ARTU Senior Honor Thesis

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ABSTRACT

We consider two sesquilinear forms $\widehat{O}_2$ and $\widehat{O}_3$ on a closed subspace of $L^2(\mathbb{R}^3)$ that appear in quantum many-body system in condensed matter physics. We begin with theorems about linear operators in Hilbert space and deduce that $\widehat{O}_3(g)$ behaves like inner product of Tg and g, where T is a linear operator. Using change of variable, we obtain an integration form of linear operator T. With further simplification, we find eigenvalues and eigenfucntions associated with T, and thus the spectrum of T. The ultimate goal is to prove a relation between $\widehat{O}_2$ and $\widehat{O}_3$, which is $\widehat{O}_2(f) + \widehat{O}_3(f) \geq \epsilon \widehat{O}_2(f)$ for some $\epsilon > 0$.

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INTRODUCTION

Motivated by a spectral gap problem from quantum many-body system in condensed matter physics, we study the following mathematics problem:

Consider the space $L^2(\mathbb{R}^3)$, it has a closed subspace

$$H = \{f \in L^2(\mathbb{R}^3) | f(x,y,z) = f(y,x,z) = f(z,y,x)\}$$

for $f \in H$, we define:

$$g(x,R) = \int_{-\infty}^{\infty} f(x, R-y, R+y) e^{-y^2} dy$$

$$\widehat{O}_2(f) = \frac{\Delta}{2} \iint_{\mathbb{R}^2} |g(x,R)|^2 dx dR, \text{ where } \Delta = \int_0^{\infty} e^{-2x^2} dx$$

$$\widehat{O}_3(f) = 4 \iiint_{\mathbb{R}^3} e^{-[(x-R)^2+(x-R')^2]} \overline{g(R'-R+x,R)} g(R-R'+x,R') dx dR dR'$$

We want to know the relation between $\widehat{O}_2$ and $\widehat{O}_3$.

Question:

Is it true that $\exists \epsilon > 0$ s.t. $\forall f \in H, \widehat{O}_2(f) + \widehat{O}_3(f) \geq \epsilon \widehat{O}_2(f)$

In this paper, we want to give an affirmative answer to the above question.

To study this problem, we need to get familiar with L^2 space and properties of linear operators. Therefore, my research is divided into two parts: reading part and research part.

In the reading part, we follow Elias M. Stein and Rami Shakarchi's *Real Analysis* closely and take John B. Conway's *A Course in Functional Analysis* as a reference.

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READING

In this section, I will recap the chapters from Elias M. Stein and Rami Shakarchi's *Real Analysis* and give two proofs. The first proof is to provide an alternative to the somewhat problematic proof provided in the book, and the second proof is an exercise in the book which becomes a key to the research problem. For the second proof, relevant theorem and Proposition will also be provided.

3.1 RECAP

We read the first four chapters in Elias M. Stein and Rami Shakarchi's *Real Analysis*, including topics on measure theory, integration theory, differentiation and integration, and Hilbert space.

Exterior measure defines a measure of set using covering of closed cubes, and it has nice properties like monotonicity, countable subadditivity, and countable additivity for almost disjoint cubes. We define Lebesgue measure of a measurable set by its exterior measure. A nice way to describe a measurable set is that any measurable set can be approximated by a sequence of open sets that contain it, or a sequence of closed sets that are contained in it. After defining Lebesgue measure, we can define simple functions and measurable functions that can be approximated by a sequence of simple functions. We conclude this chapter by Littlewood's three principles:

- Every set is nearly a finite union of intervals.
- Every function is nearly continuous.
- Every convergent sequence is nearly uniformly convergent.

The first part of integration theory is pretty routine. Lebesgue integration is introduced, and theorems related like Fatou's Lemma, monotone convergence theorem and dominated convergence theorem are also proved in detail. The climax of the chapter is the Fubini theorem and the proof of it. We conclude this chapter with the application of Fubini's theorem and Fourier inversion formula.

The objective in the third chapter, Differentiation and Integration, is to formulate and prove fundamental theorem of calculus in the setting introduced in the previous two chapters. After the Hardy-Littlewood maximal function being defined, we are able to prove the Lebesgue differentiation theorem:

Theorem 3.1.1. *If f is integrable on $\mathbb{R}^d$, then*

$$\lim_{\substack{m(B)\to 0 \\ x\in B}} \frac{1}{m(B)} \int_B f(y)dy = f(x) \text{ for a.e.} x.$$

The above theorem, together with Corollary about locally integrable functions, answers the first part in the generalized fundamental theorem of calculus.

For the second part, we introduce the definition of bounded variation and the heart of the theory of differentiation:
Definition. If F is of bounded variation on $[a, b]$, then F is differentiable almost every where.

To prove the theorem, we start with a Corollary:

Corollary 3.1.2. *If F is increasing and continuous, then F' exists almost everywhere. Moreover F' is measurable, non-negative, and*

$$\int_a^b F'(x)dx \leq F(b) - F(a)$$

We cannot establish the equality and the Cantor-Lebesgue function is a counterexample: it is increasing but fails the equality. However, if we restrict ourselves to absolutely continuous functions, then the equality will hold. By introducing differentiability of jump functions, we are able to remove the continuity assumption and prove the theorem.

The chapter that introduces Hilbert space is the last chapter we read, and it is the most important one. Properties of L^2 space which we are already familiar with are discussed first, and then the more general space, which is Hilbert space, is defined. Definitions related to Hilbert space including orthogonality, unitary mappings, and Pre-Hilbert spaces are also given. In the section of closed subspace, one important result is the theorem:

Theorem 3.1.3. *If $\mathcal{S}$ is a closed subspace of a Hilbert space $\mathcal{H}$, then $\mathcal{H}$ is the direct sum of $\mathcal{S}$ and $\mathcal{S}^\perp$.*

Then the book talks about linear transformations, and some important theorems and proofs will be included in section 3.3.

3.2 COMPLETION OF A PRE-HILBERT SPACE

When proving completion of a pre-Hilbert space, the book constructs the Hilbert space by considering the collection of all Cauchy sequences $\{f_n\}$ with $f_n \in \mathcal{H}_0, 1 \leq n < \infty$, and then the Hilbert space $\mathcal{H}$ we want is the collection of equivalence classes by setting two Cauchy sequences to be equivalent if $f_n - g_n \to 0$ as $n \to \infty$. When proving that the space is indeed complete, the argument in the book is problematic. Consider a Cauchy sequence $\{f^k\}$ in $\mathcal{H}$, and each f^k is represented by $\{f_n^k\}$, where $f_n^k \in \mathcal{H}_0$. We want to construct a function f such that f is the limit of the Cauchy sequence, and then we are done with the proof that $\mathcal{H}$ is complete. The construction in the book using f_n^n may not work because each f_n^n can be arbitrary and it is possible that $\{f_n^n\}$ is not even a Cauchy sequence. Thus, f constructed in the book may not even live in $\mathcal{H}$, so we need to seek for an alternative proof:

Proof. We adopt the same notation. Let $\{f^k\}$ be a Cauchy sequence in $\mathcal{H}$, and each f^k is represented by $\{f_n^k\}$
For $\{f_n^1\}$, there exists N_1 such that for $m, n \geq N_1$, $\|f_m^1 - f_n^1\| < \frac{1}{2}$.
Denote $i_1 = N_1$.
For $\{f_n^2\}$, there exists N_2^1 such that for $m, n \geq N_2^1$, $\|f_m^2 - f_n^2\| < \frac{1}{2^2}$.
Since $\lim_{k\to\infty} \|f_k^m - f_k^n\| = \|f^m - f^n\|$, there exists N_2^2 such that for $k \geq N_2^2$, $\|f_k^2 - f_k^1\| < 2\|f^2 - f^1\|$.
Denote $i_2 = \max\{N_2^1, N_2^2, i_1 + 1\}$
$\vdots$
For $\{f_n^M\}$, there exists N_M^1 such that for $m, n \geq N_M^1$, $\|f_m^M - f_n^M\| < \frac{1}{2^M}$,
and there exists N_M^2 such that for $k \geq N_M^2$, $\|f_k^M - f_k^1\| < 2\|f^M - f^1\|$,
and there exists N_M^3 such that for $k \geq N_M^3$,
$\|f_k^M - f_k^2\| < 2\|f^M - f^2\|$,
$\vdots$
and there exists N_M^M such that for $k \geq N_M^M$, $\|f_k^M - f_k^{M-1}\| < 2\|f^M - f^{M-1}\|$.
Denote $i_M = \max\{N_M^1, N_M^2, \cdot, N_M^M, i_{M-1} + 1\}$
We will show that the sequence $\{f_{i_n}^n\}$ is Cauchy and represents the limit of our original Cauchy sequence.
Given $\epsilon > 0$, there exists N such that for $m \geq N$, $\frac{1}{2^m} < \frac{\epsilon}{2}$ and for $m, n \geq N$, $\|f^m - f^n\| < \frac{\epsilon}{4}$ since $\{f^k\}$ is Cauchy.
For this N, take any $m, n \geq N$, assume $m < n$, so we have $i_m < i_n$ and

$$\|f_{i_m}^m - f_{i_n}^n\| \leq \|f_{i_m}^m - f_{i_n}^m\| + \|f_{i_n}^m - f_{i_n}^n\| \leq \frac{1}{2^m} + 2\|f^m - f^n\| < \epsilon/2 + \epsilon/2 = \epsilon$$

Thus, we have proved the sequence is Cauchy. Denote it as f.
Fix n. Then $\|f^n - f\| = \lim_{k\to\infty} \|f_k^n - f_{i_k}^k\|$
For $k > n$, $\|f_k^n - f_{i_k}^k\| \leq \|f_k^n - f_{i_k}^n\| + \|f_{i_k}^n - f_{i_k}^k\| \leq \frac{1}{2^n} + 2\|f^n - f^k\|$
Hence, $\lim_{n\to\infty} \lim_{k\to\infty} \|f_k^n - f_{i_k}^k\| = 0$ by sandwiching. Thus, $f^k \to f$ in $\mathcal{H}$.

□

3.3 PROOF FOR THE COROLLARY USEFUL IN THIS RESEARCH

In order to prove the Corollary, we need to state Riesz representation theorem, and the Proposition that explains why we are able to apply Riesz representation theorem in the proof for the Corollary.

Theorem 3.3.1 (Riesz representation theorem)**.** *Let ℓ be a continuous linear functional on a Hilbert space $\mathcal{H}$. Then, there exists a unique $g \in \mathcal{H}$ such that*

$$\ell(f) = (f, g).\ \textit{for all } f \in \mathcal{H}.$$

Moreover, $\|\ell\| = \|g\|$.

Proof. We consider the subspace of $\mathcal{H}$ defined by

$$\mathcal{S} = \{f \in \mathcal{H} : \ell(f) = 0\}$$

Since ℓ is continuous linear functional, the subspace $\mathcal{S}$, which is the preimage of a closed set $\{0\}$, is also closed.

If $\mathcal{S} = \mathcal{H}$, then $\ell = 0$ and we can take $g = 0$ to make $\ell(f) = (f, g)$.

Otherwise $\mathcal{S}^{\perp}$ is non-trivial because $\mathcal{H}$ is a direct sum of $\mathcal{S}$ and $\mathcal{S}^{\perp}$. Then we may pick any $h \in \mathcal{S}^{\perp}$ with $\|h\| = 1$. Now set $g = \overline{\ell(h)}h$, and we next show that this g will work.

Consider $u = \ell(f)h - \ell(h)f$. Since $\ell(u) = \ell(f)\ell(h) - \ell(h)\ell(f) = 0$, we know $u \in \mathcal{S}$, and therefore $(u, h) = 0$. Thus,

$$\begin{aligned} 0 &= (\ell(f)h - \ell(h)f, h) \\ &= (\ell(f)h, h) - (\ell(h)f, h) \\ &= \ell(f)(h, h) - \ell(h)(f, h) \\ &= \ell(f) - (f, \overline{\ell(h)}h) \end{aligned} \tag{1}$$

Hence, after moving terms, we get $\ell(f) = (f, g)$ as desired. □

We need to note that the theorem above is one version of the Riesz representation theorem, and there are other versions like the one described in terms of Lebesgue measure.

Proposition 3.3.2. *A linear operator $T : \mathcal{H}_1 \mapsto \mathcal{H}_2$ is bounded if and only if it is continuous.*

Proof. If T is bounded, then $\|T(f) - T(f_n)\|_{\mathcal{H}_2} \leq M\|f - f_n\|_{\mathcal{H}_1}$. By choosing $\delta = \epsilon/(M+1)$, we can prove that T is continuous.

Conversely, suppose that T is continuous but not bounded, and we wish to find a contradiction. Since T is not bounded, for each n, there exists $f_n \neq 0$ such that $\|T(f_n)\| \geq n\|f_n\|$. Consider elements $g_n = \frac{f_n}{n\|f_n\|}$, each g_n has norm $\frac{1}{n}$, hence $g_n \to 0$. Since T is continuous at 0, we should have $T(g_n) \to 0$, but we already have
$\|T(g_n)\| = \frac{\|T(f_n)\|}{n\|f_n\|} \geq 1$. contradiction. □

We note that any linear operator between finite-dimensional Hilbert spaces is necessarily continuous.

With the Proposition and Riesz Representation theorem, we can now look at the Corollary.

Corollary 3.3.3. *If $B(f,g)$ is linear in f, conjugate linear in g and satisfies $|B(f,g)| \leq M\|f\|\|g\|$, there is a unique linear transformation T such that*

$$B(f,g) = (Tf,g), \text{ with } M = \|T\|$$

Proof. Fix g and let $\ell(f) = B(f,g)$, then ℓ is linear. Since $|\ell(f)| \leq M\|g\|\|f\|$, ℓ is bounded. By previous Proposition, ℓ is continuous if and only if ℓ is bounded.

Hence, ℓ is continuous linear functional, and we can apply Riesz representation theorem. There exists a unique h such that $\ell(f) = (f,h)$

Let T_1 denote the mapping $g \mapsto h$. The mapping T_1 is linear and unique, and we can write $\ell(f) = (f, T_1 g)$ because $h = T_1 g$. Next we show that $\|T_1\| = M$.

For any g, let $h = T_1 g$, then $\ell(h) \leq M\|h\|\|g\|$. On the other hand, $\ell(h) = (h, T_1 g) = (h,h) = \|h\|^2$.
Therefore, $\|T_1 g\| = \|h\| \leq M\|g\|$.

Next, since M is least upper bound for $\frac{|B(f,g)|}{\|f\|\|g\|}$, $\exists f_0, g_0$ such that $|B(f_0,g_0)| = \|f_0\|\|g_0\|(M-\epsilon)$
Cauchy-Schwarz inequality gives

$$|(f_0, T_1 g_0)| \leq \|f_0\|\|T_1 g_0\|$$

Note that $|B(f_0,g_0)| = |(f_0,T_1 g_0)|$, so we have $\|f_0\|\|g_0\|(M-\epsilon) = |B(f_0,g_0)| \leq \|f_0\|\|T_1 g_0\|$

$$\|T_1 g_0\| \geq (M-\epsilon)\|g_0\|$$

Thus, $\|T_1\| = M$. now let $T = T_1^*$
Then T is unique, $\|T\| = M$ and $B(f,g) = (f, T_1 g) = (Tf, g)$

□

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RESEARCH

4.1 FINDING THE OPERATOR

To solve the problem, we first need to simplify the expression, especially the $\widehat{O}_3$ operator. we start with an operator S similar to $\widehat{O}_3$ that operates on $f, g \in L^2$:

$$S(f,g) = \iiint_{\mathbb{R}^3} e^{-[(x-R)^2+(x-R')^2]}\overline{f(R'-R+x,R)}g(R-R'+x,R')dxdRdR'$$

Since $(x-R)^2 + (x-R')^2 \geq 0$, we have $0 \leq e^{-[(x-R)^2+(x-R')^2]} \leq 1$. Therefore,

$$\begin{aligned}\iiint_{\mathbb{R}^3} e^{-[(x-R)^2+(x-R')^2]}\overline{f(R'-R+x,R)}g(R-R'+x,R')dxdRdR' \\ \leq \iiint_{\mathbb{R}^3} \overline{f(R'-R+x,R)}g(R-R'+x,R')dxdRdR'\end{aligned}$$

We can use the Cauchy-Schwartz inequality to conclude that the sesquilinear form is a bounded operator. Besides, the sesquilinear form is linear in g and conjugate linear in f, so we can use the Corollary in section 3.3 to conclude that there exists a unique bounded linear operator T such that $S(f,g) = (Tg, f)$. Observe that if we set $f = g$, then $S(g,g) = \widehat{O}_3(g)$, and therefore we can rewrite our $\widehat{O}_3$ operator using the inner product form $\widehat{O}_3(g) = (Tg, g)$. The inner product form (Tg, g) is $\iint_{\mathbb{R}^2} Tg(a,b)\overline{g(a,b)}dadb$. Apply change of variable,

$$a = R' - R + x, b = R, c = R - R' + x$$

Thus,

$$x = \frac{1}{2}(a+c), R = b, R' = \frac{1}{2}(a+2b-c)$$

$$J = \begin{vmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 1 & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

Then the integral

$$\iiint_{\mathbb{R}^3} e^{-[(x-R)^2+(x-R')^2]}\overline{g(R'-R+x,R)}g(R-R'+x,R')dxdRdR'$$

can be rewritten as

$$\iiint_{\mathbb{R}^3} \frac{1}{2} e^{-[\frac{1}{4}(a-2b+c)^2+(c-b)^2]} g(c, \frac{1}{2}(a+2b-c)) \overline{g(a,b)} dadbdc$$

By comparing it with $\iint_{\mathbb{R}^2} Tg(a,b)\overline{g(a,b)}dadb$, we have

$$Tg(a,b) = \int_{\mathbb{R}} \frac{1}{2} e^{-[\frac{1}{4}(a-2b+c)^2+(c-b)^2]} g(c, \frac{1}{2}(a+2b-c))dc$$

Hence, we can replace $\widehat{O}_3$ operator by (Tg, g), with T in the form above. Since $\hat{O}_2 = k(g,g)$, where k is a constant. To show $\widehat{O}_3(f) + \widehat{O}_2(f) \geq \epsilon \widehat{O}_2(f)$, we only need to show all eigenvalues of T satisfies $\lambda + k \geq \epsilon k$ because that will imply $(Tg, g) + k(g, g) \geq \epsilon k(g, g)$, which is equivalent to the original statement we wish to prove.

Thus, with this operator, we reduces the question to a sub question about the spectrum of T on $L^2(\mathbb{R}^2)$.

4.2 EIGENVALUES

By the property of unitary operator, the eigenvalues of new operator U^*TU would be the same as the eigenvalues of T if we construct a unitary operator U. We hope that by the construction, the integral form of the new operator will have a less complicated form and possibly a free variable so that we can further reduce the problem to the spectrum of an operator on $L^2(\mathbb{R})$ instead of $L^2(\mathbb{R}^2)$. Let U be a unitary operator on $L^2(\mathbb{R}^2)$ (change of variable for variables inside the function).

$$Ug(a,b) = g(a+2b, -\frac{1}{3}a + \frac{1}{3}b)$$

Then we have

$$U^*g(a,b) = g(\frac{1}{3}a - 2b, b + \frac{1}{3}a)$$

Remark. When calculating U^* we use the fact that jacobian of the change of variable has determinant 1.
U is unitary because $UU^* = U^*U = I$
With unitary operator U, we only need to find spectrum of the operator:

$$U^*TUg(a,b) = TUg(\frac{1}{3}a - 2b, b + \frac{1}{3}a)$$

Since $Tg(a,b) = \frac{1}{2}\int_{\mathbb{R}} e^{-[\frac{1}{4}(a-2b+c)^2+(c-b)^2]} g(c, \frac{1}{2}(a+2b-c))dc$, we have

$$TUg(\frac{1}{3}a - 2b, b + \frac{1}{3}a) = \frac{1}{2}\int_{\mathbb{R}} e^{-[\frac{1}{4}(\frac{1}{3}a-2b-2b-\frac{2}{3}a+c)^2+(c-b-\frac{1}{3}a)^2]}$$
$$Ug(c, \frac{1}{2}(\frac{1}{3}a - 2b + 2b + \frac{2}{3}a - c))dc$$

In the Ug part, we get

$$Ug(c, \frac{1}{2}(a-c)) = g(c + (a-c), \frac{\frac{1}{2}(a-c) - c}{3}) \\ = g(a, \frac{a-3c}{6})$$

Hence, we now have $g(a, \frac{a-3c}{6})$ independent from b
On the other hand, in the exponential part:

$$e^{-[\frac{1}{4}(\frac{1}{3}a-2b-2b-\frac{2}{3}a+c)^2+(c-b-\frac{1}{3}a)^2]} = e^{-[\frac{1}{4}(-\frac{1}{3}a-4b+c)^2+(c-b-\frac{1}{3}a)^2]}$$

Since c ranges from $-\infty$ to ∞, we can do a change of variable $c' = c - \frac{1}{3}a$, then $dc = dc'$
Then $U^*TUg(a,b) = TUg(\frac{1}{3}a - 2b, b + \frac{1}{3}a)$ becomes

$$\frac{1}{2}\int_{\mathbb{R}} e^{-[\frac{1}{4}(c'-4b)^2+(c'-b)^2]} g(a, -\frac{c'}{2}) dc' \quad (2)$$

Since $\frac{1}{4}(c'-4b)^2 + (c'-b)^2 = \frac{5}{4}(c' - \frac{8}{5}b)^2 + \frac{9}{5}b^2$, we can do another change of variable for the similar reason by $c'' = c' - \frac{8}{5}b$, and then $dc' = dc''$. $TUg(\frac{1}{3}a - 2b, b + \frac{1}{3}a)$ can be further simplified:

$$\frac{1}{2}\int_{\mathbb{R}} e^{-\frac{5}{4}c''^2 - \frac{9}{5}b^2} g(a, -\frac{5c''+8b}{10}) dc'' = \frac{1}{2} e^{-\frac{9}{5}b^2} \int_{\mathbb{R}} e^{-\frac{5}{4}c''^2} g(a, -\frac{5c''+8b}{10}) dc''$$

We can now guess the eigenfunction for the operator by looking at equation (2). Since a behaves like a free variable, if we can find some function of b, say $h(b)$, such that $g(a,b) = h(b)$ is the eigenfunction, then $g(a,b) = f_0(a)h(b)$ would also be an eigenfunction associated with the same eigenvalue because we can pull f_0 outside the integral. Hence, we just need to pay attention to functions with a single variable. With b^2 in the exponential part, a reasonable guess for $h(b)$ is to let $g(a,b) = h(b) = e^{-kb^2}$, then we can calculate $Tg(a,b)$ by using equation (2).

$$\begin{aligned} U^*TUg(a,b) &= \frac{1}{2}\int_{\mathbb{R}} e^{-[\frac{1}{4}(c'-4b)^2+(c'-b)^2]} e^{-\frac{kc'^2}{4}} dc' \\ &= \frac{1}{2}\int_{\mathbb{R}} e^{-5b^2+4bc'-\frac{5+k}{4}c'^2} dc' \\ &= \frac{1}{2}\int_{\mathbb{R}} e^{-\frac{5+k}{4}(c'-\frac{8b}{5+k})^2+\frac{5+k}{4}\frac{64b^2}{(5+k)^2}-5b^2} dc' \\ &= \frac{1}{2} e^{\frac{16b^2}{5+k}-5b^2} \int_{\mathbb{R}} e^{-\frac{5+k}{4}x^2} dx \text{ by appropriate change of variable} \end{aligned}$$

Hence, we want $\frac{16}{5+k} - 5 = -k$ with some $k > 0$. Solving quadratic equation $(5-x)(x+5) = 16$ gives only one real positive solution $k = 3$

Now we use the fact that $\int_{\mathbb{R}} e^{-ax^2}dx = \sqrt{\frac{\pi}{a}}$ to obtain that $\int_{\mathbb{R}} e^{-\frac{5+k}{4}x^2}dx = \sqrt{\frac{4\pi}{5+k}} = \sqrt{\frac{4\pi}{8}}$. Hence, we obtain an eigenfunction $g(a,b) = e^{-3b^2}$ of U^*TU with eigenvalue $\sqrt{\frac{\pi}{8}}$.

Next we let $g(a,b) = (x_1 b + x_2)e^{-kb^2}$ with $x_1 \neq 0$, then we can calculate $U^*TUg(a,b)$ by using equation (2).

$$\begin{aligned}
U^*TUg(a,b) &= \frac{1}{2}\int_{\mathbb{R}} e^{-[\frac{1}{4}(c'-4b)^2+(c'-b)^2]}(-\frac{x_1c'}{2}+x_2)e^{-\frac{kc'^2}{4}}dc' \\
&= \frac{x_2}{2}\int_{\mathbb{R}} e^{-5b^2+4bc'-\frac{5+k}{4}c'^2}dc' - \frac{x_1}{4}\int_{\mathbb{R}} c'e^{-5b^2+4bc'-\frac{5+k}{4}c'^2}dc' \\
&= \frac{x_2}{2}e^{\frac{16b^2}{5+k}-5b^2}\int_{\mathbb{R}} e^{-\frac{5+k}{4}x^2}dx \\
&\quad - \frac{x_1}{4}\int_{\mathbb{R}} c'e^{-\frac{5+k}{4}(c'-\frac{8b}{5+k})^2+\frac{5+k}{4}\frac{64b^2}{(5+k)^2}-5b^2}dc' \\
&= \frac{x_2}{2}e^{\frac{16b^2}{5+k}-5b^2}\int_{\mathbb{R}} e^{-\frac{5+k}{4}x^2}dx \\
&\quad - \frac{x_1}{4}e^{\frac{16b^2}{5+k}-5b^2}\int_{\mathbb{R}}(y+\frac{8b}{5+k})e^{-\frac{5+k}{4}y^2}dy \\
&= e^{\frac{16b^2}{5+k}-5b^2}(\frac{x_2}{2}\int_{\mathbb{R}} e^{-\frac{5+k}{4}x^2}dx - \frac{x_1}{4}\int_{\mathbb{R}} ye^{-\frac{5+k}{4}y^2}dy \\
&\quad - \frac{x_1}{4}\frac{8b}{5+k}\int_{\mathbb{R}} e^{-\frac{5+k}{4}y^2}dy)
\end{aligned}$$

Formula inside the parenthesis is a linear polynomial of b and we still want $\frac{16}{5+k} - 5 = -k$ with some $k > 0$. The only real positive solution is $k = 3$ Now we use the fact that $\int e^{-ax^2}dx = \sqrt{\frac{\pi}{a}}$ to obtain that $\int e^{-\frac{5+k}{4}x^2}dx = \sqrt{\frac{4\pi}{5+k}} = \sqrt{\frac{4\pi}{8}}$ and we know $\int ye^{-ay^2}dy = 0$. Hence, we have:

$$U^*TUg(a,b) = e^{-3b^2}(\frac{x_2}{2}\sqrt{\frac{\pi}{2}} - \frac{bx_1}{4}\sqrt{\frac{\pi}{2}})$$

In order to let $U^*TUg(a,b) = \lambda g(a,b) = \lambda(x_1b + x_2)e^{-3b^2}$, we need

$$\frac{x_2}{2}\sqrt{\frac{\pi}{2}} - \frac{bx_1}{4}\sqrt{\frac{\pi}{2}} = \lambda x_1 b + \lambda x_2$$

$x_1 \neq 0$ implies $\lambda = -\frac{1}{4}\sqrt{\frac{\pi}{2}}$ and $x_2 = 0$.

We continue by trying functions in the form $p_n(x)e^{-kx^2}$ where $p_n(x)$ is a degree n polynomial of x. After some efforts, we reach the conclusion that U^*TU has eigenfunctions $b^n e^{-3b^2} + e^{-3b^2}p_{n-1}(b)$ with eigenvalues $\frac{(-1)^n}{2^{n+1}}\sqrt{\frac{\pi}{2}}$. Since $\{x^n e^{-3x^2}\}$ forms a basis for $L^2(\mathbb{R})$, we have exhausted all eigenfunctions and eigenvalues. Hence, The spectrum of U^*TU on domain $L^2(\mathbb{R})$ would be $\{\frac{(-1)^n}{2^{n+1}}\sqrt{\frac{\pi}{2}} | n = 0,1,2,...\}$.

We conclude this section with the theorem about the eigenvalues of operator T on domain $L^2(\mathbb{R}^2)$ by applying the fact that U^*TU and T have the same spectrum:

Theorem 4.2.1. *The operator U^*TU on $L^2(\mathbb{R}^2)$ given by*

$$U^*TUg(a,b) = \frac{1}{2}\int_{\mathbb{R}} e^{-[\frac{1}{4}(c'-4b)^2+(c'-b)^2]} g(a, -\frac{c'}{2})dc'$$

has spectrum $\{\frac{(-1)^n}{2^{n+1}}\sqrt{\frac{\pi}{2}}|n = 0, 1, 2, ...\}$ *and associated eigenfunctions*

$$\{A_0(a)e^{-3b^2}, A_1(a)be^{-3b^2}, ..., A_n(a)(b^n e^{-3b^2} + e^{-3b^2}p_{n-1}(b)), ...\}$$

which becomes a basis for $L^2(\mathbb{R}^2)$

Remark.1 each $p_{n-1}(b)$ is a fixed polynomial with degree $\leq n-1$ and we can denote $e_n(b) = b^n e^{-3b^2} + e^{-3b^2}p_{n-1}(b))$, while $A_i(a)$ can be an arbitrary function of a such that $A_i(a)e_i(b)$ belongs to $L^2(\mathbb{R}^2)$. Remark.2 the operator U^*TU is Hermitian, so the basis we get can be made into orthornormal basis by multiplying appropriate constant in $A_i(a)$.

4.3 ELIMINATING SPECIAL CASE

As illustrated in section 4.1, To show $\widehat{O}_3(f) + \widehat{O}_2(f) \geq \epsilon\widehat{O}_2(f)$, we only need to show all eigenvalues of T satisfies $\lambda + k \geq \epsilon k$, where $k = \frac{\Delta}{2} = \frac{1}{4}\sqrt{\frac{\pi}{2}}$. We note that the only eigenvalue that contradicts $\lambda \geq (\epsilon - 1)k$ is $-\frac{1}{4}\sqrt{\frac{\pi}{2}}$. This may be because we are working on a larger function space than we should have. If we use the condition that f is in the subspace $H = \{f \in L^2(\mathbb{R}^3)|f(x,y,z) = f(y,x,z) = f(z,y,x)\}$ instead of in $L^2(\mathbb{R}^3)$ which we previously assumed, we may be able to prove that the eigenspace associated with the problematic eigenvalue is orthogonal to the subspace H, and hence the problematic eigenvalue can be eliminated.

We start with the previously discussed T satisfying $\widehat{O}_3(g) = (Tg, g)$

$$Tg(a,b) = \frac{1}{2}\int_{\mathbb{R}} e^{-[\frac{1}{4}(a-2b+c)^2+(c-b)^2]} g(c, \frac{a+2b-c}{2})dc$$

Using the unitary transformation $Ug(a,b) = g(a+2b, \frac{b-a}{3})$, the eigenvalues of U^*TU are the same as the eigenvalues of T.

Following previous result, the only problematic eigenvalue is $-\frac{1}{4}\sqrt{\frac{\pi}{2}}$, and the corresponding eigenfunction for operator U^*TU is

$$g(a,b) = be^{-3b^2}.$$

Hence, the eigenfunction for operator T corresponding to the same eigenvalue is

$$Ug(a,b) = g(a+2b, \frac{b-a}{3}) = \frac{b-a}{3}e^{-3(\frac{b-a}{3})^2} = \frac{b-a}{3}e^{-\frac{(a-b)^2}{3}}$$

To prove that the subspace which f lives in is orthogonal to the eigenspace corresponding to $-\frac{1}{4}\sqrt{\frac{\pi}{2}}$, we only need to show that any function $g(x,R) = \int_{\mathbb{R}} f(x, R-y, R+y)e^{-y^2}dy$, where f belongs to the subspace, is orthogonal to the eigenfunction $\frac{b-a}{3}e^{-\frac{(a-b)^2}{3}}$. We will state this as a Proposition and prove it.

Proposition 4.3.1. *The inner product* $\iint_{\mathbb{R}^2} g(a,b)(a-b)e^{-\frac{(a-b)^2}{3}}dadb = 0$, *where* $g(x,R) = \int_{\mathbb{R}} f(x, R-y, R+y)e^{-y^2}dy$

Proof. By replacing g by f using definition, we only need to show

$$\iiint_{\mathbb{R}^3}(a-b)f(a,b-c,b+c)e^{-c^2-\frac{(a-b)^2}{3}}dadbdc = 0$$

It is equivalent to show

$$\iiint_{\mathbb{R}^3} af(a,b-c,b+c)e^{-c^2-\frac{(a-b)^2}{3}}dadbdc = \iiint_{\mathbb{R}^3} bf(a,b-c,b+c)e^{-c^2-\frac{(a-b)^2}{3}}dadbdc$$

by change of variable $a' = b+c, b' = \frac{a+b-c}{2}, c' = \frac{b-c-a}{2}$ that has determinant of jacobian 1, we have

$$a = b'-c', b-c = b'+c', b+c = a'$$

$$\begin{aligned}
&\iiint_{\mathbb{R}^3} af(a,b-c,b+c)e^{-c^2-\frac{(a-b)^2}{3}}dadbdc \\
&= \iiint_{\mathbb{R}^3}(b'-c')f(b'-c',b'+c',a')e^{-(\frac{a'-b'-c'}{2})^2-\frac{1}{3}(b'-c'-\frac{a'+b'+c'}{2})^2}da'db'dc' \\
&= \iiint_{\mathbb{R}^3}(b'-c')f(a',b'-c',b'+c')e^{-(c')^2-\frac{1}{3}(a'-b')^2}da'db'dc' \\
&= \iiint_{\mathbb{R}^3}(b-c)f(a,b-c,b+c)e^{-c^2-\frac{1}{3}(a-b)^2}dadbdc \\
&= \iiint_{\mathbb{R}^3} bf(a,b-c,b+c)e^{-c^2-\frac{1}{3}(a-b)^2}dadbdc \\
&\quad - \iiint_{\mathbb{R}^3} cf(a,b-c,b+c)e^{-c^2-\frac{1}{3}(a-b)^2}dadbdc
\end{aligned}$$

Since

$$\iiint_{\mathbb{R}^3} cf(a,b-c,b+c)e^{-c^2-\frac{(a-b)^2}{3}}dadbdc = \iint_{\mathbb{R}^2}(\int_{\mathbb{R}} cf(a,b-c,b+c)e^{-c^2}dc)e^{-\frac{1}{3}(a-b)^2}dadb$$

and the integrand for the integral is an odd function:

$$-[cf(a,b-c,b+c)e^{-c^2}] = (-c)f(a,b-(-c),b+(-c))e^{-(-c)^2}$$

We have that

$$\iiint_{\mathbb{R}^3} cf(a,b-c,b+c)e^{-c^2-\frac{1}{3}(a-b)^2}dadbdc = \iint_{\mathbb{R}^2} 0dadb = 0.$$

Hence,

$$\iiint_{\mathbb{R}^3} af(a,b-c,b+c)e^{-c^2-\frac{(a-b)^2}{3}}dadbdc = \iiint_{\mathbb{R}^3} bf(a,b-c,b+c)e^{-c^2-\frac{(a-b)^2}{3}}dadbdc$$

□

Now we can safely remove the eigenvalue $-\frac{1}{4}\sqrt{\frac{\pi}{2}}$ from the spectrum of T restricted on the subspace, and hence establish the validity of formula $\widehat{O}_3(f)+\widehat{O}_2(f)\geq \epsilon\widehat{O}_2(f)$. We conclude with the theorem:

Theorem 4.3.2. *$\widehat{O}_3(f)+\widehat{O}_2(f)\geq \epsilon\widehat{O}_2(f)$ for f in*

$$H = \{f \in L^2(\mathbb{R}^3) | f(x,y,z) = f(y,x,z) = f(z,y,x)\}$$

Proof. For any given f, we have $g(x,R)=\int_{\mathbb{R}} f(x,R-y,R+y)e^{-y^2}dy$, and then we can expand U^*g using the orthonormal basis of eigenfunctions in Theorem 4.2.1:

$$U^*g = \sum_{i=0}^{\infty} A_i(a)e_i(b).$$

By Proposition 4.3.1, inner product $(g, Ue_1) = (U^*g, e_1) = 0$, so

$$U^*g = \sum_{i\neq 1} A_i(a)e_i(b)$$

$$\begin{aligned} U^*Tg &= U^*TUU^*g \\ &= U^*TU(\sum_{i\neq 1} A_i(a)e_i(b)) \\ &= \sum_{i\neq 1} A_i(a)U^*TUe_i(b) \\ &= \sum_{i\neq 1} A_i(a)\lambda_i e_i(b) \end{aligned}$$

Then, we have

$$\begin{aligned} (Tg,g) &= (U^*Tg, U^*g) \\ &= (\sum_{i\neq 1} A_i(a)\lambda_i e_i(b), \sum_{i\neq 1} A_i(a)e_i(b)) \\ &= \sum_{i\neq 1} \lambda_i(A_i(a), A_i(a)) \end{aligned}$$

Similarly, we have $(g, g) = (U^* g, U^* g) = \sum_{i \neq 1} (A_i(a), A_i(a))$
For any $i \neq 1$, λ_i satisfies $\lambda_i + \frac{\Delta}{2} \geq \epsilon \frac{\Delta}{2}$, where $\Delta = \frac{1}{2}\sqrt{\frac{\pi}{2}}$, if we pick $\epsilon = \frac{3}{4}$. $\lambda_i + \frac{\Delta}{2} \geq \epsilon \frac{\Delta}{2}$ for all $i \neq 1$ imply that

$$\sum_{i \neq 1} \lambda_i (A_i(a), A_i(a)) + \frac{\Delta}{2} \sum_{i \neq 1} (A_i(a), A_i(a)) \geq \epsilon \frac{\Delta}{2} \sum_{i \neq 1} (A_i(a), A_i(a))$$

substitute the sum by (Tg, g) and (g, g), we have

$$(Tg, g) + \frac{\Delta}{2}(g, g) \geq \epsilon \frac{\Delta}{2}(g, g)$$

which is equivalent to

$$\widehat{O}_3(f) + \widehat{O}_2(f) \geq \epsilon \widehat{O}_2(f)$$

$\square$

5

CONCLUSION AND ACKNOWLEDGEMENT

In this paper, we have successfully established the relation $\widehat{O}_3(f) + \widehat{O}_2(f) \geq \epsilon\widehat{O}_2(f)$ by reducing the original problem to a simpler problem which is to find spectral gap of a linear operator T on $L^2(\mathbb{R})$, but we need to note that the operator discussed here is for 3 particles (as the domain of $\widehat{O}_2$ and $\widehat{O}_3$ operators are $L^2(\mathbb{R}^3)$). We also simplify the case by working on continuous functions rather than discrete ones. The topic discussed in this paper is part of the project on spectral gap for n particles conducted by Professor Alexander Seidel in Department of Physics and Professor Xiang Tang in Department of Mathematics. I would like to acknowledge helpful guidance and advising from Professor Xiang Tang and Professor Alexander Seidel. I would also like to acknowledge the financial support and supervision of the Advanced Research Training for Undergraduates program and Office of Undergraduate Research.

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