Credit crunches as Markov equilibria

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A B S T R A C T

We explain the large observed volatility of commercial and industrial loans as a Markov equilibrium of an economy with limited commitment in which all credit is unsecured and self-enforcing. Aggregate income growth shocks affect gains from future asset market trading, inducing fluctuations in credit limits. The economy alternates between a high state of well diversified idiosyncratic risks and a “credit crunch” state of low debt limits and poor diversification.

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1. Introduction

Lending by commercial banks to non-financial institutions fluctuates about trend five times more than U.S. GDP1, with deviations often exceeding 20 Asset backed and non-asset backed loans appear to move in tandem since the 1990s. This paper views fluctuations in unsecured loans as endogenous changes in debt limits brought about by the interaction of two factors:

1. An exogenous shock to the growth rate of aggregate income which affects borrowers’ gains from trading in asset markets.
2. An endogenous regime switch from a nearly Arrow-Debreu state of smoothly functioning loan markets to a Diamond–Dybvig state that resembles a financial panic.

The combined force of these shifts shocks the economy from a desirable aggregate state of highly valued reputations and generous credit limits to a poor state of stringent lending which everyone calls a “credit crunch” The nature of this equilibrium is periodic, described by a simple two-state Markov process whose transition probability matrix coincides with, and is triggered by, the aggregate growth shock.

Aggregate shocks in our economy are equivalent to shifts in the rate of time preference. Assuming that dated consumption goods are normal and gross substitutes makes people more (less) patient when they expect high (low) growth rates in income because high (low) incomes raise (reduce) gains from future trading in asset markets and improve (diminish) the value of solvency and of a good reputation for borrowers. Good shocks favor self-enforcement and bad ones hinder it.

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1 Good data sources are the Board of Governors for aggregate series and Compustat for firm-level data; see, for example, Herrera et al. (2011).
As debt limits fluctuate, equilibrium switches between two states: a good state of high income, generous debt limits and well-diversified idiosyncratic income risks; and a bad one of reduced total income, severe credit rationing and poorly diversified individual shocks. This movement resembles what would occur in economies where all lending is backed by collateral, like those studied in Kiyotaki and Moore (2008) and Kocherlakota (2009), if the maximum permissible leverage changes exogenously.

Unbacked lending, however, is literally a bubble that depends crucially on the borrower’s “reputation,” that is, on the difference between the payoff for solvency from that of default. The Markov equilibria we examine in this paper are simply movements from a highly inflated reputation bubble in the good state to a partially deflated bubble in the bad one. The difference between our economy and related work on financial bubbles and housing bubbles is that we analyze a single bubble that shrinks and grows while the literature typically looks at bursting bubbles that vanish asymptotically and are replaced by fresh ones.

Closest in spirit to our work is the literature on lending with commitment in environments with hidden action, studied in a long literature that stretches from Suarez and Sussman (1997) to Myerson (2012), or with hidden information studied in Azariadis and Smith (1993), Reichlin and Siconolfi (2004) and elsewhere. In all these environments, credit is rationed in order to induce borrowers truthfully to reveal private information. Debt limits simply reflect the severity of the agency problem and the rate of interest. High market yields attract deposits, loosening borrowers’ incentives to withhold information while low yields lead to disintermediation and tighten debt limits.

A key feature of dynamic models with private information is the richness and complexity of the equilibrium set which includes multiple steady states, as well as limit cycles \(^2\), and other outcomes in which credit constraints alternate between tightness and slackness. Our economy removes the private information friction and replaces it with a limited commitment friction.

In what follows, Section 2 describes an exchange economy with aggregate and idiosyncratic shocks and defines dynamic equilibria. Section 3 provides examples of deterministic periodic cycles that alternate between normal and “credit crunch” states; Section 4 extends results to stochastic economies and Section 5 discusses extensions and conclusions.

2. A dynamic exchange environment

2.1. The environment

To set the stage for studying credit crunches, we start from the limited commitment model proposed in Kehoe and Levine (1993) and elaborated since in Kocherlakota (1996), Alvarez and Jermann (2000) and others. All these studies are concerned about an infinitely-lived exchange economy populated by a continuum of infinitely-lived households with unit measure. Self-enforcing loans are secured by the threat of excluding defaulters from further asset trading in perpetuity. \( t = 0, 1, \ldots \) and there are fluctuations in both aggregate income and individual household shares of total income.

Households are indexed by their state at time \( t \) and, better yet, by the history of that state. All state histories are public information. Let

\[ s_t = (\theta_t, y_t) \in \{ \theta, y \} \times \{ y_L, y_H \} \]

describe an individual state that consists of an idiosyncratic random variable \( \theta \) and an aggregate random variable \( y \). Each variable takes on two possible values, with \( y \) denoting average income, and \( \theta \) describing an individual share of average income. In short, the endowment of a household in state \((\theta, y)\) is simply

\[ \omega(\theta, y) = \theta y \]  \hspace{1cm} (1)

Let \( s^t = (s_0, \ldots, s_t) = ((\theta_0, y_0), \ldots, (\theta_t, y_t)) \) be the state history of a typical household. We assume that the share variable \( \theta_t \in \{ \theta_L, \theta_H \} \) is independent and identically distributed with probabilities \( \phi \theta_H + (1 - \phi) \theta_L = 1 \) that is

\[ \theta_H > 1, \phi \theta_H + (1 - \phi) \theta_L = 1 \]  \hspace{1cm} (2)

We further assume that the aggregate income variable \( y_t \in \{ y_L, y_H \} \) follows a Markov process with a unitary long-term average value. In particular, denote

\[ p_t = \text{Pr}(y_{t+1} = y_t | y_t = y_t), \ t \in \{ L, H \} \]

and assume

\[ p_H y_H + (1 - p_H) y_L = 1 \]  \hspace{1cm} (3)

\[ \pi_L \geq \pi_H \]

\[ p_H y_H + (1 - p_H) y_L = 1 \]  \hspace{1cm} (4)

\(^2\) The earliest investigations are reported by Wallace (1980) and Tirole (1985). Examples of more recent work, some of it occasioned by the 2008 financial panic, are Caballero and Krishnamurthy (2006), Farhi and Tirole (2012) and Martin and Ventura (2012).

\(^3\) See Azariadis and Smith (1998) for an attempt to characterize the set of competitive equilibria in a dynamic economy with adverse selection.
where
\[
\pi^*_H := \frac{1 - \pi_t}{2 - \pi_t - \pi_H}
\]
is the long term frequency of state \(y_H\).

Common isoelastic flow utilities
\[
u(c) = (c^{1-\gamma} - 1)/(\gamma - 1), \ \gamma \geq 0
\]
help describe continuation payoffs starting at time \(t\) for any history up to point \(t\), that is,
\[
v(s^t) = E \left\{ \sum_{j=0}^{\infty} \beta^j u(c(s^{t+j})) | s^t \right\}
\]
where \(\beta \in (0, 1)\) is a common discount rate and \(c(s^t)\) is consumption at \(t\) for each particular history up to that point.

We also denote by
\[
w(s^t) = E \left\{ \sum_{j=0}^{\infty} \beta^j u(o(s^{t+j})) | s^t \right\}
\]
the continuation payoff of autarky at point \(t\). The difference between these two payoffs,
\[
p(s^t) = v(s^t) - w(s^t)
\]
captures the value of reputation, that is, the gains from trading in asset markets. Payoffs from autarky depend on the current state alone, that is, on the realized value of the individual endowment \(\theta y \in \{\theta y_H, \theta y_L, \theta y_H, \theta y_L\}\).

Autarky payoffs are convex combinations of utility flows in the two aggregate states; they are fully described in the following result whose proof is in Appendix A.

**Lemma 1.** Continuation values in autarky starting from any state \((\theta, y)\) satisfy (a)
\[
w(\theta_H, y_H) = \frac{1}{1 - \beta} \left[ \pi_H \theta_H + (1 - \pi_H) \theta_L \right]
\]
\[
w(\theta_L, y_L) = \frac{1}{1 - \beta} \left[ (1 - \pi_L) \theta_H + \pi_L \theta_L \right]
\]
\[
w(\theta_H, y_H) = w(\theta_H, y_H) - \beta y_H^{1-\gamma}
\]
\[
w(\theta_L, y_L) = w(\theta_L, y_L) - \beta y_L^{1-\gamma}
\]
where \(\gamma\) is the constant elasticity of marginal utility,
\[
\pi_H = \frac{1 - \beta \pi_L}{1 - \beta}, \ \pi_H = \frac{1 - \beta \pi_H}{1 - \beta}\rho
\]
\[
\rho = \pi_H + \pi_L - 1
\]
\[
\delta = u(\theta_H) - u(\theta_L)
\]
\[
A_H = y_H^{1-\gamma} u(\theta_H) - \beta (1 - \phi) \delta \left[ \pi_H y_H^{1-\gamma} + (1 - \pi_H) y_L^{1-\gamma} \right]
\]
and
\[
A_L = y_L^{1-\gamma} u(\theta_L) - \beta (1 - \phi) \delta \left[ (1 - \pi_L) y_H^{1-\gamma} + \pi_L y_L^{1-\gamma} \right]
\]
(b) In the absence of idiosyncratic shocks, \(\theta_H = \theta_L = 1\) and \((A_H, A_L) = (u(y_H), u(y_L))\).

Starting from any current endowment \(\theta y\), autarky pays the discounted flow of alternating between the high and low aggregate states, \(A_H(1 - \beta)\) and \(A_L(1 - \beta)\) with state-dependent frequencies.

2.2. Perfect enforcement

If loans were made with a binding commitment to repay, individual consumption would be free from all idiosyncratic risk. If, in addition, all households start from the same asset positions, then everyone would be consuming the average income in equilibrium, i.e., for all states \((\theta, y)\) we would have an allocation that corresponds to an equal-treatment optimum (E-TO)
\[
c(\theta, y) = y
\]
Proceeding analogously with Lemma 1, it is easy to see how continuation payoffs in aggregate state $y$ depend only on current aggregate income. Then

$$v(y_H) = \frac{1}{1-\beta}[(1-\varepsilon)u(y_H) + (1-\delta)u(y_L)]$$

(10)

$$v(y_L) = \frac{1}{1-\beta}[(1-\varepsilon)u(y_H) + \delta u(y_L)]$$

(11)

and ($\varepsilon, \delta$) are defined in Lemma 1, and payoffs are again discounted values of spending part of one’s time in state $y_H$ and the remainder in state $y_L$.

2.3. Self-enforcement

Is the equal-treatment optimum we just described feasible in an economy with limited commitment? That’s possible whenever the value of reputation, defined in Eq. (8), is non-negative, that is, if and only if the market payoff $v$ is at least as large as the autarky payoff for all borrowers. Borrowers are households who sell claims on their income is state $\theta_H$ and buy debt that pays off in state $\theta_L$.

The ideal allocations of an equal-treatment optimum are then consistent with self-enforcement if they obey the following incentive constraints:

$$v(y_H) \geq w(\theta_H, y_H)$$

(12)

$$v(y_L) \geq w(\theta_H, y_L)$$

(13)

Alvarez and Jermann (2000) prove that both of these constraints are satisfied and the E-TO allocation is self-enforcing if equilibrium gains from trade are sufficiently high, that is, if one of the following requirements are met:

i. $\beta$ is sufficiently close to 1;

ii. the risk aversion coefficient $\gamma$ is high enough

iii. idiosyncratic shocks are big enough, i.e., the value of $\theta_H/\theta_L$ is large.

In what follows we will suppose that both incentive constraints (12) and (13) fail at the E-TO. The following section gives an example showing that self-enforcing periodic equilibria with a low state similar to a credit crunch are possible when perfect financial markets are incompatible with limited commitment.

3. Periodic credit crunches

3.1. Equilibrium defined

What equilibria are possible when complete smoothing of idiosyncratic risks clashes with self-enforcement? To allow for the possibility of partial consumption smoothing, we use the entire state vector $s = (\theta, y)$ to condition consumption $c(\theta, y)$, asset holdings $b(\theta, y)$ and prices $q_c(\theta', y')$ for securities purchased in current state $y$ which deliver one unit of consumption in future state $(\theta', s')$, and nothing otherwise. Potential equilibria are described in the following.

**Definition 1.** Suppose the equal-treatment optimal allocation violates both self-enforcement constraints (12) and (13). Then a periodic equilibrium with rationing is a list ($c(\theta, y), b(\theta, y), q_c(\theta', y')$) of consumption allocations, security holdings and security prices with the following properties

i. For state history $s'$, households maximize the discounted payoff $\nu(s')$ s.t. the budget constraint

$$c(\theta, y) + \sum_{(\theta', y')} q_c(\theta', y')b(\theta', y') = \theta y + b(\theta, y)$$

(14)

and the debt constraint

$$b(\theta, y) + L(\theta, y) \geq 0$$

(15)

taking as given prices $q_c(\theta', y')$, and debt limits $L(\theta, y)$.

ii. Markets for goods and securities clear, that is,

$$c(\theta_H, y) + (1 - \phi)c(\theta_L, y) = y$$

(16)

$$\phi b(\theta_H, y) + (1 - \phi)b(\theta_L, y) = 0$$

(17)

for all $y \in \{y_H, y_L\}$.

iii. Self-enforcement constraints are satisfied with equality for all borrowers who are about to repay, that is,
\[ v[c(\theta_H, y_H)] = w(\theta_H, y_H) \]
\[ v[c(\theta_L, y_L)] = w(\theta_L, y_L) \]

iv. In all current and future states, security prices reflect the valuations and first-order conditions of unconstrained households, that is,
\[ q_x(\theta', y') u'[c(\theta_H, y)] = \beta \pi(\theta', y') [v[c(\theta', y')]] \]

where \( \pi(\theta', y') \) is the transition probability from the current state \((\theta_H, y)\) to the future state \((\theta', y')\) for any \(\theta' \in \{\theta_H, \theta_L\}\) and any \((y, y') \in \{y_H, y_L\} \times \{y_H, y_L\}\).

3.2. A deterministic example

The simplest way to illustrate credit crunches is to describe them in a deterministic economy in which all aggregate and idiosyncratic shocks are known in advance. Suppose, in particular, that there are two population groups \( i = 1, 2 \) of equal size with periodic income shares and periodic total income. In particular, for some \( A > 1 \) and \( a \in [0,1] \), we have individual endowments
\[ (\omega_1^t, \omega_2^t) = \begin{cases} ((1-a)A, (1+a)A), & \text{if } t = 0, 2, \ldots \\ (1+a, 1-a), & \text{if } t = 1, 3, \ldots \end{cases} \]

where average income is
\[ y_t = \begin{cases} A, & \text{if } t = 0, 2, \ldots \\ 1, & \text{if } t = 1, 3, \ldots \end{cases} \]

Let \((c_1^t, c_2^t) = (x_H y_t, (2-x_H) y_t)\) be the consumption of the high and low-share households. If low-share households are always credit rationed, then everyone is rationed periodically and self-enforcement constraints bind every two periods. For isoelastic utility functions, these constraints take the form
\[ u(x_t) + \delta_t u(2 - x_{t+1}) = u(1 + x) + \delta_t u(1 - x) \]

and equate the two-period payoffs from solvency and default when we use the discount rate
\[ \delta_t = \beta (y_{t+1}/y_t)^{1-\gamma} = \begin{cases} \beta A^{1-\gamma}, & \text{if } y_t = 1 \\ \beta A^{-\gamma}, & \text{if } y_t = A > 1 \end{cases} \]

In particular, for \( \gamma \in (0,1) \), we denote
\[ \delta_H := \beta A^{1-\gamma} > \delta_L := \beta A^{-\gamma} \]

Then a periodic equilibrium \( x_t \in \{x_L, x_H\} \) is a solution to Eq. (21) written out for odd and even time periods and solved for some \( x_L \in [1, 1 + a] \) and \( x_H \in [1, x_L] \). Specifically we look for a solution to

Fig. 1. Periodic states.
The solution, shown in Fig. 1, exists if, and only if, the desired gains from asset trading are always large enough to keep unsecured credit from vanishing altogether. The following result is proved in Appendix A.

**Lemma 2.** A necessary and sufficient condition for the existence of a deterministic equilibrium \( x_t \in \{x_L, x_H\} \) with period 2 is

\[
\frac{u'(1 + \alpha)}{\delta_L u'(1 - \alpha)} < \frac{\alpha}{\delta_H (1 - \alpha)}
\]

and

\[
u(1 + \alpha) + \delta_H u(1 - \alpha) > (1 + \delta_H) u(1)
\]

Looking at Fig. 1, it is reasonable to guess that the amplitude of the credit cycle, that is, the difference \( |x_H - x_L| \), depends on the size of the effective discount rate \((y_H y_L)^{1-\gamma}\). Since \((y_H y_L)\) cannot be much above one and \(\gamma\) cannot be too different from one, deterministic credit cycles are unlikely to show large amplitude. To correct this undesirable feature, we must allow aggregate states to be persistent. Persistent aggregate states lower the conditional value of autarky and improve debt limits when idiosyncratic shocks are white noise. In particular, a very persistent aggregate state increases fluctuations in individual incomes and may drastically improve borrowing constraints.

4. **Markov equilibrium**

We turn now to stochastic economies in which individual incomes reflect both idiosyncratic and common, economy-wide shocks. In these environments we seek to identify repetitive behavioral patterns that switch from states of light credit rationing with generous debt limits to states of severe rationing worthy of the moniker “credit crunch”. This pattern is easiest to envision when income shocks are persistent: gains from trade and credit limits are high in a persistent state of high aggregate income and low in a persistent state of low aggregate income.

Proper stochastic periodic equilibria are Markov processes defined over two aggregate states \( y_t \in \{y_L, y_H\} \) in which the consumption allocations \( c(0, y) \) of households in state \((0, y)\) are defined as

\[
c(H, H) = x_H y_H, \quad c(L, H) = \frac{1 - \phi}{1 - \phi} y_H
\]

\[
c(H, L) = x_L y_L, \quad c(L, L) = \frac{1 - \phi}{1 - \phi} y_L
\]

for some \((x_L, x_H) \in [0, \theta_L] \times [1, \theta_H]\), in a manner that equates aggregate consumption with aggregate income.

Equilibrium values for the consumption shares \((x_L, x_H)\) of the high-income share households in the aggregate state \((y_L, y_H)\) are again conjectured to obey tight self-enforcement constraints, as specified in Eqs. (18) and (19), that is,

\[
v(x_H y_H) = \omega(\theta_H, y_H)
\]

\[
v(x_L y_L) = \omega(\theta_L, y_L)
\]

**Lemma 3** explains the detailed structure of self-enforcement constraints, with proof in Appendix A.

**Lemma 3.** Define the parameter \( m := (y_H y_L)^{1-\gamma} > 1 \) and the functions

\[
q(x; \pi) := \beta(1 - \phi)(1 - \pi) u(x) - u(1 - \phi x) \quad Q(x; \pi) := [1 - \beta (1 - \phi) \pi u(x) + \beta (1 - \phi) \pi u(1 - \phi x)]
\]

Then Eqs. (28) and (29) are equivalent to

\[
mQ(x_L; \pi_H) - q(x_L; \pi_L) = mQ(\theta_H; \pi_H) - q(\theta_H; \pi_H)
\]

\[
Q(x_L; \pi_L) - mQ(x_H; \pi_H) = Q(\theta_L; \pi_L) - mQ(\theta_H; \pi_H)
\]

The easiest economy to analyze is the symmetric structure \( \pi_H = \pi_L = \pi \in (0, 1) \) and \( \phi = 1/2 \), even though this structure is not as relevant for equilibria with large fluctuations in lending as would be an asymmetric economy with persistent state, e.g., \( \pi_H > \pi_L > 1/2 \). In this symmetric economy, we have

\[
q := \beta(1 - \pi)[u(x) - u(2 - x)]
\]
and
\[ Q := (2 - \beta \pi)u(x) + \beta \pi u(2 - x) \]

Then Eqs. (30) and (31) become
\[ \begin{align*}
    mQ(x_H; \pi) - q(x_L; \pi) &= mQ(1 + x; \pi) - q(1 + x; \pi) \\
    Q(x_L; \pi) - mq(x_H; \pi) &= Q(1 + x; \pi) - mq(1 + x; \pi)
\end{align*} \tag{32, 33} \]

Our last result, proved in Appendix A, describes in detail the solution to Eqs. (32) and (33).

**Theorem 1.** Define the parameter \( \lambda = (\beta/2)(\pi + m(1 - \pi)) \) and suppose \( m = (y_H/y_L)^{1-\gamma} > 1 \). In addition, assume

1. \( A1: \pi + m(1 - \pi) < 2/\beta \)
2. \( A2: (2 - \beta \pi)u'(1 + x) < \beta \pi u'(1 - x) \)
3. \( A3: \lambda u'(1 - x) + (1 - \lambda)u(1 + x) > u(1) \)

Then

(a) Eqs. (32) and (33) have two solutions: an autarkic one at \( x_L = x_H = 1 + \alpha \) and a “credit crunch” one with \( 1 < x_L < x_H < 1 + \alpha \).

(b) The periodic credit crunch equilibrium becomes a steady state with rationing as \( m \to 1 \).

Assumptions (A1, A2, A3) require aggregate shocks, measured by the parameter \( m \), to be persistent and of moderate size; idiosyncratic risks to be larger but not too large; and households to be “not too patient”. Solutions are shown in Fig. 2 where the credit crunch corresponds to the parameter value \( x_H \), and the good state coincides with \( x_L \), that is, with the best possible consumption smoothing.

The amplitude of the cycle, as measured by the ratio \( (x_H / x_L) \) is tied up directly to the parameter \( m = (y_H/y_L)^{1-\gamma} \), which captures the growth rate of aggregate output. This is already obvious in Fig. 1. Low values of \( \gamma \) or, equivalently, large values of the intertemporal substitution elasticity, cause large cycles because they encourage borrowers to accept lower consumption smoothing today for higher consumption tomorrow. That simply means a tolerance to trade-off tight debt limits today for generous ones in the future (see Fig. 2).

5. Conclusions and extensions

This paper studies periodic equilibria in a stochastic economy without commitment or collateral where all loans must be self-enforcing. We find that periodic equilibria with fluctuating credit exist if the effective rate at which households discount future utility is sensitive to aggregate income shocks. In that case, expectations of high future income improve patience and debt limits; periodic equilibria are saddles with periodicity tied to that of aggregate shocks. Financial cycles and real cycles move in tandem. The amplitude of the cycle depends on the elasticity of substitution in consumption and also, on the relative persistence of the high and low aggregate states.
For those interested in quantitative aspects of business cycles, the class of models outlined in this paper offers a rich menu of possibilities for quantitative work once the income process is modeled with more realism than we have achieved here. A minimum dose of realism would include persistent idiosyncratic shocks and production, as suggested in a related paper by Azariadis and Kaas (2013). Introducing capital to this economy would inevitably bring to the fore issues of capital mobility and capital misallocation which are central to a voluminous literature \(^4\) but of secondary concern here \(^5\).

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Appendix A

A.1. Proof of Lemma 1

We start from the recursive relations

\[
\begin{align*}
&w(\theta_H, y_H) = \{u(\theta_H y_H) + \beta \pi_H [\phi w(\theta_H, y_H) + (1 - \phi) w(\theta_L, y_H)] + \beta (1 - \pi_H) [\phi w(\theta_H, y_L) + (1 - \phi) w(\theta_L, y_L)] \} \quad (A.1) \\
&w(\theta_L, y_L) = \{u(\theta_L y_L) + \beta \pi_L [\phi w(\theta_H, y_L) + (1 - \phi) w(\theta_L, y_L)] + \beta (1 - \pi_L) [\phi w(\theta_H, y_H) + (1 - \phi) w(\theta_L, y_H)] \} \quad (A.2)
\end{align*}
\]

These two equations specify the continuation value of autarky in the two high-income share states \((\theta_H, y_H)\) and \((\theta_L, y_L)\) as the sum of current utility flows plus discounted expected continuation values from autarky in all possible states. Because idiosyncratic risks are independently distributed, the advantage of being in a good idiosyncratic state is simply the flow of current endowment. Thus

\[
\begin{align*}
u(\theta_H y_H) - u(\theta_L y_H) &= w(\theta_H, y_H) - w(\theta_L, y_H) \\
u(\theta_L y_L) - u(\theta_L y_H) &= w(\theta_L, y_L) - w(\theta_L, y_H)
\end{align*}
\]

Next we use (A.3) and (A.4) to eliminate \(w(\theta_L y_H)\) and \(w(\theta_L, y_L)\) from (A.1) and (A.2) which then become a system of two linear equations with two unknowns. Solving that system gives the values listed in Lemma 1. Note, in addition, that homothetic utility implies that the right-hand side of (A.3) and (A.4) equal \(\delta y_H^{1-\gamma}\) and \(\delta y_L^{1-\gamma}\), respectively where \(\delta = u(\theta_H) - u(\theta_L) > 0\).

A.2. Proof of Lemma 2

The proof relies on Fig. 1 which shows a periodic equilibrium to exist if two indifference curves for \((\delta_H, \delta_L)\) lie above the perfect diversification outcome \((x_H, x_L) = (1, 1)\), and if both of them are less steep than the budget line at \((x_H, x_L) = (1 + z, 1 - z)\). In particular, it is easy to show from Fig. 1 that there is a continuous function \(f : [1, 1 + z] \times \mathbb{R}_+ \to [1, 1 + z]\) such that

(a) \(x_H = f(x_H, \delta_H)\) solves Eq. (24).
(b) \(x_L = f(x_L, \delta_L)\) solves Eq. (25).
(c) \(f(1, \delta)\) is increasing, concave in \(x \in [1, 1 + z]\), and increasing in \(\delta\).
(d) \(f(1 + z, \delta) = 1 + z, \forall \delta > 0\).
(e) \(f(1, \delta) < 1\), implied by \(f\) lying above the point \((1, 1)\).
(f) the slope of \(f\) w.r.t. \(x\) at \(x = 1 + z\) equals \(f'(1 + z, \delta) = u'(1 + z) / [\delta u'(1 - z)] < 1\) by assumption.

Then the periodic equilibrium corresponds to the point \((x_H, x_L)\) in Fig. 3.

A.3. Proof of Lemma 3

As shown in Lemma 1, we define \(z_H = (1 - \phi x_H) / (1 - \phi)\) and \(z_L = (1 - \phi x_L) / (1 - \phi)\), and start with the recursive relations

\[
\begin{align*}
v(x_H y_H) &= \{u(x_H y_H) + \beta \pi_H [\phi v(x_H y_H) + (1 - \phi) v(z_H y_H)] + \beta (1 - \pi_H) [\phi v(x_H y_L) + (1 - \phi) v(z_H y_L)] \} \\
v(x_L y_L) &= \{u(x_L y_L) + \beta \pi_L [\phi v(x_L y_L) + (1 - \phi) v(z_L y_L)] + \beta (1 - \pi_L) [\phi v(x_H y_L) + (1 - \phi) v(z_L y_L)] \}
\end{align*}
\]

\(^4\) Classic treatments are Hsieh and Klenow (2009) and Buera and Shin (2013).
\(^5\) Credit crunches bring with them last-resort lending and retailed issues in central bank policy which economists have puzzled over since Thornton (1802) and Bagehot (1873); See Azariadis (forthcoming) for a modern survey.
Proceeding as in Lemma 1, we derive continuation solvency values for the high-share agent in high and low aggregate states

\[ v(x_h y_H) = \frac{1}{1 - \beta} |x_h y_H + (1 - \alpha) B_L | \]  
(A.7)

\[ v(x_L y_L) = \frac{1}{1 - \beta} |(1 - \alpha) B_H + \alpha L B_L | \]  
(A.8)

In these equations \((x_H, x_L)\) are as defined in Lemma 1; the parameters \((B_H, B_L)\) are utility flows defined by the following equations

\[ B_H = \{ y_H^{1, \gamma} [ (1 - \beta (1 - \phi) \pi_H) u(x_H) + \beta (1 - \phi) \pi_H u(z_H) ] - y_H^{1, \gamma} \beta (1 - \phi) (1 - \pi_H) |u(x_L) - u(z_L)| \} \]  
(A.9)

\[ B_L = \{ y_L^{1, \gamma} [ (1 - \beta (1 - \phi) \pi_L) u(x_L) + \beta (1 - \phi) \pi_L u(z_L) ] - y_L^{1, \gamma} \beta (1 - \phi) (1 - \pi_L) |u(x_H) - u(z_H)| \} \]  
(A.10)

Comparing Eqs. (A.9) and (A.10) with the top two equations in Lemma 1, it is easy to see that \((A_H, A_L) = (B_H, B_L)\) when \((x_H, x_L) = (1 + \alpha, 1 - \alpha)\). In addition, the self-enforcement equilibrium conditions (28) and (29) are equivalent to

\[ A_H = B_H, \ A_L = B_L \]  
(A.11)

Next we define \(m = (y_H y_L)^{1, \gamma}\) and \((B_H, B_L)\) with \((A_H, A_L)\) as defined in Lemma 1. The outcome is the two equations listed as (30), (31) in the main text.

A.4. Proof of the main theorem

Since \(\pi_H = \pi_L = \pi\) and \(\phi = 1/2\) in our example, we suppress \(\pi\) as an argument in the functions \((q, Q)\) defined in Lemma 3, and redefine those functions as

\[ q(x) := \beta (1 - \pi) [u(x) - u(2 - x)] \]  
(A.12)

\[ Q(x) := (2 - \beta \pi) u(x) + \beta \pi u(2 - x) \]  
(A.13)

Then the self-enforcement constraints (28) and (29) in the main text become

\[ q(x_L) = q(1 + \alpha) + m [Q(x_H) - Q(1 + \alpha)] \]  
(A.14)

\[ q(x_H) = q(1 + \alpha) + (1/m) [Q(x_H) - Q(1 + \alpha)] \]  
(A.15)

By inspection, we have that \(q\) is increasing with \(q(1) = 0\) and \(Q\) is concave. Assumption A2 in the theorem implies two things: one, there exists a unique \(\bar{x} \in (1, 1 + \alpha)\) that solves the equation

\[ (2 - \beta \pi) u'(x) = \beta \pi u'(2 - x) \]

and two, the concave function \(\bar{Q}\) is increasing for \(x \in [1, \bar{x}]\), decreasing for \(x \in [\bar{x}, 1 + \alpha]\).
It follows that Eq. (A.14) has a solution \( x_L = f(x_H) \) for each \( x_H \in [1, 1 + x] \); similarly, Eq. (A.15) has a solution \( x_H = g(x_L) \) for each \( x_L \in [1, 1 + x] \). Both \((f, g)\) are increasing for \( x \in [1, 1 + x] \), decreasing for \( x \in [1, 1 + x] \) with slopes \( f'(1 + x) < g'(1 + x) < 0 \). In addition, they have \( x = 1 + x \) as a common fixed point, and \( f(x) < g(x) \) for all \( x \in [1, 1 + x] \). Continuing, it is easy to show that \( f(1) < 1 \) and \( g(1) < 1 \) if Assumption A1 and A3 hold. This leads us directly to Fig. 2 which completes the proof of the theorem.

References