ARTU Progress Report

Zhengdao Chen

September 2016

The primary aim of my project is to improve the numerical method for integrating the Hodgkin-Huxley equations, a system of 4 ordinary differential equations that models the activities of 4 biophysical variables of a neuron, which is a fundamental model for neuronal modeling. If we use some common Runge-Kutta methods - a family of widely used single-step numerical methods, such as Euler's method or the 4th order Runge-Kutta method (RK4), we sometimes need to take very small time steps for the simulation in order to ensure that the numerical integration remains stable, i.e. does not blow up to infinity. Hence, my goal is to find a method with a higher stability than the current methods in simulating Hodgkin-Huxley equations.

The first attempt is to adapt the Stoermer-Verlet method (sometimes called the 'leapfrog' method) used for integrating Hamiltonian systems to a method applicable to the Hodgkin-Huxley equations. For a Hamiltonian system, the Stoermer-Verlet method updates the position variables and the velocity variables at interleaved time steps, which achieves second order accuracy. The original Hodgkin-Huxley system contains 4 variables: V, m, n and h. To apply a Stoermer-Verlet-like method, we first group m, n and h as one vector variable by letting x = [m, n, h] to form a 2-variable system, and then integrate the two variables V and x at interleaved time steps in a way equivalent to the composition of the two symplectic Euler methods, which results in a trapezoidal update for each variable at each step. Thanks to the conditional linearity of the Hodgkin-Huxley system, which means the ODE for each of the four variables is linear if the other three variables are held constant, all the updates are explicit rather than implicit.

We found that this method indeed allows a larger time step in simulation while still remaining stable than Euler's method and RK4 given the same parameters and initial conditions. However, we later discovered that Mascagni and Sherman have proposed the same method as a modified version of the trapezoidal/Crank-Nicolson method. Our work thus offers a new perspective to understand this method.

Taking a step further, we noticed that thanks to the conditional linearity of the Hodgkin-Huxley system again, each of the two variables V and x could be solved exactly if the other variable is held constant, in which case the solution is an exponential function. Hence, instead of updating V and x using the trapezoidal rule in the modified trapezoidal method, we can update each of them with the exact exponential solution of the corresponding ODE with the other variable held constant at interleaved time steps. The resulting numerical method is a kind of splitting method, and can be showed to also have second order accuracy. In numerical experiments, this method allows larger time steps while still remaining stable than the modified trapezoidal method, which is the main result of my project so far.

To test the stability of the modified trapezoidal method and the splitting method further, we implemented them to solve the van der Pol system numerically, which is a simpler 2-dimensional system. The results showed that, under the constraint of stability, the modified trapezoidal method allows a larger time step than Euler's method, and the splitting method allows an even larger time step than the modified trapezoidal method.

Another question we have been investigating is how the choice of the time step of integration affects the system-level behavior when we simulate the activity of a group of interconnected neurons using the Hodgkin-Huxley equations with external inputs designed to resemble a real-world case. We found that, at least with the modified trapezoidal method, while the time step of integration is always kept below a certain threshold to make sure that the integration is stable, changing the time step could change the duration of synchronized firing activities of the neurons. Hence, when a network of neurons is simulated, the numerical method may affect the emergent properties such as synchronization of the population, which we cannot observe from only simulating one neuron's activity. I think this is worthy of further exploring.