorphic to the permutation representation corresponding to X.*

Proof. By the orbit-stabilizer theorem, the G-action on X is isomorphic to the G-action on the (left) cosets of H. So it suffices to show the result for the permutation representation action of G on the cosets of H. Fix $T = \{t_1, \dots, t_l\}$ a transversal. Let $\mathfrak{Y} = (y_{ij})$ be the representation $(1_H)\uparrow^G$, and let $\mathfrak{X} = (x_{ij})$ be the permutation representation. Note that for a fixed $g \in G$, both $\mathfrak{Y}(g)$ and $\mathfrak{X}(g)$ are matrices consisting entirely of zeros and ones. So because

$$\begin{aligned} y_{ij}(g) = 1 &\iff t_i^{-1} g t_j \in H \\ &\iff g t_j H = t_i H \iff x_{ij}(g) = 1, \end{aligned}$$

the matrices $\mathfrak{Y}(g)$ and $\mathfrak{X}(g)$ are equal, and we are done. □

We now consider the characteristic 0 case, and look at the character of an induced representation.

Proposition 3.15. *Say $\mathbb{F}$ has characteristic 0. If $\mathfrak{X}$ is an $\mathbb{F}$-representation of $H \leq G$ that gives the character χ, then the character afforded by $\mathfrak{X}\uparrow_H^G$ is given by $\chi\uparrow_H^G(g) := \frac{1}{H}\sum_{x\in G}\chi(x^{-1}gx)$ where we define $\chi(g) = 0$ for $g \notin H$.*

Proof. Note that because χ is a class function on H, for any $h \in H$ and t_i in the transversal T we have that $\chi((t_i h)^{-1} g t_i h) = \chi(h^{-1} t_i^{-1} g t_i h) = \chi(t_i^{-1} g t_i)$. Recall that t_i runs through T and h runs through H the product $t_i H$ runs through all of the elements of G exactly once. So taking the trace of the matrix in Definition 3.13, we have that the induced representation gives the character

$$\begin{aligned} \sum_i \operatorname{tr}(\mathfrak{X}(t_i^{-1} g t_i)) &= \frac{1}{|H|}\sum_{h\in H}\sum_i \operatorname{tr}(\mathfrak{X}((t_i h)^{-1} g t_i h)) \\ &= \frac{1}{|H|}\sum_{h\in H}\sum_i \chi((t_i h)^{-1} g t_i h) \\ &= \frac{1}{|H|}\sum_{x\in G}\chi(x^{-1}gx). \end{aligned}$$

□

In this case we say that $\chi\uparrow_H^G(g) := \frac{1}{H}\sum_{x\in G}\chi(x^{-1}gx)$ is the character induced from χ. This is a specific case of Definition 5.1 in [3] which uses the same formula to define induced class functions.

4 The Symmetric Groups

Recall that a permutation $\pi \in S_n$ can be written uniquely as a product of disjoint cycles in S_n, up to rearrangement. If $\mu_1 \geq \cdots \geq \mu_l$ are the lengths of these cycles written in non-decreasing order, then the partition $\mu = (\mu_1, \dots, \mu_l) \vdash n$ is called the *cycle type* of π. The conjugacy classes in S_n are the sets $K_\mu = \{\pi \in S_n | \pi \text{ has cycle type } \mu\}$ of permutations of a fixed cycle type $\mu \vdash n$. For $\pi \in S_n$, we will also use K_π to denote its conjugacy class.

By the orbit-stabilizer theorem, for any $\pi \in K_\mu$ $|K_\mu| = \dfrac{|S_n|}{|C_G(\pi)|}$. So $|C_G(\pi)|$ is the same for all $\pi \in K_\mu$, and we let z_μ denote this shared value. It is not hard to see (Proposition 1.1.1 in [12]) that if $\mu = (1^{m_1}, \ldots, n^{m_n})$, then $z_\mu = \prod_{j=1}^{n} j^{m_j}(m_j!)$.

We continue this pattern of notations as follows. If χ is any class function of S_n, or even a function that is defined on the union of some conjugacy classes and is constant on each conjugacy class, then we will write $\chi(\mu)$ to denote the value of χ on elements of cycle type $\mu \vdash n$. In particular, we will use this notation for characters and Brauer characters of S_n.

There is also another class of subsets of S_n parameterized by partitions of n. For $\lambda = (\lambda_1, \ldots, \lambda_l) \vdash n$, the corresponding *Young Subgroup* of S_n is

$$S_\lambda = S_{\{1,2,\ldots,\lambda_1\}} \times S_{\{\lambda_1+1,\lambda_1+2,\ldots,\lambda_1+\lambda_2\}} \times \cdots \times S_{\{n-\lambda_l+1,n-\lambda_l+2,\cdots,n\}}.$$

We now move to the representation theory of the symmetric groups. Fix $\mathbb{F}$, an algebraically closed field.

Definition 4.1. Say $\lambda \vdash n$. The *permutation module corresponding to* λ is the $\mathbb{F}[S_n]$-permutation module for the action of S_n on the set of λ-tabloids by permuting the entries.

Let $\lambda = (\lambda_1, \ldots, \lambda_l)$. First note that S_n obviously acts transitively on the set of λ-tabloids. Also, the λ-tabloid

1	2	$\cdots$	λ_1
λ_1+1	λ_1+2	$\cdots$	$\lambda_1+\lambda_2$
	$\vdots$		
$n-\lambda_l+1$	$\cdots$	n	

has stabilizer S_λ. So Proposition 3.14 gives us that $M^\lambda \cong_{\mathbb{F}[S_n]} (1_{S_\lambda}) \uparrow^{S_n}$.

Another class of S_n-representations parameterized by partitions $\mu \vdash n$ are the *Specht modules* S^μ. Section 2.3 of [12] gives their construction, and the following sections prove many of its properties. In particular (Theorem 2.4.6 in [12]), if the characteristic of $\mathbb{F}$ is 0, then the S^μ for $\mu \vdash n$ are, up to isomorphism, all of the irreducible $\mathbb{F}[S_n]$-modules. Another major result is

Theorem 4.2 (**Young's Rule**). *Fix $\lambda \vdash n$. If $\mathbb{F}$ has characteristic 0, the permutation module corresponding to λ decomposes as $M^\lambda \cong \bigoplus_{\mu \vdash n} K_{\mu\lambda} S^\mu$. More generally, for any field $\mathbb{F}$, M^λ has a series of successive submodules such that all of the quotients are isomorphic to Specht modules, and for all $\mu \vdash n$, S^μ appears in the list of quotients with multiplicity $K_{\mu\lambda}$.*

Proof. The characteristic 0 case is Theorem 2.11.2 in [12]. The general case is one case of Theorem 17.13, part (iv) in [4], but the notation for that theorem can be hard to follow, so we also note that it follows immediately from the characteristic 0 case and Corollary 17.14 in [4]. $\square$

5 Grothendieck Groups

Fix a ring R. The group $\mathcal{VM}(R)$ or virtual R-modules gives us a nice way of adding and subtracting R-modules, with the addition corresponding to taking the direct sum of R-modules. But $\mathcal{VM}(R)$ is not particularly useful if we need to also work with quotients of R-modules. This motivates the following definition.

Definition 5.1. The *Grothendieck group* $K(R)$ is the quotient of the group $\mathcal{VM}(R)$ of virtual R-modules by the subgroup generated by $[M] - [N] - [M/N]$ whenever M is an R-module and $N \leq_R M$ is a submodule of M.

As before, we abuse notation and write elements of $\mathcal{VM}(R)$ for their images in $K(R)$. That is, elements of $K(R)$ will be written as $[V] - [W]$ where V and W are R-modules.

In the case of $\mathbb{F}[G]$, it is clear that the map from $\mathcal{VM}(\mathbb{F}[G])$ to $\mathcal{VC}(G, \mathbb{F})$ assigning to a virtual $\mathbb{F}[G]$-module its virtual (Brauer) character induces a map from $K(\mathbb{F}[G])$ to $\mathcal{VC}(G, \mathbb{F})$. As such, we will talk about the virtual (Brauer) character afforded by an element of $K(\mathbb{F}[G])$. Further, the map assigning an element of $K(\mathbb{F}[G])$ to its virtual (Brauer) character is a group isomorphism. This is not hard to see because for a given $\mathbb{F}[G]$-module M, its representative $[M]$ in the Grothendieck group and its (Brauer) character are uniquely determined by the composition factors of M.

For any ring R, we say that an element of $K(R)$ is an *actual module* if it can be written as $[M]$ for some R-module M. Note that any positive integer combination of actual modules is an actual module. Also, in the case $R = \mathbb{F}[G]$ the virtual (Brauer) character afforded by an actual module is a (Brauer) character of G.

If R is semisimple, then $K(R)$ is just $\mathcal{VM}(R)$. This is true because of the following fact: If M is a completely reducible R-module and $N \leq_R M$ is a submodule, then N is a direct summand of M (See Chapter XVII, §2 of [6]). So, in particular, if R is semisimple, M is an R-module, and $N \leq_R M$, then $M \cong N \oplus (M/N)$, so the relation $[M] - [N] - [M/N] = 0$ is already true in $\mathcal{VM}(R)$.

6 Chain Complexes

Chain complexes are the main object of study in homological algebra. While they originally arose in the context of algebraic topology, chain complexes and homological algebra now have applications in many areas of math. See [14] for a thorough introduction to homological algebra, and [9] for many uses of chain complexes in algebraic topology. While chain complexes can be defined in the abstract setting of an abelian category, we only define it in a much more concrete case, though it is a case which is still quite general and in which much of homological algebra can be developed.

Fix a ring R.

Definition 6.1. A *chain complex* (C, ∂) of R-modules is a doubly infinite sequence

$$\cdots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \xrightarrow{\partial_{-1}} \cdots$$

consisting of a collection $C = \{C_n | n \in \mathbb{Z}\}$ of R-modules, sometimes called the *chain spaces*, and R-module homomorphisms $\partial = \partial_n : C_n \to C_{n-1}$ such that each composition $\partial^2 = \partial \circ \partial : C_n \to C_{n-2}$ is zero. We will often drop the "∂" and refer to C itself as the chain complex. The maps ∂ are called the *boundary maps* of C. The module of *n-cycles* of C is $Z_n = Z_n(C) := \ker(\partial : C_n \to C_{n-1})$. The module of *$n$-boundaries* of C is $B_n = B_n(C) := \mathrm{Im}(\partial : C_{n+1} \to C_n)$. The condition $\partial^2 = 0$ says exactly that $B_n \subset Z_n$ for all $n \in \mathbb{Z}$. The n^{th} *homology module* of C is the quotient module $H_n = H_n(C) := \dfrac{Z_n}{B_n}$.

Note that one way to look at a chain complex is as a $\mathbb{Z}$-graded R-module C together with a degree -1 endomorphism $\partial : C \to C$ satisfying $\partial^2 = 0$. For more on this view and its applications, see Chapter VI, and specifically Sections 2 and 7, of [7].

We say that a chain complex C is bounded if only finitely many of the C_ns are nonzero.

Proposition 6.2. *Say C is a bounded chain complex of R-modules. Then as elements of $K(R)$,*

$$\sum_{i\in\mathbb{Z}} (-1)^i [C_i] = \sum_{i\in\mathbb{Z}} (-1)^i [H_i].$$

Proof. Note that for any $i \in \mathbb{Z}$, $\partial_i : C_i \to B_{i-1}$ is a surjective R-module homomorphism with kernel Z_i, so by the first isomorphism theorem $C_i/Z_i \cong_R B_{i-1}$. So in the Grothendieck group we have $[C_i] = [Z_i] + [B_{i-1}]$. Also, because $H_i = Z_i/B_i$ by definition, in the Grothendieck group we have $[H_i] = [Z_i] - [B_i]$. So now

$$\begin{aligned}\sum_i (-1)^i [C_i] &= \sum_i (-1)^i ([Z_i] + [B_{i-1}]) = \sum_i (-1)^i [Z_i] + \sum_i (-1)^i [B_{i-1}] \\ &= \sum_i (-1)^i [Z_i] + \sum_i (-1)^{i+1} [B_i] = \sum_i (-1)^i ([Z_i] - [B_i]) \\ &= \sum_i (-1)^i ([H_i]).\end{aligned}$$

□

Proposition 6.2 is a generalization of the Euler-Poincaré formula, which, in one form, says that for a bounded chain complex C of $\mathbb{F}$-vector spaces for a field $\mathbb{F}$, $\sum_{i\in\mathbb{Z}} (-1)^i \dim_{\mathbb{F}}(C_i) = \sum_{i\in\mathbb{Z}} (-1)^i \dim_{\mathbb{F}}(H_i(C))$. This is a specific case of our formula because if $R = \mathbb{F}$ is a field then $K(\mathbb{F})$ is isomorphic to $\mathbb{Z}$ by the map $[V] \mapsto \dim_{\mathbb{F}}(V)$. This formula, or one like it, is used in algebraic topology to show that the Euler characteristic of spaces is a topological invariant. Indeed Theorem 22.2 in [9], which is used there to show the topological invariance of the Euler characteristic, is a specific case of the result that if C is a chain complex of abelian groups (i.e. $\mathbb{Z}$-modules) with all of the chain spaces free abelian groups, and if $T(A)$ denotes the torsion subgroup of an abelian group A, then $\sum_{i\in\mathbb{Z}} (-1)^i \mathrm{rank}(C_i) = \sum_{i\in\mathbb{Z}} (-1)^i \mathrm{rank}(H_i(C)/T(H_i(C)))$. This result is again an instance of our formula, because $K(\mathbb{Z})$ is isomorphic to $\mathbb{Z}$ by the map sending a finitely generated abelian group A to the rank of its free part. In [9], however, this result is proved by the Hopf trace theorem (Theorem 22.1 there) which states that if C is a chain complex of free abelian groups and if $\phi : C \to C$ is a chain map (i.e. a sequence of maps $\phi : C_i \to C_i$ satisfying $\partial\phi = \phi\partial$), in which case ϕ induces maps ϕ_* on the homology of C, then $\sum_{i\in\mathbb{Z}}(-1)^i \operatorname{tr}(\phi, C_i) = \sum_{i\in\mathbb{Z}}(-1)^i \operatorname{tr}(\phi_*, H_i(C)/T(H_i(C)))$. It is interesting to note that the Hopf trace theorem can be used to show a specific case of our formula, though one has to use a form of the theorem for chain complexes of vector spaces, for which is proved by the same proof given in [9]. Given a bounded chain complex C of $\mathbb{C}[G]$-modules, and considering the traces of the maps $(v \mapsto gv)$ on each C_i and H_i, the Hopf trace formula gives us that the alternating sum of the character values on the chain spaces and on the homology are equal, and then Proposition 6.2 follows from the isomorphism $K(\mathbb{C}[G]) \cong \mathcal{VC}(G, \mathbb{C})$. In this author's personal opinion, though, the proof given here gives more intuition than the technical computations with traces needed to prove the Hopf trace formula.

In light of Proposition 6.2 we make the following definition.

Definition 6.3. The *Lefschetz module* of a bounded chain complex C of R-modules is the element $\tilde{L}(C, R) = \sum_{i\in\mathbb{Z}}(-1)^i[C_i] = \sum_{i\in\mathbb{Z}}(-1)^i[H_i]$ of the Grothendieck group $K(R)$.

Now considering representations of a group, we are led to a corresponding definition.

Definition 6.4. Say C is a bounded chain complex of $\mathbb{F}[G]$-modules. The *Lefschetz (Brauer) character* of C, denoted $\tilde{\Gamma}(C, \mathbb{F})$, is the virtual (Brauer) character of G afforded by the Lefschetz module $\tilde{L}(C, \mathbb{F}[G])$.

Note that $\tilde{\Gamma}(C, \mathbb{F}) = \sum_i(-1)^i\psi_i = \sum_i(-1)^i\chi_i$ where ψ_i is the (Brauer) character afforded by C_i and χ_i is the (Brauer) character afforded by H_i.

7 q-Analogs

In dealing with q-analogs we can start by fixing q to be some element of a ring, but this is not necessary. Rather, we can do our original computations in the polynomial ring $\mathbb{Z}[q]$ (or the field of rational functions $\mathbb{Q}(q)$). Then to consider q an element of a ring R we take the tensor product $R \otimes \mathbb{Z}[q] \cong R[q]$ and map the variable q to that element of R.

For any (positive/nonnegative) integer n, we define the q-number of n to be $[n]_q := \dfrac{1-q^n}{1-q} = \sum_{j=0}^{n-1} q^j$. Now define the q-factorial $[n]_q! = \prod_{j=1}^{n}[j]_q$. Note that as it is the empty product, $[0]_q! = 1$ Finally, for non-negative integers $k \le n$, define the

q-binomial coefficient $\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$. If $k < 0$ or $k > n$, we let $\binom{n}{k}_q = 0$. We now prove several well-known facts about the q-binomial coefficients.

Lemma 7.1. *For any $k, n \in \mathbb{Z}$,*

(a) $\binom{n}{k}_q = \binom{n}{n-k}_q$.

(b) $\binom{n}{k}_q = q^k \binom{n-1}{k}_q + \binom{n-1}{k-1}_q$.

(c) $\binom{n}{k}_q = \binom{n-1}{k}_q + q^{n-k}\binom{n-1}{k-1}_q$.

Proof. (a) This is obvious.
(b) If $k = n$, $\binom{n}{k}_q = 1 = q^k \cdot 0 + 1 = q^k\binom{n-1}{k}_q + \binom{n-1}{k-1}_q$. If $k < 0$ or $k > n$ this is obvious because all terms are 0. And now for $0 \leq k < n$,

$$\begin{aligned} q^k\binom{n-1}{k}_q + \binom{n-1}{k-1}_q &= q^k \frac{[n-1]_q!}{[k]_q![n-k-1]_q!} + \frac{[n-1]_q!}{[k-1]_q![n-k]_q!} \\ &= \frac{[n-1]_q!}{[k-1]_q![n-k-1]_q!}\left(q^k \frac{1}{[k]_q} + \frac{1}{[n-k]_q}\right) \\ &= \frac{[n-1]_q!}{[k]_q![n-k]_q!}\left(q^k[n-k]_q + [k]_q\right) \\ &= \frac{[n-1]_q!}{[k]_q![n-k]_q!}\left(q^k \sum_{i=0}^{n-k-1} q^i + \sum_{j=0}^{k-1} q^j\right) \\ &= \frac{[n-1]_q!}{[k]_q![n-k]_q!}\left(\sum_{j=k}^{n-1} q^j + \sum_{j=0}^{k-1} q^j\right) \\ &= \frac{[n-1]_q!}{[k]_q![n-k]_q!}[n]_q = \binom{n}{k}_q. \end{aligned}$$

(c) $\binom{n}{k}_q = \binom{n}{n-k}_q = \binom{n-1}{n-k-1}_q + q^{n-k}\binom{n-1}{n-k}_q = \binom{n-1}{k}_q + q^{n-k}\binom{n-1}{k-1}_q$.

□

Corollary 7.2. *Say $0 \leq k \leq n$. Then the q-binomial coefficient $\binom{n}{k}_q$ is a polynomial in q.*

Proof. Induction on n. The base case of $n = 1$ is trivial: $\binom{1}{0}_q = \binom{1}{1}_q = \frac{[1]_q!}{[1]_q![0]_q!} = \frac{1}{1} = 1$. Inductive step: We have $\binom{n}{0}_q = \binom{n}{n}_q = \frac{[n]_q!}{[n]_q![0]_q!} = 1$. For $0 < k < n$ we have $\binom{n}{k}_q = q^k\binom{n-1}{k}_q + \binom{n-1}{k-1}_q$, which, by the inductive hypothesis, is a polynomial. □

8 N-Complexes

Definition 8.1. An *N-complex* (C, ∂) of R-modules is a doubly infinite sequence

$$\cdots \xrightarrow{\partial} C_k \xrightarrow{\partial} C_{k-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0 \xrightarrow{\partial} C_{-1} \xrightarrow{\partial} \cdots$$

consisting of a collection $C = \{C_k | k \in \mathbb{Z}\}$ of R-modules, sometimes known as the *chain spaces*, and R-module homomorphisms $\partial : C_k \to C_{k-1}$ such that each composition $\partial^N : C_k \to C_{k-N}$ is zero. We will often drop the "∂" and refer to C itself as the N-complex. The maps ∂ are called the *boundary maps* of C. For $1 \leq i \leq N-1$, let $\ker(\partial^i) \cap C_k$ denote $\ker(\partial^i : C_k \to C_{k-i})$. The condition $\partial^N = 0$ says exactly that $\partial^{N-i}(C_{k+N-i}) \subset \ker(\partial^i) \cap C_k$. The *homology modules* of C are the quotient modules $H_{k,i} = H_{k,i}(C) := \dfrac{\ker(\partial^i) \cap C_k}{\partial^{N-i}(C_{k+N-i})}$ for $k \in \mathbb{Z}$ and $1 \leq i \leq N-1$.

Clearly a 2-complex is the same as a chain complex.

In Section 1 of [5], Kapranov proves several general results, and gives constructions such as the hom complex and tensor product complex of two N-complexes. We, however, will only be concerned with those results that impact our specific cases.

A *bounded* N-complex is one in which all but finitely many of the chain spaces are 0.

A *subcomplex* B of an N-complex C is a doubly infinite sequence of submodules $B_k \subset C_k$ such that $\partial(B_k) \subset B_{k-1}$. So B, together with the restrictions of ∂ to the B_k's, is itself an N-complex.

An N-complex C is called *N-exact* if all of its homology modules $H_{k,i}(C) = 0$.

Let (C, ∂) be an N-complex. For $k \in \mathbb{Z}$ and $1 \leq i \leq N-1$, let $C[k, i]$ denote the chain complex

$$\cdots \xrightarrow{\partial^{N-i}} C_{k+N} \xrightarrow{\partial^i} C_{k-i+N} \xrightarrow{\partial^{N-i}} C_k \xrightarrow{\partial^i} C_{k-i} \xrightarrow{\partial^{N-i}} C_{k-N} \xrightarrow{\partial^i} \cdots,$$

called the *k, i-section of C* or the *section of C at k with step size i*. In particular, $C[k, i]_{2l} = C_{k+lN}$ and $C[k, i]_{2l-1} = C_{k-i+lN}$. So $H_0(C[k, i]) = H_{k,i}(C)$, and more generally, $H_{2l}(C[k, i]) = H_{k+lN,i}$ and $H_{2l-1}(C[k, i]) = H_{k-i+lN,N-i}$.

Note that when we talk about the collection of sections of a given N-complex, we often identify, for a fixed k, i all $C[k + lN, i]$ and $C[k - i + lN, N - i]$ for $l \in \mathbb{Z}$, which are all the same up to re-indexing.

If C is a bounded N-complex of R-modules, then for $k \in \mathbb{Z}$ and $1 \leq i \leq N$, we let $\tilde{L}_{k,i}(C, R) := \tilde{L}(C[k, i], R)$, the Lefschetz module of the section $C[k, i]$. If $R = \mathbb{F}[G]$, then we let $\tilde{\Gamma}_{k,i}(C, \mathbb{F}) := \tilde{\Gamma}(C[k, i], \mathbb{F})$, the Lefschetz (Brauer) character of the section $C[k, i]$.

8.1 Δ-Sets

One of the typical constructions of chain complexes comes from simplicial sets, defined in Chapter 8 of [14] (and Section 3 of [2], and Section 2 of [11]). In [5] Kapranov shows, given a simplicial set, how to construct a corresponding N-complex of $\mathbb{C}$-vector spaces. We generalize this construction to get N-complexes of modules over more general rings. Towards doing this, we first define a Δ-set.

Definition 8.2. A Δ-*set* X consists of a sequence of sets $X_0, X_1, \dots$ and, for all $n \geq 1$ and $0 \leq i \leq n$, functions $d_i : X_n \to X_{n-1}$ satisfying

$$d_i d_j = d_{j-1} d_i \quad \text{for} \quad i < j, \tag{1}$$

called the *face maps*.

In particular, every simplicial set is a Δ-set. Our Δ-sets are referred to as Delta sets in [2] and as semi-simplicial sets in [14]. In those cases the names avoiding the use of the capital Greek Δ are preferred due to the ubiquity of the "Δ" symbol in the study of simplicial sets. Because we are not using any other properties of simplicial sets, we hope our notation will cause no confusion.

Fix a ring R and an element $q \in R$. For any Δ-set X we define a sequence

$$\cdots \xrightarrow{\partial} R[X]_n \xrightarrow{\partial} R[X]_{n-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} R[X]_0 \xrightarrow{\partial} R[X]_{-1} \xrightarrow{\partial} \cdots$$

of R-modules and R-module homomorphisms as follows. The R-modules are given by $R[X]_n = \begin{cases} R[X_n] & n \geq 0 \\ 0 & n < 0 \end{cases}$ where $R[X_n]$ denotes the free R-module generated by the set X_n. To define the homomorphisms, first extend each face map $d_i : X_n \to X_{n-1}$ to a map $d_i : R[X_n] \to R[X_{n-1}]$ by R-linearity, where if d_i is the empty map (i.e. $X_n = \emptyset$), this extension is the zero map. The homomorphisms for $R[X]$ are now given by

$$\partial = \partial_q : R[X_n] \to R[X_{n-1}] \text{ is defined as } \partial_q = \sum_{i=0}^{n} q^i d_i. \tag{2}$$

Proposition 8.3. *Let R be a ring and say $q \in R$ is such that $1 + q + \cdots + q^{N-1} = 0$ for some $N \in \mathbb{N}$. Then for any Δ-set X, $(R[X], \partial)$ is an N-complex.*

Proof. Recalling that $1 + q + \cdots + q^{N-1} = [N]_q$, the following lemma shows us that $\partial^N = 0$, i.e. $(R[X], \partial)$ is an N-complex. ☐

Lemma 8.4. *Say R is a ring, $q \in R$, and X is a Δ-set. Then for any $x \in R[X_m]$ and $N \leq m$*

$$\partial_q^N(x) = [N]_q! \sum_{i_1 \geq \cdots \geq i_N} q^{i_1 + \cdots + i_N} d_{i_1} \cdots d_{i_N}(x).$$

Proof. First, note that it is not important to specify further limits on what the i_js can be, because we can define $d_i : R[X]_n \to R[X]_{n-1}$ to be the zero map if $i < 0$ or $i > n$, so including extra terms will not change the sum. Using this idea we also see that we don't need to consider some specific $x \in R[X_m]$ nor do we need $N \leq m$.
Note that for $N = 1$, $\partial_q^1 = [1]_q! \sum_{i_1} q^{i_1} d_{i_1}$ is true by definition and because $[1]_q! = 1$. Proceeding by induction on N, suppose $\partial_q^N = [N]_q! \sum_{i_1 \geq \cdots \geq i_N} q^{i_1 + \cdots + i_N} d_{i_1} \cdots d_{i_N}$.

Then letting $i_0 = \infty$ and $i_{N+1} = -\infty$, we have

$$\begin{aligned}
\partial_q^{N+1} &= \sum_k q^k d_k \left([N]_q! \sum_{i_1 \geq \cdots \geq i_N} q^{i_1 + \cdots + i_N} d_{i_1} \cdots d_{i_N} \right) \\
&= [N]_q! \sum_{i_1 \geq \cdots \geq i_N} \sum_{j=0}^{N} \sum_{k=i_{j+1}}^{i_j - 1} q^k q^{i_1 + \cdots + i_N} d_k d_{i_1} \cdots d_{i_N} \\
&= [N]_q! \sum_{i_1 \geq \cdots \geq i_N} \sum_{j=0}^{N} \sum_{k=i_{j+1}}^{i_j - 1} q^j q^{(i_1 - 1) + \cdots + (i_j - 1) + k + i_{j+1} + \cdots + i_N} d_{i_1 - 1} \cdots d_{i_j - 1} d_k d_{i_{j+1}} \cdots d_{i_N} \\
&= [N]_q! \sum_{j=0}^{N} q^j \sum_{*} q^{(i_1 - 1) + \cdots + (i_j - 1) + k + i_{j+1} + \cdots + i_N} d_{i_1 - 1} \cdots d_{i_j - 1} d_k d_{i_{j+1}} \cdots d_{i_N} \\
&= [N]_q! \sum_{j=0}^{N} q^j \sum_{l_1 \geq \cdots \geq l_{N+1}} q^{l_1 + \cdots + l_{N+1}} d_{l_1} \cdots d_{l_{N+1}} \\
&= [N]_q! [N+1]_q \sum_{l_1 \geq \cdots \geq l_{N+1}} q^{l_1 + \cdots + l_{N+1}} d_{l_1} \cdots d_{l_{N+1}} \\
&= [N+1]_q! \sum_{l_1 \geq \cdots \geq l_{N+1}} q^{l_1 + \cdots + l_{N+1}} d_{l_1} \cdots d_{l_{N+1}}
\end{aligned}$$

(where * stands for $(i_1 - 1) \geq \cdots \geq (i_j - 1) \geq k \geq i_{j+1} \geq \cdots \geq i_N$), completing the inductive step. □

Corollary 8.5. *Let X be a Δ-set.*

(a) If R is an integral domain or, more generally, a ring with no zero divisors, and $q \neq 1$ is such that $q^N = 1$, then $(R[X], \partial_q)$ is an N-complex.

(b) If R has characteristic N and $q = 1$, then $(R[X], \partial_q)$ is an N-complex.

Proof. By Proposition 8.3, it suffices in each case to show that $q^{N-1} + \cdots + q + 1 = 0$.

(a) In this case $0 = q^N - 1 = (q-1)(q^{N-1} + \cdots + q + 1)$, so because $q - 1 \neq 0$ and there are no zero divisors in R, $q^{N-1} + \cdots + q + 1 = 0$.

(b) Because R has characteristic N, $q^{N-1} + \cdots + q + 1 = \underbrace{1 + \cdots + 1}_{N} = 0$.

□

If X is a Δ-set then a *sub-Δ-set* of X is a sequence Y of subsets $Y_n \subset X_n$ for $n \geq 0$ such that $d_i(Y_n) \subset Y_{n-1}$ for $0 \leq i \leq n$, where the d_i are the face maps of X. In this case Y is itself a Δ-set with face maps $d_i|_{Y_n}$. Note that if Y is a sub-Δ-set of X, and R and $q \in R$ are such that $(R[X], \partial_q)$ is an N-complex, then $(R[Y], \partial_q)$ is a subcomplex of $(R[X], \partial_q)$.

While the definition of a Δ-set requires the sequence of sets to start at X_0, all of our constructions will work in another case as well. Let a *Δ^--set* be the same as a Δ-set, except there is also a set X_{-1} and a single face map $d_0 : X_0 \to X_{-1}$. It is clear that all of the results of this section hold for Δ^--sets.

9 Word Complexes

Fix $n \in \mathbb{N}$ and let X_k ($k \geq -1$) be the set of words of length $k+1$ on an alphabet of n letters. That is, for a set Ω with $|\Omega| = n$, X_k is the set of sequences of elements of Ω with length $k+1$. Define the face maps $d_i : X_k \to X_{k-1}$ ($0 \leq i \leq k$) by letting d_i be the function removing the $(i+1)^{\text{st}}$ letter of a word. It is clear that $d_i d_j = d_{j-1} d_i$ for $i < j$, so this defines a Δ^--set X.

So if R is a ring and $q \in R$ is such that $[N]_q = 0$, then $(R[X], \partial = \partial_q)$ is an N-complex. The *(total) word complex (on n letters)*, denoted $W = W^\Omega$, is this N-complex with its indexing increased by one, i.e. $W_k = R[X]_{k-1}$. In particular, W_k has a basis given by the words of length k.

Now let $\tilde{X}$ be the sub-Δ^--set of X consisting of the injective words, i.e. $\tilde{X}_k$ is the set of words of length $k+1$ with no letter repeated and the face maps are the restrictions of those of X. So $R[\tilde{X}]$ is subcomplex of $R[X]$. The *injective word complex*, denoted $\tilde{W}$, is the subcomplex of W corresponding to $R[\tilde{X}]$, i.e. $\tilde{W}_k = R[\tilde{X}]_{k-1}$.

If G is a group acting on Ω, then G acts on X_{k-1} and $\tilde{X}_{k-1}$, and so on $W_k = W_k^\Omega$ and $\tilde{W}_k = \tilde{W}_k^\Omega$, by $g(\omega_1, \ldots, \omega_k) = (g\omega_1, \ldots, g\omega_k)$. Note that this action commutes with the face maps d_i, so it commutes with the boundary maps ∂, so W and $\tilde{W}$ are chain complexes of $R[G]$-modules. In particular, $W^{\{1,\ldots,n\}}$ and $\tilde{W}^{\{1,\ldots,n\}}$ are chain complexes of $R[S_n]$-modules.

The word complex can be given a graded algebra structure in a natural way. Specifically, if w is a word of length l and z is word of length k, then $w * z$, given by concatenating the two words, is a word of length $l + k$, and this product is then extended by linearity to the rest of W. In order for this product to be useful to us, we need to know how it behaves with respect to the boundary maps.

Let $A = \bigoplus_i A_i$ be a graded algebra over a ring R, and let $q \in R$. $T \in \mathrm{End}_R(A)$ is said to satisfy the q-Leibniz rule if for all $w \in A_l$ and $z \in A_k$, $T(wz) = T(w)z + q^l w T(z)$.

Lemma 9.1. *Say $w \in W_l$ and $z \in W_k$. Then*

(a) $d_i(w * z) = \begin{cases} (d_i w) * z & \text{if } i < l \\ w * (d_{i-l} z) & \text{if } l \leq i \end{cases}$,

(b) ∂ satisfies the q-Leibniz rule, and

(c) $\partial^m(w * z) = \sum_{j=0}^{m} q^{(m-j)(l-j)} \binom{m}{j}_q \partial^j w * \partial^{m-j} z.$

Proof. (a) By linearity it suffices to prove this in the case where w and z are actually words. But in this case it is obvious.

(b)

$$\begin{aligned}
\partial(w*z) &= \sum_{i=0}^{l+k-1} q^i d_i(w*z) = \sum_{i=0}^{l-1} q^i d_i(w*z) + \sum_{i=l}^{l+k-1} q^i d_i(w*z)\\
&= \sum_{i=0}^{l-1} q^i d_i(w)*z + \sum_{i=l}^{l+k-1} q^i w * d_{i-l}(z)\\
&= \left(\sum_{i=0}^{l-1} q^i d_i(w)\right) * z + \sum_{i=0}^{k-1} q^{l+i} w * d_i(z)\\
&= \partial(w)*z + q^l w * \partial(z).
\end{aligned}$$

(c) This follows from (b) and the following lemma. □

Lemma 9.2. *Let R be a ring and $A = \bigoplus_i A_i$ be a graded R-algebra. If $\partial : A \to A$ is an R-linear map of degree -1 that satisfies the q-Leibniz rule for some $q \in R$, then for all $m \geq 0$, $\partial^m(wz) = \sum_{j=0}^{m} q^{(m-j)(l-j)} \binom{m}{j}_q \partial^j(w)\partial^{m-j}(z)$.*

Proof. Induction on m. The $m = 0$ case is trivial, and the $m = 1$ case is the q-Leibniz rule. So assume the statement is true for m. Then

$$\begin{aligned}
\partial^{m+1}(wz) &= \partial\left(\sum_j q^{(m-j)(l-j)} \binom{m}{j}_q \partial^j(w)\partial^{m-j}(z)\right)\\
&= \sum_j q^{(m-j)(l-j)} \binom{m}{j}_q \left(\partial^{j+1}(w)\partial^{m-j}(z) + q^{l-j}\partial^j(w)\partial^{m-j+1}(z)\right)\\
&= \sum_j q^{(m-j)(l-j)} \binom{m}{j}_q \partial^{j+1}(w)\partial^{m-j}(z) + \sum_j q^{(m-j+1)(l-j)} \binom{m}{j}_q \partial^j(w)\partial^{m-j+1}(z)\\
&= \sum_j q^{(m-j+1)(l-j+1)} \binom{m}{j-1}_q \partial^j(w)\partial^{m-j+1}(z) + \sum_j q^{(m-j+1)(l-j)} \binom{m}{j}_q \partial^j(w)\partial^{m-j+1}(z)\\
&= \sum_j q^{(m-j+1)(l-j)} \left(q^{m-j+1}\binom{m}{j-1}_q + \binom{m}{j}_q\right) \partial^j(w)\partial^{m-j+1}(z)\\
&= \sum_j q^{(m+1-j)(l-j)} \binom{m+1}{j}_q \partial^j(w)\partial^{m+1-j}(z)
\end{aligned}$$

which completes the inductive step. □

10 Homology of the Total Word Complex

We consider the total word complex. Specifically, we fix a ring R and an element $q \in R$, and let (W, ∂) be defined as in the previous section. In this section and the next, much of our work parallels parts of the discussion in sections 2 and 4 of [8].

Lemma 10.1. *Say $m > k$ are integers such that $[m-k]_q!$ is invertible in R. Then for all $z \in \ker(\partial) \cap W_k$, there is some $F \in W_m$ such that $\partial^{m-k} F = z$. That is,* $\ker(\partial) \cap W_k \subset \partial^{m-k}(W_m)$.

Proof. If $k < 0$, then $C_k = 0$, and setting $F = 0$ we are done. So say $k \geq 0$. Let $l = m - k$, and let $w_l \in W_l$ be any word of length l. Now

$$\partial^l(w_l * z) = \sum_{j=0}^{l} q^{(l-j)^2} \binom{l}{j}_q \partial^j w * \partial^{l-j} z = x^0 \binom{l}{l}_q \partial^l w_l * \partial^0 z = [l]_q! z$$

where the second to last equality holds because $\partial^{l-j} z = 0$ for $j < l$, and the last equality holds because $\partial^l w_l = [l]_q! \emptyset$ (which follows from Lemma 8.4). So letting $F = \frac{1}{[m-k]_q!} w_l * z$, we have that $\partial^{m-k} F = z$. □

Proposition 10.2. *Say $m > k$ are integers such that $[m-k-1]_q!$ is invertible in R. Then for all $0 < i < m-k$,* $\ker(\partial^i) \cap W_{k+i} \subset \partial^{m-k-i}(W_m)$.

Proof. Induction on i. The base case $i = 1$ is exactly the previous lemma with k substituted by $k+1$. So assume $\ker(\partial^i) \cap W_{k+i} \subset \partial^{m-k-i}(W_m)$, and consider any $f \in \ker(\partial^{i+1}) \cap W_{k+i+1}$. Then $\partial(f) \in \ker(\partial^i) \cap W_{k+i}$, so there is some $F \in W_m$ such that $\partial^{m-k-i} F = \partial f$. Now $\partial(\partial^{m-k-i-1}(F) - f) = 0$, so by the previous lemma, there is some $H \in W_m$ such that $\partial^{m-k-i-1} H = \partial^{m-k-i-1}(F) - f$. So $f = \partial^{m-k-i-1}(F - H) \in \partial^{m-k-i-1}(W_m)$. □

Theorem 10.3. *If R is a ring and $q \in R$ is such that $[N-1]_q!$ is a unit while $[N]_q = 0$, then (W, ∂) is N-exact.*

Proof. $[N]_q = 0$ gives us that (W, ∂) is an N-complex. So consider $H_{k,i}(W)$. Because $[N-1]_q!$ is a unit we can apply the previous proposition with $k - i$ substituted for k and $k + N - i$ substituted for m to get that $\ker(\partial^i) \cap W_k \subset \partial^{N-i}(W_{k+N-i})$, so $H_{k,i}(W) = 0$. □

Corollary 10.4. *If $R = \mathbb{F}$ is a field and $q \in \mathbb{F}$ is a primitive N^{th} root of unity, or if R is ring of prime characteristic $N = p$ and $q = 1$, then the N-complex (W, ∂) is N-exact.*

Proof. In either of these cases, $[N]_q = 0$ but $[l]_q \neq 0$ for $l < N$. In the $R = \mathbb{F}$ case this immediately shows that $[N-1]_q! \neq 0$ is a unit. For the case where R is a ring of characteristic $N = p$ and $q = 1$, begin by recalling that in such a ring, the subring generated by 1 is isomorphic to $\mathbb{Z}_p$. Because the q-analog numbers are polynomials in q, we know that $[l]_{q=1}$ lies in this copy of $\mathbb{Z}_p$. So because $[l]_{q=1} \neq 0$ for $l < N$ and $\mathbb{Z}_p$ is a field, this shows that $[N-1]_{q=1}! \neq 0$ is a unit in R. So in either case, Theorem 10.3 gives us that (W, ∂) is N-exact. □

11 The Injective Word Complex

We begin investigating the injective word complexes $\tilde{W}$. Fix $\mathbb{F}$ an algebraically closed field and $q \in \mathbb{F}$ such that $[N]_q = 0$ but $[N-1]_q! \neq 0$. For a natural number $n \in \mathbb{N}$ we consider $\tilde{W} = \tilde{W}^n := \tilde{W}^{\{1,\dots,n\}}$, the injective word complex on n letters. We know that this is an N-complex of $\mathbb{F}[S_n]$-modules. Now for $1 \leq k \leq n$, $\tilde{W}_k$ is the permutation module for the action of S_n on sequences of length k made of distinct elements from $\{1,\dots,n\}$. Thus, it is clear that $\tilde{W}_k \cong_{\mathbb{F}[S_n]} M^{\lambda^k}$, where $\lambda^k = (n-k, 1^k)$.

Towards understanding these modules better, let $\mathbb{F} = \mathbb{C}$ and consider M^λ for $\lambda \vdash n$. Defining ϕ^λ to be the character corresponding to M^λ, we know that $\phi^\lambda = (1_{S_\lambda}) \uparrow^{S_n}$. For any $\pi \in S_n$ with cycle type $\mu \vdash n$ we have

$$\begin{aligned}\phi^\lambda(\mu) = \phi^\lambda(\pi) = (1_{S_\lambda}) \uparrow^{S_n} (\pi) &= \frac{1}{|S_\lambda|} \sum_{\sigma \in S_n} 1_{S_\lambda}(\sigma^{-1}\pi\sigma) \\ &= \frac{1}{|S_\lambda|} |C_{S_n}(\pi)| \cdot |S_\lambda \cap K_\pi| = \frac{1}{|S_\lambda|} z_\mu \cdot |S_\lambda \cap K_\mu|.\end{aligned}$$

Assume now that $\mathbb{F}$ has characteristic $p > 0$. Consider the Brauer character ϕ_p^λ of the $\mathbb{F}[G]$-permutation module M^λ. It follows from Proposition 3.10 that ϕ_p^λ is the restriction of ϕ^λ to the set of p-regular elements of S_n. So we will write ϕ^λ to also denote ϕ_p^λ in what follows.

We return to the specific case of M^{λ^k}. Note that because $\lambda^{n-1} = \lambda^n$, we lose no generality by only considering $k \neq n-1$. Writing $\mu = (1^{m_1}, \dots, n^{m_n})$, we want to find the value $\phi^k(\mu) = \phi_n^k(\mu) := \phi^{\lambda^k}(\mu)$. Note that any $\pi \in S_{\lambda^k}$ fixes (at least) the k numbers $n-k+1, \dots, n$. Also, any $\pi \in K_\mu$ must have exactly m_1 fixed points. So if $m_1 < k$, $|S_{\lambda^k} \cap K_\mu| = 0$. On the other hand, if $m_1 \geq k$, then $S_{\lambda^k} \cap K_\mu$ consists of those $\pi \in S_n$ with cycle type μ fixing each of $n-k+1, \dots, n$. So by considering the action of such π on $\{1, \dots, n-k\}$, it is clear that we obtain a bijection between $S_{\lambda^k} \cap K_\mu$ and $K_{\tilde{\mu}^k}$, the conjugacy class of S_{n-k} consisting of those elements of cycle type $\tilde{\mu}^k = (1^{m_1-k}, 2^{m_2}, \dots, (n-k)^{m_{n-k}})$. Because also $|S_{\lambda^k}| = |S_{n-k}|$,

$$\begin{aligned}\phi^k(\mu) &= \frac{1}{|S_{\lambda^k}|} z_\mu \cdot |S_{\lambda^k} \cap K_\mu| = \frac{1}{|S_{n-k}|} z_\mu \cdot |K_{\tilde{\mu}^k}| \\ &= \frac{z_\mu}{z_{\tilde{\mu}^k}} = \frac{\prod_{j \geq 1} j^{m_j}(m_j)!}{1^{m_1-k}((m_1-k)!) \prod_{j \geq 2} j^{m_j}(m_j)!} \\ &= \frac{m_1!}{(m_1-k)!}.\end{aligned}$$

Proposition 11.1. *Let $\phi^k = \phi_n^k$ denote the (Brauer) character afforded by the $\mathbb{F}[G]$-module $\tilde{W}_k = \tilde{W}_k^n$. If $\mu = (1^{m_1}, \dots, n^{m_n}) \vdash n$, then* $\phi^k(\mu) = \begin{cases} 0 & \textit{if } k < 0 \\ \frac{m_1!}{(m_1-k)!} & \textit{if } 0 \leq k \leq m_1 \\ 0 & \textit{if } m_1 < k \end{cases}$.

Proof. First, note that our notation is consistent with that above, because for $1 \leq k \leq n$, $\tilde{W}_k \cong_{\mathbb{F}[S_n]} M^{\lambda^k}$. Now the above discussion proves this result except when $k = n-1$. Because $\lambda^{n-1} = \lambda^n$, we have

$$\begin{aligned}\phi^{n-1}(\mu) = \phi^n(\mu) &= \begin{cases} \frac{m_1!}{(m_1-n)!} & \text{if } m_1 \geq n \\ 0 & \text{if } m_1 < n \end{cases} = \begin{cases} n! & \text{if } \mu = (1^n) \\ 0 & \text{if } \mu \neq (1^n) \end{cases} \\ &= \begin{cases} \dfrac{m_1!}{(m_1-(n-1))!} & \text{if } m_1 \geq n-1 \\ 0 & \text{if } m_1 < n-1 \end{cases},\end{aligned}$$

because the only partition of n having at least $n-1$ ones is (1^n), and in that case $(m_1 - (n-1))! = 1! = 1$. □

Towards understanding the Lefschetz modules of $\tilde{W}$, we develop notation to pick out the nonzero sections of $\tilde{W}$ in a unique way, where we consider shifts of the same section as equivalent.

Definition 11.2. For fixed n, N, a pair (k, i) with $0 \leq k \leq n$ and $1 \leq i \leq N-1$ is called a *top index* if $n < k + N - i$, or equivalently if $n - N + i < k$.

It is clear that for $k \in \mathbb{Z}$ and $1 \leq i \leq N-1$, (k, i) is a top index if and only if $\tilde{W}_k \neq 0$ but $\tilde{W}_{k+N-i} = 0$, i.e. $\tilde{W}_k$ is the highest nonzero module of $\tilde{W}[k, i]$. Thus, it is clear that each section of $\tilde{W}$ (up to equivalence by shift) that is not identically zero has a unique representation as $\tilde{W}[k, i]$ with (k, i) a top index.

For convenience, throughout the rest of the section we let $L_{k,i} = L^n_{k,i} = L^{n,N}_{k,i} := \tilde{L}_{k,i}(\tilde{W}^n, \mathbb{F}[S_n])$, and similarly for the Lefschetz (Brauer) characters.

Proposition 11.3. *Fix n, N. If (k, i) is a top index, then $L_{k,i}$ is an actual module.*

Proof. By definition $L^{n,N}_{k,i} = \sum_{j \in \mathbb{Z}} \left(\left[\tilde{W}^n_{k+jN} \right] - \left[\tilde{W}^n_{k-i+jN} \right] \right)$, and because (k, i) is a top index, all the terms with j positive vanish. So it suffices to show that each $[\tilde{W}_m] - [\tilde{W}_l]$ with $n \geq m \geq l$ is an actual module, because $k \leq n$. If $l < 0$ this is obvious because $\tilde{W}_l = 0$, so we need only consider $n \geq m \geq l \geq 0$. In this case

$$\left[\tilde{W}_m\right] - \left[\tilde{W}_l\right] = \left[M^{\lambda^m}\right] - \left[M^{\lambda^l}\right] = \sum_{\mu \vdash n} (K_{\mu\lambda^m} - K_{\mu\lambda^l}) \left[S^\mu\right],$$

so the following combinatorial lemma completes the proof. □

Lemma 11.4. *For all $\mu \vdash n$, if $n \geq m \geq l \geq 0$ then $K_{\mu\lambda^m} \geq K_{\mu\lambda^l}$.*

Proof. It is clear that we need only consider the case $m = l + 1$, and because $\lambda^n = \lambda^{n-1}$, we can further assume that $l \neq n-1$. We prove the result directly, by exhibiting an injection f from $SSYT(\mu, \lambda^l)$, the set of semistandard μ-tableaux of content $\lambda^l = (n-l, \underbrace{1, \ldots, 1}_{l})$, into $SSYT(\mu, \lambda^{l+1})$, the set of semistandard μ-tableaux of content

$\lambda^{l+1} = (n-l-1, \underbrace{1,\dots,1}_{l+1})$. Say T is a semistandard Young tableau of shape μ and content λ^l, so the entries of T are $n-l$ ones and one each of $2,\dots,l+1$. Because T is semistandard, all of the 1s must be in the first row. Now let $f(T)$ be the generalized Young tableau defined by increasing the last 1 to a 2 and each of $2,\dots,l,l+1$ to $3,\dots,l+1,l+2$, respectively. Then it is obvious that $f(T)$ is weakly increasing across rows and strictly increasing down columns, i.e. is semistandard, because T is. Finally, f is an injection because $g : \mathrm{Im}(f) \to SSYT(\mu, \lambda^l)$, given by subtracting 1 from each entry of a given T' that is > 1, is clearly a left inverse for f. □

We now introduce another way of indexing the nonzero sections of $\tilde{W}$.

Definition 11.5. For fixed n, N, a pair (r,i) with $0 \le r \le n$ and $1 \le i \le N-1$ is called a *bottom index* if $r < i$.

It is clear that for $r \in \mathbb{Z}$ and $1 \le i \le N-1$, (r,i) is a bottom index if and only if $\tilde{W}_r \ne 0$ but $\tilde{W}_{r-i} = 0$, i.e. $\tilde{W}_r$ is the lowest nonzero module of $\tilde{W}[r,i]$. Thus, it is clear that each section of $\tilde{W}$ (up to equivalence by shift) that is not identically zero has a unique representation as $\tilde{W}[r,i]$ with (r,i) a bottom index. So for any bottom index, there is a unique top index, denoted $\Psi(r,i) = \Psi_n(r,i) = \Psi_{n,N}(r,i)$, such that $\tilde{W}[\Psi(r,i)]$ is a shift of $\tilde{W}[r,i]$. Clearly (for fixed n and N) Ψ is a bijection from the set of bottom indices to the set of top indices.

Fix N. Note that if (r,i) is a bottom index for some $n \ge r$, then (r,i) is a bottom index for all $n \ge r$, and this occurs if and only if $0 \le r < i < N$. So if $0 \le r < i < N$, let $\chi_n^{r,i} = \chi_{n,N}^{r,i} := \Gamma_n^{\Psi_n(r,i)}$ for all $n \ge r$.

In problem 7.65 of [13], Richard Stanley defines a sequence of characters ($\psi_n : S_n \to \mathbb{C}$) to be *elementary* if for all $\pi \in S_n$, $\psi_n(\pi)$ equals either $\pm\deg(\psi_m)$ for some $m \le n$ or 0. After giving an example in part (a) of the problem, in part (b) he poses the open questions: "What other 'interesting' elementary sequences are there? Can all elementary sequences be completely classified?"([13], page 470)

We expand Stanley's definition of elementary sequences to include sequences of Brauer characters and sequences starting at some arbitrary $n > 1$.

Theorem 11.6. *Fix N. For any integers r, i such that $0 \le r < i < N$, $(\chi_n^{r,i})_{n \ge r}$ is an elementary sequence.*

Proof. First, note that for any $n \ge r$, $\chi_n^{r,i} = \Gamma_n^{\Psi_n(r,i)} = \pm\Gamma_n^{r,i}$ because $\tilde{W}[\Psi_n(r,i)]$ is a shift of $\tilde{W}[r,i]$. Also, because $\Psi_n(r,i)$ is, by definition, a top index, Proposition 11.3 gives us that $\chi_n^{r,i}$ is a (Brauer) character of S_n. Now consider any

$\mu = (1^{m_1}, \ldots, n^{m_n}) \vdash n$. We have

$$\begin{aligned}
\Gamma_n^{r,i}(\mu) &= \sum_{k \equiv r (\bmod N)} \phi_n^k(\mu) - \sum_{k \equiv r-i (\bmod N)} \phi_n^k(\mu) \\
&= \sum_{\substack{k \equiv r (\bmod N) \\ 0 \le k \le m_1}} \frac{m_1!}{(m_1-k)!} - \sum_{\substack{k \equiv r-i (\bmod N) \\ 0 \le k \le m_1}} \frac{m_1!}{(m_1-k)!} \\
&= \sum_{k \equiv r (\bmod N)} \deg\left(\phi_{m_1}^k\right) - \sum_{k \equiv r-i (\bmod N)} \deg\left(\phi_{m_1}^k\right) \\
&= \deg\left(\Gamma_{m_1}^{r,i}\right).
\end{aligned}$$

which is 0 if $m_1 < r$. So

$$\chi_n^{r,i}(\mu) = \pm \Gamma_n^{r,i}(\mu) = \pm \deg\left(\Gamma_{m_1}^{r,i}\right) = \pm \deg\left(\chi_{m_1}^{r,i}\right). \tag{3}$$

□

These elementary sequences directly generalize Stanley's example from part (a) of the problem. In particular, his example is the case where $N = 2$, $r = 0$, and $i = 1$, i.e. $(\chi_{n,2}^{0,1})_{n \in \mathbb{N}}$.

On the other hand, this result may be indication of a negative answer to Stanley's second question. John Shareshian has hypothesized that one of the main ways that elementary sequences might be classified is if it were true that for a given elementary sequence it was uniquely determined after some finite number of terms, up to some conditions on nontriviality (personal communication, spring 2014). That is, if it were true that for any elementary sequence (ψ_n) there were some $M \in \mathbb{N}$ such that (ψ_n) would be the only elementary sequence satisfying the nontriviality conditions and starting as $(\psi_1, \ldots, \psi_M)$.

The most basic nontriviality condition would be that after some n, none of the characters could be a multiple of the trivial character. A more selective nontrivialty condition would be that after some n, every one of the ψ_n's must take some negative value. This condition obviously implies the first, and also rules out the possibility of replacing ψ_n with a permutation character of S_n acting on a set of size $\deg(\psi_n)$ with carefully chosen fixed points. The sequences $(\chi_n^{r,i})_{n \geq r}$ satisfy this second nontriviality condition, because in equation (3) the first two $\pm$s are determined by n, r, i, N, but the third $\pm$ is determined by whether the first component of $\Psi_{m_1,N}(r,i) = (k,j)$ satisfies $k \equiv r \mod N$ or $k \equiv r - i \mod N$. So for sufficiently large n both signs occur, because as m grows the length of $\tilde{W}^m[r,i]$ eventually grows by one at a time, so the length of $\tilde{W}^m[r,i]$ is even for some m and odd for other m.

We see that even with such nontriviality conditions, for any elementary sequence (ψ_n) and any $M \in \mathbb{N}$, every choice of $r < i < N$ such that $M < r$ gives us a distinct elementary sequence $\psi_1, \ldots, \psi_{r-1}, \chi_{r,N}^{r,i}, \chi_{r+1,N}^{r,i}, \ldots$ which agrees with (ψ_n) on the first M terms and satisfies the nontriviality condition. In fact, this construction seems to show that no "reasonable" nontriviality conditions will work, because $(\chi_{n,2}^{0,1})$, being Stanley's example of an "interesting" elementary sequence, should presumably not be excluded.

Knowing that the Lefschetz modules of the sections of $\tilde{W}^n$ form interesting representations, we are naturally led to ask whether these representations are actually realized on the homology modules. As mentioned in the introduction, in the case $N = 2$ considerations from topological combinatorics show that the only nontrivial homology module of $\tilde{W}^n$ is $H_n(\tilde{W}^n)$, and so this module gives us a "realization" of the interesting representations. Unfortunately this does not generalize to arbitrary N, nor have we been able to prove what the homology modules of $\tilde{W}^n$ are in general. However, based on computations of small cases over various fields obtained using the SageMath mathematics software system, we have arrived at the following conjecture.

Conjecture 11.7. *Let $\mathbb{F}$ be a field, and let $q \in \mathbb{F}$ be a primitive N^{th} root of unity. Let (k,i) be a top index for n and N. If $(k,i) \neq (n,1)$ then the only nontrivial homology module of $\tilde{W}^n[k,i]$ is $H_{k,i}(\tilde{W}^n)$. For $(k,i) = (n,1)$, all homology modules of $\tilde{W}^n[n,1]$ except $H_{n,1}(\tilde{W}^n)$ and $H_{n-1,N-1}(\tilde{W}^n)$ are trivial.*

One nice thing about this conjecture is that it would still imply explicit constructions of all of the Lefschetz modules. Fix n, N and consider a top index (k,i). For $(k,i) \neq (n,1)$, the conjecture and Proposition 6.2 would immediately imply that $\tilde{L}_{k,i}(\tilde{W}^n, \mathbb{F}[S_n]) = [H_{k,i}(\tilde{W}^n)]$, so $H_{k,i}(\tilde{W}^n)$ is an $\mathbb{F}[S_n]$-module that affords the corresponding Lefschetz (Brauer) character $\tilde{\Gamma}_{k,i}(\tilde{W}^n, \mathbb{F})$.

Now consider the case $(k,i) = (n,1)$. $\tilde{W}^n[n,1]$ looks like

$$\cdots \xrightarrow{\partial} 0 \xrightarrow{\partial^{N-1}} \tilde{W}^n_n \xrightarrow{\partial} \tilde{W}^n_{n-1} \xrightarrow{\partial^{N-1}} \tilde{W}^n_{n-N} \xrightarrow{\partial} \tilde{W}^n_{n-N-1} \xrightarrow{\partial^{N-1}} \cdots .$$

Now recalling that $\tilde{W}^n_n \cong_{\mathbb{F}[S_n]} M^{(1^n)} \cong_{\mathbb{F}[S_n]} \tilde{W}^n_{n-1}$, we see that the "truncated" chain complex

$$\cdots \xrightarrow{\partial} 0 \xrightarrow{\partial^{N-1}} 0 \xrightarrow{\partial} 0 \xrightarrow{\partial^{N-1}} \tilde{W}^n_{n-N} \xrightarrow{\partial} \tilde{W}^n_{n-N-1} \xrightarrow{\partial^{N-1}} \cdots$$

has the same Lefschetz module. It is not hard to see that the conjecture would imply that the only nonzero homology module of this truncated chain complex occurs at $\tilde{W}^n_{n-N}$, and this is just $\ker(\partial : \tilde{W}^n_{n-N} \to \tilde{W}^n_{n-N-1})$. So the Lefschetz module of the truncated chain complex would be

$$\tilde{L}_{n,1}(\tilde{W}^n, \mathbb{F}[S_n]) = \tilde{L}_{n,1}(\tilde{W}^n, \mathbb{F}[S_n]) - [\tilde{W}^n_n] + [\tilde{W}^n_{n-1}] = [\ker(\partial : \tilde{W}^n_{n-N} \to \tilde{W}^n_{n-N-1})].$$

So $\ker(\partial : \tilde{W}^n_{n-N} \to \tilde{W}^n_{n-N-1})$ would be an explicitly constructed $\mathbb{F}[S_n]$-module affording the Lefschetz (Brauer) character $\tilde{\Gamma}_{n,1}(\tilde{W}^n, \mathbb{F})$.

References

[1] K.S. Brown. *Cohomology of Groups.* Number 87 in Graduate Texts in Mathematics. Springer, 1982.

[2] Greg Friedman. An elementary illustrated introduction to simplicial sets. *Rocky Mountain Journal of Mathematics*, 42(2):353–424, 2012.

[3] I.M. Isaacs. *Character Theory of Finite Groups.* Pure and applied mathematics. Academic Press, 1976.

[4] G.D. James. *The Representation Theory of the Symmetric Groups*, volume 682 of *Lecture Notes in Mathematics.* Springer-Verlag Berlin Heidelberg, 1978.

[5] M.M. Kapranov. On the q-Analog of Homological Algebra. arXiv:q-alg/9611005, Preprint 1996.

[6] S. Lang. *Algebra.* Graduate Texts in Mathematics. Springer New York, 2002.

[7] S. MacLane. *Homology.* Number 114 in Die Grundlehren Der Mathematischen Wissenschaften in Einzeldarstellungen. Springer-Verlag, 1963.

[8] Valeriy B. Mnukhin and Johannes Siemons. On modular homology in the boolean algebra. *Journal of Algebra*, 179(1):191–199, 1996.

[9] J.R. Munkres. *Elements of Algebraic Topology.* Advanced book classics. Perseus Books, 1984.

[10] Victor Reiner and Peter Webb. The combinatorics of the bar resolution in group cohomology. *J. Pure Appl. Algebra*, 190:291–327, 2004.

[11] Emily Riehl. A leisurely introduction to simplicial sets, 2008. http://www.math.harvard.edu/∼eriehl/ssets.pdf.

[12] Bruce E. Sagan. *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions.* Graduate Texts in Mathematics. Springer, 2001.

[13] Richard P. Stanley. *Enumerative Combinatorics, Volume 2.* Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1999.

[14] Charles A. Weibel. *An Introduction to Homological Algebra.* Number 38 in Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1994.