Local Equilibrium Equivalence in Probabilistic Voting Models

John Wiggs Patty*
Department of Social and Decision Sciences
Carnegie Mellon University

December 6, 2004

Abstract

Electoral equilibria depend upon candidates’ motivations. Maximization of expected vote share may not lead to the same behavior as maximization of the probability of winning the election. Accordingly, it is desirable to understand when electoral equilibria are insensitive to the choice of candidate motivations. This paper examines sufficient conditions for local equilibrium equivalence between expected vote share maximization and maximization of probability of victory in the spatial model of elections with probabilistic voters.

Journal of Economic Literature Classification Numbers: C72, D72.

*This paper has benefited enormously from discussions with Jeff Banks, Kim Border, Mark Fey, Richard McKelvey, Tom Palfrey, and especially John Duggan. In addition, the helpful comments of an anonymous referee and financial support of the Alfred P. Sloan Foundation are both gratefully acknowledged.
1 Introduction

In order to offer any normative or positive predictions, a theory of electoral politics must include assumptions about the goals of candidates. Politicians may seek various things, including good policy, advancement in elected office, stature in their party, or simply (re-)election. While the exact nature of politicians’ motivations is an interesting question in its own right, these motivations may or may not have an impact on political outcomes as well. Obviously, if the motivations of individual politicians have no impact on the policies implemented, then the need to study them is commensurately reduced.

In this paper, the area of study is restricted to electoral politics. In particular, the principal question in this paper is whether populist motivations by a political candidate – such as seeking to maximize the number of votes received – will lead to different political platforms than pure majoritarian motivations – such as simply seeking to secure more votes than any other candidate. Thus, this paper adds to our knowledge of whether the choice between two plausible political motivations affect the positive predictions of models of electoral politics.

Substantively, this paper attempts to sketch the theoretical lines along which positive theories of behavior by vote-seeking and office-seeking candidates converge. The two objectives are clearly different normatively: every potential voter counts for a vote-seeker, while only a plurality of the electorate are of importance to an office-seeker. The positive differences between these motivations (in terms of the predictions that result from their assumption) are the focus of this paper.
Formally, this paper is concerned with a question of equivalence between vote-seeking and office-seeking candidates. In the context of electoral politics, two motives are equivalent if they lead to identical observed behavior on the part of the candidates. Equivalent objectives are observationally equivalent. Therefore, applied modelers have no reason to fret about which of two equivalent objectives they base a model on: the predictions each objective generates will be identical. We provide conditions under which this is the case for the choice between vote-seeking and office-seeking candidates.

1.1 Related Literature

The question of equivalence between candidate objectives in non-cooperative models of elections had long been of interest to scholars in political economy. Previous authors (including Aranson, Hinich, and Ordeshook [1974], Hinich [1977], Ledyard [1984], Patty [2002], and Duggan [2000]) have examined the question of equivalence between maximization of expected vote share, maximization of expected plurality, and maximization of probability of victory. This body of work examines the robustness of best response functions and equilibria to perturbations in the preferences of office-seeking politicians.\(^1\)

As discussed by Patty [2000], there are two principal notions of equivalence, best response equivalence and equilibrium equivalence. Best response equivalence - exhibited by two objectives that imply identical best response correspon-

\(^1\)Wittman [1983], Calvert [1985] and Duggan and Fey [2004] have examined the implications of policy motivations entering the candidates’ decision calculus.
dences - is essentially a decision-theoretic concern, as it is defined to hold regardless of the opponents’ strategies, while equilibrium equivalence - exhibited by two objective functions that generate identical sets of equilibria - is a game-theoretic (i.e., strategic) concern.

In this paper, I examine strict local Nash equilibria - profiles of candidate strategies in which no candidate has an arbitrarily small deviation which leads to a weakly greater payoff. My motivation for examining local equilibria is two-fold: of course, any global equilibrium is also a local equilibrium, so if some form of equivalence holds at all local equilibria, then it necessarily holds at all global equilibria and, second, local equilibria have historically been, and continue to be, of interest in the study of electoral politics (see, for example, Schofield and Sened [2002] and Schofield [2004]). Specifically, the results provided here apply to the models studied by Coughlin [1992], Coughlin and Nitzan [1981a, 1981b], Ledyard [1984], Schofield [2004], and McKelvey and Patty [2003].

More generally, the topic of local equilibrium equivalence represents a question of stability or robustness of predictions in models of electoral strategy. Accordingly, the study of local equilibrium equivalence speaks to the differential

\[\text{\footnotesize\textsuperscript{2}}\text{In particular, Theorem 1 of this paper, when combined with Corollary 1 in McKelvey and Patty [2003], implies that the equilibrium characterized in that paper for two party elections without abstention (in which both parties locate at the policy that maximizes the sum of voters’ expected utilities) is the only possible pure strategy equilibrium under maximization of probability of victory.}\]
incentives faced by candidates who seek popularity (or mandates) versus those whose goal is simply winning (or remaining in) office. In examining the question of equivalence, this paper focuses on what types of conditions are necessary to ensure that both types of objectives lead to identical behavior. Put another way, the results presented in this paper highlight conditions under which it is impossible to use observed platforms to infer whether different candidates have office-seeking and vote-seeking motivations.

This paper attempts to add to recent work by Duggan [2000] regarding local equilibrium equivalence in models of probabilistic voting. The main results (Theorems 1 and 2) extend Theorems 1 and 4, respectively, in recent work by Duggan [2000]. Duggan examines the question of equilibrium equivalence between maximization of probability of victory and expected vote share maximization in 2 candidate elections without abstention. In particular, Duggan shows that, in such elections, strict interior Nash equilibria under probability of victory maximization are also equilibria under maximization of expected vote share when the voters’ types are independent. As for the converse, Duggan examines a general model of probabilistic behavior known as the additive bias model (as utilized in Banks and Duggan [2003]), in which voters’ types are represented by a utility bias in favor of one candidate or the other. He proves that the negative definiteness of the matrix of second derivatives of the sum of the voters’ utility functions is a sufficient condition for a strict interior Nash equilibrium under maximization of expected vote share to be a Nash equilibrium under maximization of probability of victory.

This paper extends Duggan’s work by examining elections with more than
2 candidates and allowing for a larger set of models of voter behavior. Similar to Duggan’s results, this paper derives first and second order conditions for local equilibrium equivalence that represent a “minimal check” on the similarity of candidates’ incentives. By this, I mean that the satisfaction of this paper’s conditions guarantees that any candidate examining small changes in his or her platform will not find a better alternative platform. If the conditions are not met at some policy profile $x$ that maximizes the number of votes received by a candidate (for example), then a candidate seeking to maximize the probability of winning office will disagree about the optimality of $x$.

One of the main lessons of this paper is that even local equilibrium equivalence is difficult to obtain.\(^3\) Even with heroic assumptions about voter behavior and the nature of the equilibrium policy being considered, the fact that a policy profile is a local Nash equilibrium under maximization of vote share does not necessarily imply that it is a local Nash equilibrium under maximization of probability of victory.

In many ways, the logic underlying the paper’s results is as important as the results themselves.\(^4\) This paper demonstrates that popularity motivations and office motivations often lead to different incentives for candidates because a candidate’s probability of victory is a complicated and nonlinear function of the voters’ behaviors, whereas that candidate’s expected vote share is a simple summation of each voter’s expected behavior. Put another way, the marginal impact of any voter’s be-

\(^3\)This point is also made in a more abstract setting by Patty [2003].

\(^4\)I thank a referee for suggesting this emphasis.
behavior on a candidate’s expected vote share is constant, whereas the voter’s effect on that candidate’s probability of winning of the election can depend sensitively on the expected behaviors of the other members of the electorate. It is this difference between the two candidate objectives that differentiates the incentives faced under each.

Aside from the larger implications of the paper’s findings, the paper does provide some positive results. In particular, the paper provides a set of sufficient conditions for local equilibria to be invariant to whether candidates are assumed to maximize their vote shares or their probabilities of victory when voters’ preferences are unknown. These conditions essentially amount to symmetry and concavity of voter behavior at an equilibrium policy profile, given two assumptions: (1) voters are *ex ante* indistinguishable by their expected behavior and (2) their vote choices depend to some degree on independent, idiosyncratic random perturbations.\(^5\)

Theorem 1 in the paper states that local equilibria under maximization of probability of victory are also equilibria under maximization of expected vote share so long as the expected vote of each candidate is strictly concave with respect to her own policy choice in a neighborhood of the equilibrium policy profile. The utility of this result is best seen in light of previous work in probabilistic voting models of candidate competition: the results of this paper can be applied in a positive sense to a broad set of probabilistic voting models. For instance, Coughlin and

---

\(^5\)I remain agnostic about the source of these perturbations. They may result from errors (“trembles”) or from privately observed payoff disturbances. This foundational issue of models of probabilistic voting is discussed in more detail in McKelvey and Patty [2003].
Nitzan [1981a, 1981b] examine local Nash equilibria for two candidate elections under a probabilistic voting model, and this paper’s results apply in much of their framework, as they require voters “make independent voting decisions,” which corresponds to Assumption 1 in this paper. Additionally, the primitives of this model are congruent with those assumed in the frameworks studied by Hinich [1977], Ledyard [1984], and McKelvey and Patty [2003].

I formally define the model in the next section (Section 2). I then present the main results of the paper in Section 3 and offer a brief conclusion in Section 4.

2 The Model

Let $\mathcal{J}$, with $|\mathcal{J}| = J$, denote the set of candidates and let $\mathcal{N}$, with $|\mathcal{N}| = N$, denote the set of voters. Each candidate simultaneously chooses a point in some compact, convex policy space $X \subset \mathbb{R}^M$ with $M$ finite. I denote the policy announced by candidate $j$ by $x_j$ and the space of all $J$-dimensional vectors of policy proposals by $Y$.

The vote of voter $i$ is denoted by $a_i \in \mathcal{J}$ (abstention is not allowed). The vector of $a_i$ for all voters $i$ is denoted by $a$ and the space of all such vectors is denoted by $A$. Each $a_i$ is a random variable, potentially conditional on $y \in Y$, the vector of policies announced by the candidates. Thus, when useful, I will write $a_i(y)$ to make the dependence of $i$’s vote on the candidates’ policy announcements explicit, and similarly write $a(y)$ to denote the dependence of the vector of all voters’ votes on the candidates’ policy choices. I denote the probability that voter
votes for candidate $j$ at $y \in Y$ by $p_i^j(y)$ and the vector of all $p_i^j(y)$, for some candidate $j$ and all voters $i$, by $p^j_i(y)$. I refer to $p_i$ as a \textit{response function} for voter $i$ and assume that, conditional on a vector policy announcements $y \in Y$, the voters’ vote choices are mutually independently distributed.

\textbf{Assumption 1 (Independence)} \textit{Conditional on a vector of policy proposals, $y \in Y$, the set of $a_i(y)$ are mutually independent random variables, each distributed according to $p_i(y)$, respectively, for all $i \in N$.}

Assumption 1 is tantamount to assuming that, once the effect of the policy announcements have been taken into account, there is no coordination, either explicit or implicit, between the voters when choosing their votes. Explicit coordination might entail public deliberation before votes are cast, for example. Such coordination, if feasible, should be modeled separately in its own right. “Extra-policy” coordination by voters is plausible and an interesting topic for future research. However, for the purposes of this paper, I assume that the each candidate believes that such coordination will not occur.

Throughout the paper I assume that $p_i : Y \to \Delta(J)$, where $\Delta(J)$ denotes the $J - 1$ dimensional simplex, is at least twice continuously differentiable. In other words, $\Delta(J)$ is the space of lotteries over $J$ or, equivalently, the space of mixed voting strategies.

While Assumption 1 and the assumption that voters’ behaviors are continuous

---

\footnote{In other words, voters may act similarly due to the similarity of their underlying (and unmodeled) policy preferences. Assumption 1 implies that, once this source of correlated behavior is taken into account, voters’ vote choices are independent of one another.}
functions of the candidates’ locations may seem restrictive, both of these conditions are satisfied, for example, in the model of voting studied by McKelvey and Patty [2003]. In that model, voting behavior is a quantal response equilibrium (McKelvey and Palfrey [1995], [1998]), meaning that voters vote strategically, though the voting appears probabilistic to an outside observer.\footnote{Formally, Assumption 1 and continuity of behavior are satisfied in a Bayesian equilibrium of any voting game with what McKelvey and Patty term an admissible type distribution.} It is also satisfied in the (nonstrategic) probabilistic voting models of Coughlin [1992].

The number of votes received by candidate $j$ is denoted by $v_j(a) = |\{i \in N : a_i = j\}|$. When the context is obvious, I abbreviate this by $v_j = v_j(a)$. The set of winning candidates, given a vector of votes $a$, is denoted by $W(a)$. I assume that the electoral rule is simple plurality, so that $W(a) = \{j \in J : \forall k \in J, v_j(a) \geq v_k(a)\}$. In the case of ties (i.e., $W(a)$ is not a singleton), I assume that a fair lottery is used to select the winner from $W(a)$; I denote the winner by $w(a)$.\footnote{Of course, in most cases $w(a) = W(a)$. This notation makes it possible to deal with ties in a concise fashion.}

### 2.1 Objective Functions

This paper examines two objective functions, expected vote share and probability of victory. Conditional on a policy profile $y \in Y$, candidate $j$’s expected vote share is given by

$$V_j(y) = \frac{1}{N} \sum_{i=1}^{N} p_i^j(y), \quad (2.1)$$

The number of votes received by candidate $j$ is denoted by $v_j(a) = |\{i \in N : a_i = j\}|$. When the context is obvious, I abbreviate this by $v_j = v_j(a)$. The set of winning candidates, given a vector of votes $a$, is denoted by $W(a)$. I assume that the electoral rule is simple plurality, so that $W(a) = \{j \in J : \forall k \in J, v_j(a) \geq v_k(a)\}$. In the case of ties (i.e., $W(a)$ is not a singleton), I assume that a fair lottery is used to select the winner from $W(a)$; I denote the winner by $w(a)$.
and candidate $j$’s probability of victory is given by

$$R_j(y) = \sum_{a \in A} \left( \frac{1}{|W(a)|} 1[j \in W(a)] \prod_{i \in \mathcal{N}} p_i^{ja}(y) \right). \quad (2.2)$$

Since I will deal with first order conditions under the two objectives, it is useful to note that, for any electoral game with differentiable response functions $p$, any candidate $j \in \mathcal{J}$, and any policy profile $y \in Y$,\(^9\)

$$D_x V_j(y) = \frac{1}{N} \sum_{i \in \mathcal{N}} D_x p_i^j(y), \quad (2.3)$$

where $D_x p_i^j(y)$ is a $M \times 1$ vector. Similarly, by the chain rule,

$$D_x R_j(y) = \sum_{i \in \mathcal{N}} D_x p_i(y) \cdot D_p R_j(y), \quad (2.4)$$

where $D_p R_j(y)$ is a $J \times 1$ vector and $D_x p_i(y)$ is a $M \times J$ matrix.

I now define $p$-symmetric, interior, and $p$-interior vectors of response functions. In words, these conditions are fairly straightforward: $p$-symmetry holds at a policy profile $y$ whenever all voters behave identically in expectation when the candidates choose $y$; a policy profile $y \in Y$ is interior to $Y$ if it is not on the boundary of $Y$; and a policy profile $y \in Y$ is $p$-interior if no voter is voting for any candidate with probability zero, conditional upon $y$ being the announced policy positions.

\(^9\)Throughout this paper, for any function $f$, the notation $D_t f$ represents the gradient of $f$ with respect to $t$ and $D_t^2 f$ represents the Hessian of $f$ with respect to $t$. 

11
My definition of $p$-symmetry is satisfied when, conditional on the policy announced, each voter’s expected behavior is identical.

**Definition 1** Given a vector of response functions, $p$, a policy profile $y \in Y$ is $p$-symmetric if, for all $i, j \in \mathcal{N}$ and all $k \in J$,

$$p^k_i(y) = p^k_j(y).$$

The methods of proof used in this paper utilize the necessary first order and sufficient second order conditions for optimization. In order to simplify the analysis, I examine interior equilibria - equilibria for which the necessary first order conditions are that the gradients vanish. Perhaps obviously, an interior equilibrium is any equilibrium in pure strategies, $y^*$, where $y^*$ is in the interior of $Y$.

A vector $y \in Y$ is in the interior of $Y$, written $y \in \text{Int}(Y)$, if, for all $z \in \mathbb{R}^M$ and some $\varepsilon > 0$,

$$||z - y|| \leq \varepsilon \Rightarrow z \in Y.$$  

This paper’s analysis restricts attention to policy profiles in which no voter is voting for any candidate with probability zero. Such a policy profile is referred to as $p$-interior.\(^{10}\) The reason for this restriction is that the partial derivatives of a candidate’s probability of victory with respect to any voter’s response function are functions of the other voters’ response functions.

**Definition 2** Given a vector of response functions $p$, a policy profile $y \in Y$ is

\(^{10}\)Note that strong $p$-symmetry at a policy $x$ implies that $x$ is $p$-interior.
$p$-interior if, for all voters $i \in \mathcal{N}$ and candidates $j \in \mathcal{J}$, $p_i^j(y) > 0$.

An electoral game is a quintuple, $G = (\mathcal{J}, X, \mathcal{N}, p, u)$, where the first four components have been previously defined and $u$ is one of the two candidate objective functions being considered in this paper, probability of victory (i.e., $R$) or expected vote share ($V$). I now define a strict local equilibrium of an electoral game.

**Definition 3** Let $G = (\mathcal{J}, X, \mathcal{N}, p, u)$ be an electoral game. A policy profile $y = (x_1, \ldots, x_J)$ is a strict local equilibrium of $G$ if there exists $\varepsilon > 0$ such that, for all $j \in \mathcal{J}$ and for all $x_j'$ which are within $\varepsilon$ of $x_j$ (i.e., $||x_j' - x_j|| < \varepsilon$),

$$u_j(y) \geq u_j(x_j', y_{-j}),$$

with the inequality being strict whenever $x_j' \neq x_j$ and $||x_j' - x_j|| < \varepsilon$.

To simplify exposition, whenever I am comparing two electoral games $G = (\mathcal{J}, X, \mathcal{N}, p, u)$ and $G' = (\mathcal{J}, X, \mathcal{N}, p, u')$, I will refer to the set of strict local equilibria of $G$ and $G'$ as being strict local equilibria under $u$ and strict local equilibria under $u'$, respectively. Finally, two electoral games $G = (\mathcal{J}, X, \mathcal{N}, p, u)$ and $G' = (\mathcal{J}, X, \mathcal{N}, p, u')$ exhibit strict local equilibrium equivalence if the sets of strict local equilibria in the two games are identical.
3 Results

In this section, I provide two results which, when taken together, provide an insight into when local equilibria under maximization of expected vote share and probability of victory are equivalent. In essence, $p$-symmetric local equilibria in which voter behavior is “concave enough” are equivalent. I describe these results in reverse order, as the second result requires the most restrictive conditions.

The second of the two results (Theorem 2) states that, if $x^* \in Y$ is a $p$-symmetric strict local equilibrium under either expected vote share or probability of victory maximization, then the negative definiteness of the matrix of second order partial derivatives of each candidate $j$’s expected vote share at $x^*$, $\frac{1}{N} \sum_{i \in N} D_x^2 p_i^j(x^*)$, is a sufficient condition for equilibrium equivalence.

The first result (Theorem 1) establishes that local strict concavity of each candidate $j$’s expected vote share as a function of her own policy position for strict local equilibria under probability of victory maximization to also be strict local equilibria under expected vote share maximization. I now state this result formally. The proof is presented in the appendix.

**Theorem 1** Let $x^* \in \mathbb{R}^M$ be an interior, $p$-interior, $p$-symmetric strict local equilibrium under probability of victory maximization such that, for each candidate $j \in J$, $V_j$ is strictly concave with respect to $x_j$ in an open neighborhood of $x_j^*$. Then $x^*$ is also a strict local equilibrium under maximization of expected vote share.
For the next result, define the following matrix:

\[
\Phi_{i,k}^j (x) = \begin{bmatrix}
\frac{\partial^2 R_j(x)}{\partial p_i \partial p_k^2} & \cdots & \frac{\partial^2 R_j(x)}{\partial p_i \partial p_k^2} \\
\vdots & \ddots & \vdots \\
\frac{\partial^2 R_j(x)}{\partial p_i \partial p_k^2} & \cdots & \frac{\partial^2 R_j(x)}{\partial p_i \partial p_k^2}
\end{bmatrix} = \begin{bmatrix}
\phi^j_{1,1}(x) & \cdots & \phi^j_{1,J}(x) \\
\vdots & \ddots & \vdots \\
\phi^j_{1,J}(x) & \cdots & \phi^j_{J,J}(x)
\end{bmatrix}.
\]

It can be verified that if \( x \) is \( p \)-symmetric, then \( \Phi_{i,k}^j (x) \) is a symmetric matrix and that \( \Phi_{i,k}^j (x^*) = \Phi^j (x^*) \) for all voters \( i, k \). The next result (the proof of which is contained in the appendix) uses \( \Phi^j (x^*) \) to state sufficient conditions for a strict local equilibrium under expected vote share maximization to be a strict local equilibrium under maximization of probability of victory. Notice that the sufficient conditions now require that the matrix of second derivatives of each candidate’s expected vote with respect to that candidate’s policy choice must be negative definite, which is almost equivalent to, but slightly stronger than, strict concavity.\(^{11}\)

**Theorem 2** Let \( x^* \in \mathbb{R}^M \) be an interior, \( p \)-interior, \( p \)-symmetric strict local equilibrium under expected vote maximization. If, for each candidate \( j \), the Hessian of \( V_j (x^*) \) with respect to \( j \)'s policy is negative definite and \( \Phi^j (x^*) \) is positive semidefinite, then \( x^* \) is also a strict local equilibrium under probability of victory maximization.

Before proceeding, it is useful to consider how one might go about verifying the conditions for Theorem 2. While the interior, \( p \)-interior, and \( p \)-symmetry condi-

\(^{11}\) To see this, consider the function \( y = -x^4 \), the second derivative of \( y \) with respect to \( x \) (i.e., \( -12x^2 \)) is only negative semidefinite at \( x = 0 \), even though \( y \) is a strictly concave function of \( x \) at \( x = 0 \).
tions are simple to verify, checking the final two conditions of Theorem 2 may be very difficult. Therefore, it would be helpful to know what conditions on voter response functions would ensure that $D^2_{x_j} V_j(x^*)$ is negative definite for all $j \in J$ as well as what assumptions are sufficient to guarantee that $\Phi^j(x^*)$ is positive semidefinite for all $j \in J$.

**Negative Definiteness of $D^2_{x_j} V_j(x^*)$ for all Candidates $j$.** It is possible to show that one sufficient condition is that the following hold for each candidate $j$. For all voters $i$, $p^j_i(x^*)$ is locally concave in $x_j$ and, for at least one voter $i^j$, $p^j_{i^j}(x^*)$ has a negative definite matrix of second partial derivatives with respect to that candidate’s policy choice, $x_j$.\(^\text{12}\)

**Positive Semidefiniteness of $\Phi^j(x^*)$ for all Candidates $j$.** The requirement that $\Phi^j(x)$ be positive semidefinite is admittedly difficult to verify in practice. However, it is necessary so long as we do not restrict the first derivatives of the voters’ response functions. There are few \textit{a priori} restrictions on these derivatives that seem plausible. However, there is one restriction, which I refer to as \textit{unbiasedness}, that is satisfied in many probabilistic voting models (for example, it is satisfied in all of the probabilistic voting models discussed in the introduction). In words, this restriction requires that when a candidate $j$ unilaterally deviates from a symmetric policy profile $x$, every candidate other than $j$ gains or loses equally.

\(^{12}\)The qualifications of this condition must be carefully considered: for each candidate $j$, the concavity and negative definiteness of the Hessian are each with respect to each candidate’s \textit{own} policy choice, $x_j$. This qualification is necessary because the components of each voter’s response function must necessarily sum to one for any policy profile. I thank a referee for pointing out the potential for slipperiness here.
from a given voter.

**Definition 4** A response function $p_i$ is unbiased at a policy profile $y$ if, for all candidates $j$, it is the case that $D_{x_j}p_i^k(y) = -Dp_i^j(y)/(J - 1)$.

To understand this definition, consider unbiasedness of $p_i$ at a symmetric policy profile (i.e., when all candidates announce the same policy). Intuitively, this represents a weak requirement that voters not play “favorites” with respect to candidates with identical policy positions.\(^{13}\) Whereas unbiasedness is a restriction on the derivatives of the response function at $x^*$, the following strengthening of $p$-symmetry, which I refer to as strong $p$-symmetry, is a restriction on the evaluations of the vector of response functions at $x^*$. Strong $p$-symmetry requires that all candidates are equally likely to receive any given voter’s vote. In other words, each voter votes for a given candidate with probability $1/J$.

**Definition 5** Given a vector of response functions, $p$, a policy profile $y \in Y$ is strongly $p$-symmetric if, for all $i, j \in N$ and all $k, l \in J$,

$$p_i^k(y) = p_j^l(y).$$

The next result (Proposition 1) demonstrates that unbiasedness, strong $p$-symmetry and an algebraic relation between four conditional probabilities at a policy profile

\(^{13}\)It is weak, of course, because it does not restrict the value of $p_i$ at a symmetric policy position. One could, for example, restrict the vector of response functions to be $p$-symmetric at all symmetric policy profiles or even require that $p_i^j(y) = p_i^k(y)$ for all policy profiles $y$ in which $x_j = x_k$.\(^{17}\)
are sufficient for equilibrium equivalence at $x^*$. First, however, I describe the matrix $\Phi_j(x^*)$ whenever $x^*$ is strongly $p$-symmetric.

For any policy profile $x$, the entry in the $l^{th}$ row and $m^{th}$ column of $\Phi_{i,k}^j(x)$ corresponds to the probability that candidate $j$ wins the election, conditional on voter $i$ voting for candidate $m$ and voter $k$ voting for candidate $l$. If $x$ is strongly $p$-symmetric, it can be demonstrated that $\Phi^j(x^*)$ contains at most four distinct values. In particular, considering candidate 1 for the moment,

$$
\Phi^1(x^*) = \begin{bmatrix}
\alpha & \beta & \ldots & \beta \\
\beta & \delta & \ldots & \gamma \\
\ldots & \ldots & \ldots & \ldots \\
\beta & \gamma & \ldots & \delta 
\end{bmatrix},
$$

with $\alpha > \beta > \gamma \geq \delta \geq 0$. The ordering of $\alpha$, $\beta$, $\gamma$, and $\delta$ then correspond to the ordering of the following conditional probabilities of victory:

- $\alpha$: conditional upon both voters $i$ and $k$ voting for candidate 1,
- $\beta$: conditional upon voter $i$ (or $k$) voting for candidate 1 and voter $k$ (respectively, $i$) voting for a candidate other than 1,

---

14 As an aside, it can be shown that unbiasedness at $x^*$ implies that local equilibrium equivalence at $x^*$ can be obtained without requiring that $\Phi^j(x^*)$ be positive semidefinite. This follows because unbiasedness essentially restricts the set of vectors of response functions that can result from one candidate deviating “locally.”

15 Of course, as argued in the proof of Theorem 2, the role of $i$ and $k$ in this definition is trivial if $x$ is $p$-symmetric.

16 For candidates other than 1, $\Phi^j(x^*)$ can be similarly expressed, though the ordering of the rows and columns is altered.

17 These inequalities are strict whenever $n > 6$ and $J > 2$. 

18
• $\gamma$: conditional upon voters $i$ and $k$ voting for two distinct candidates other than 1,

• $\delta$: conditional upon voters $i$ and $k$ both voting for the same candidate other than 1.

The next result (the proof of which is contained in the appendix) provides a simple algebraic condition which, if satisfied by $\Phi^j(x^*)$ for all candidates $j$, implies equilibrium equivalence in unidimensional policy spaces when voters’ response functions are unbiased.

**Proposition 1** Let $x^* \in \mathbb{R}$ be an interior, symmetric, strongly $p$-symmetric strict local equilibrium under expected vote maximization and suppose that $p_i$ is unbiased at $x^*$ for all voters $i$. If, for each candidate $j$, the Hessian of $V_j(x^*)$ with respect to $j$’s policy is negative definite and

$$\alpha - 2\beta + \frac{(J - 1)\gamma + \delta}{J} \geq 0,$$ (3.1)

then $x^*$ is also a strict local equilibrium under probability of victory maximization.

Unfortunately, the immediate extension of Proposition 1 to multidimensional policy spaces is not necessarily true, as I have not been able to verify that the cross multiplication of the gradients of response functions – even unbiased ones – through $\Phi^j(x^*)$ will result in a negative semidefinite matrix. Nevertheless, Proposition A.2 does provide a useful result, since many models of electoral politics do assume a unidimensional policy space and, in addition, satisfy both unibasedness
and strong symmetry at convergent policy profiles.

4 Conclusion

This paper provides sufficient conditions for local equilibrium equivalence between maximization of expected vote share and maximization of probability of victory in simple plurality elections with any number of candidates and continuous probabilistic voting behavior in finite electorates. The results of this paper help illuminate the differences between maximization of expected vote share and maximization of probability of victory. In particular, even when voters’ behaviors are independent of one another, conditional on the policies announced by the candidates, heterogeneous voter response functions interact in nonlinear ways when considering any candidate’s probability of victory, while these interactions are not present when considering a candidate’s expected vote.

References


### A Proofs

**Proof of Theorem 1**
Proof: Since $x^*$ is an interior local equilibrium under maximization of probability of victory, then the following condition must hold for any candidate $j \in J$.

$$D_{x_j} R_j(x^*) = \sum_{i \in N} [D_{x_j} p_i(x^*) \cdot D_{p_i} R_j(x^*)] = (0, \ldots, 0).$$

The first order conditions for an interior local equilibrium under maximization of expected vote share are given by the following.

$$D_{x_j} V_j(x^*) = \frac{1}{N} \sum_{i \in N} D_{x_j} p_i^j(x^*) = (0, \ldots, 0).$$

By the fact that $x^*$ is $p$-interior and $p$-symmetric, it follows that, for all voters $i$ and $k$ and all candidates $j$, $D_{p_i} R_j(x^*) = D_{p_k} R_j(x^*) \gg 0$. Therefore, the first order conditions for local equilibrium under expected vote share are satisfied at $x^*$.

By the hypothesis that $\sum_{i \in N} p_i^j(x^*)$ is strictly concave in $x_j$ in an open neighborhood of $x^*$, $x_j^*$ is a strict local maximizer of $V_j(x^*)$ in an open neighborhood of $x^*$ for all candidates $j$, implying the result.

Proof of Theorem 2

Proof: The first order condition for an interior local equilibrium under expected vote maximization is $D_{x_j} V_j(x^*) = (0, \ldots, 0)$, for all $j \in J$. To see that this implies the first order condition for local equilibrium under probability of victory

---

18The notation $\gg 0$ indicates that each element of the $J \times 1$ matrix is strictly positive. This fact may seem strange at first glance. To verify that it is true, note that the gradient in question, $D_{p_i} R_j$, is not constrained by the identity $\sum_{k=1}^J p_i^k = 1$. Similarly, it is not the case that $\sum_{j \in J} D_{p_i} R_j = 0$. 

23
maximization, \( D_{x_j}R_j(x^*) = (0, \ldots, 0) \) for all \( j \in \mathcal{J} \), recall that
\[
D_{x_j}R_j(x^*) = \sum_{i=1}^{n} D_{x_j}p_i(x^*) \cdot D_{p_i}R_j(x^*),
\]
and
\[
D_{x_j}V_j(x^*) = 1/N \sum_{i=1}^{n} D_{x_j}p_i^j(x^*).
\]
Since \( x^* \) is \( p \)-symmetric, it follows that \( D_{p_i}R_j(x^*) = D_{p_k}R_j(x^*) \) for all voters \( i \) and \( k \) and candidates \( j \). Also, since \( x^* \) is \( p \)-interior, it follows that \( D_{p_i}R_j(x^*) > 0 \) for all \( i \). Therefore, the following are equivalent:

- \( D_{x_j}V_j(x^*) = 0 \),
- \( \sum_{i=1}^{n} D_{x_j}p_i^j(x^*) = 0 \),
- \( \sum_{i=1}^{n} D_{x_j}p_i(x^*) \cdot D_{p_i}R_j(x^*) = 0 \), and
- \( D_{x_j}R_j(x^*) = 0 \),

where 0 represents the \( M \times 1 \) zero vector in each case. Thus, the necessary first order conditions for maximization of probability of victory are satisfied for each candidate at \( x^* \).

A sufficient second order condition for local equilibrium under maximization of \( R \) at \( x^* \) is the negative definiteness of
\[
D^2_{x_j}R_j(x^*) = \sum_{i=1}^{n} D^2_{x_j}p_i(x^*) \cdot D_{p_i}R_j(x^*) + \sum_{i=1}^{n} D_{x_j}D_{p_i}R_j(x^*) \cdot D_{x_j}p_i(x^*)^T,
\]
(A.1)
where the dimensions of the matrices involved are as follows:

- $D^2_{x_j} p_i(x^*)$ is $M \times M \times J$,
- $D_{p_i} R_j(x^*)$ is $J \times 1 \times 1$,
- $D_{x_j} D_{p_i} R_j(x^*)$ is $M \times J$, and
- $D_{x_j} p_i(x^*)$ is $M \times J$.

Note that the right hand side of equation A.1 is a $M \times M$ matrix, since a 3-dimensional $M \times M \times 1$ matrix is equivalent to a 2-dimensional $M \times M$ matrix.

Applying the chain rule, it follows that

$$D_{x_j} D_{p_i} R_j(x^*) = \sum_{k \neq i} D_{x_j} p_k \cdot \Phi^j(x^*).$$

Now, substituting in the first order conditions for $x^*$ to be a local equilibrium under maximization of expected vote share (namely, that $\sum_{k \neq i} D_{x_j} p_k(x^*) = -D_{x_j} p_i(x^*)$), we obtain

$$D_{x_j} D_{p_i} R_j(x^*) = -D_{x_j} p_i(x^*) \cdot \Phi^j(x^*).$$

Thus, the second order conditions expressed in equation A.1 can be rewritten as

$$D^2_{x_j} R_j(x^*) = \sum_{i=1}^{n} D^2_{x_j} p_i(x^*) \cdot D_{p_i} R_j(x^*) + \sum_{i=1}^{n} -D_{x_j} p_i(x^*) \cdot \Phi^j(x^*) \cdot D_{x_j} p_i(x^*)^T.$$

By hypothesis, the first sum is negative definite. The second sum is guaranteed to
be negative semidefinite since $B \cdot C \cdot B^T$ is positive semidefinite for any real matrix $B$ so long as $C$ is positive semidefinite, the negative of a positive semidefinite matrix is negative semidefinite, and the sum of negative semidefinite matrices is negative semidefinite. Finally, since the sum of a negative definite matrix and a negative semidefinite matrix is negative definite, it follows that $D_{x_j}^2 R_j(x^*)$ is negative definite. Therefore, the first and second order sufficient conditions for local equilibrium under maximization of probability of victory are satisfied.

Proof of Proposition 1

Proof: I omit the derivation of the first order conditions for equilibrium equivalence as it is identical to the proof of Theorem 2. I skip to the verification that the unbiasedness of $p_i$ at $x^*$ and condition 3.1 ensures that

$$-D_{x_j} p_i(x^*) \cdot \Phi^j(x^*) D_{x_j} p_i(x^*)^T \leq 0.$$ 

Unbiasedness of $p_i$ at $x^*$ implies that $D_{x_j} p_i(x^*)$ can be parameterized by a single real number, $q$. In particular, considering candidate 1’s policy:

$$D_{x_j} p_i(x^*) \cdot \Phi^j(x^*) D_{x_j} p_i(x^*)^T = \begin{bmatrix} (J - 1)q \\ -q \\ \vdots \\ -q \end{bmatrix}^T \begin{bmatrix} \alpha & \beta & \ldots & \gamma \\ \beta & \delta & \ldots & \gamma \\ \vdots & \vdots & \ddots & \vdots \\ \beta & \gamma & \ldots & \delta \end{bmatrix} \begin{bmatrix} (J - 1)q \\ -q \\ \vdots \\ -q \end{bmatrix} = q^2 (J^2 \alpha - 2J^2 \beta + J(J - 1)\gamma + J\delta),$$
which is nonnegative so long as

\[ J^2 \alpha - 2J^2 \beta + J(J - 1)\gamma + J\delta \geq 0, \]

or

\[ \alpha - 2\beta + \frac{(J - 1)\gamma + \delta}{J} \geq 0. \]

Thus, the second term on the righthand side of equation A.1 is negative semidefinite (as a nonpositive real number is a negative semidefinite $1 \times 1$ matrix). Thus, the sufficient second order condition for local equilibrium equivalence is satisfied.

\[ \blacksquare \]