A spatial network approach to urban configurations

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Abstract. We propose a spatial network approach to studying urban configurations by modelling explicitly knowledge transmission, aggregation and spillovers via links formed, as well as by allowing the population and the locality role ('core' versus 'periphery') in each location to be endogenously determined in equilibrium. We establish conditions for the commonly assumed monocentric configuration to arise as the unique and efficient equilibrium outcome. We also illustrate under what circumstances a spatial equilibrium may feature multicentric, urban-rural, or multiple (disconnected) urban areas and under what conditions each configuration is socially efficient. We further characterize the spatial equilibrium by performing various comparative statics. JEL classification: R11, R23, D85

Une approche en termes de réseaux spatiaux aux configurations urbaines. On propose une approche en termes de réseaux spatiaux pour étudier les configurations urbaines en modélisant explicitement la transmission, l'agrégation et les débordements de l'information via les liens formés, et en permettant aux rôles de la population et de la localité (cœur versus périphérie) dans chaque localisation d'être déterminés de façon endogène en équilibre. On établit les conditions d'émergence de la configuration monocentrique communément assumée en tant que résultat d'équilibre unique et efficient. On montre aussi dans quelles circonstances un équilibre spatial peut être multicentrique, urbain-rural, ou composé d'aires urbaines multiples déconnectées, et dans quelles conditions chaque configuration est socialement efficiente. On aide à préciser les caractéristiques de l'équilibre spatial en faisant certains exercices de statique comparative.

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> Creative effort flourishes in a dense intellectual atmosphere . . . the possibility of more intensive intellectual contact . . . afforded by greater numbers may be an important factor in stepping up the rate of additions to new knowledge.
>
> Kuznets 1962, 328–9

1. Introduction

We propose a new approach to city formation by extending conventional network games in three important dimensions to suit the study of spatial configurations and urban structures. We explicitly model knowledge transmission, aggregation, and spillovers via links formed in our spatial networks. We also allow the population in each node or location to be endogenously determined in equilibrium based on the locational choice of freely mobile players and permit agents at a location to decide collectively whether to take the locality role as a knowledge-aggregating and transmitting core or to play passively as a periphery.

While there has been a global trend of urbanization over the past century, the spatial configurations of different urban areas have been rather divergent. In some regions there is a single dominant core city (e.g., Boston, Calgary, and Chicago), whereas in some others there are two or more cores (e.g., Minneapolis-St. Paul, Ottawa-Gatineau, and Vancouver-Surrey). While some core cities serve many satellite or peripheral cituies (i.e., featuring a core-periphery configuration such as Los Angeles, New York, and Toronto), some others serve none (i.e., featuring an urban-rural configuration such as Austin, Edmonton, and Salt Lake City).[1] Two natural questions arise. Why would different urban configurations form, particularly monocentric vs. multicentric and core-periphery vs. urban-rural? Why would different locations play distinct locality roles in various regional economies, particularly core vs. non-core? Our paper addresses these interesting issues using a new network approach that is particularly designed to suit for studying the patterns of urban configurations. Such an endeavour is valuable because the provision of a more thorough analysis of plausible forces behind the formation of cities of various patterns can help guide urban policies concerning land, housing, transportation system, and local public amenities.

It has been almost half a century since Kuznets (1962) and Jacobs (1969) stressed that knowledge spillovers in forms of production externality are crucial for geographical concentration of intellectual activities, firm clustering, and city formation. There has been abundant empirical evidence supporting the role of geographical concentration in facilitating knowledge transmission (e.g., see Saxenian 1994). Being inspired by these compelling arguments, we propose knowledge creation, exchanges and spillovers as the primary driving forces of spatial agglomeration. We model such agglomrative activities using the network approach. Network formation is of particular relevance because of its wide application to

1 See McMillen and Smith (2003) for documenting various U.S. urban structures.

analyses of transportation system and information processing, which are essential for urban trade and exchanges of productive knowledge. While locations are nodes on the spatial network, links represent potential route of transportation, communication, and trade between locations. Because knowledge can be transmitted only when the parties involved are linked as a network, network formation with endogenously determined equilibrium link patterns is a natural framework within which spatial equilibrium can be intuitively defined and the associated urban configurations explicitly analyzed. The main challenge is that we must deviate from the conventional network games in two key dimensions: one is to allow an endogenous determination of the size of each node that represents local population; another is to permit the locality role of each node to be chosen endogenously. Specifically, each worker possesses one unit of knowledge and is freely mobile to choose a location to work. Each location can play as a core or a periphery, where a core serves to facilitate knowledge aggregation and spillover.[2] Each periphery, upon paying a link maintenance cost, is served by the closest cores with knowledge spillovers. While core and all connected periphery locations are urban, periphery locations disconnected from cores are rural. So the two primary trade-offs are (i) the benefit of connection with an urban area to take advantage of knowledge spillovers versus the cost of maintaining such a network link, (ii) the benefit of forming a core city to enjoy the entirety of aggregated knowledge inflows without decays versus receiving spillovers as a periphery. Since workers are homogeneous and freely mobile, per capita income is equalized across all locations in equilibrium.

Concerning the network formation game established herein, we employ an equilibrium concept that is a combination of standard Nash equilibrium and pairwise stability commonly seen in the network literature, to suit our study of the formation of spatial networks. In particular, the spatial configuration can change in any possible way via severing/establishing links, switching roles between core and periphery, or combinations of any of them. A location is allowed to sever links, to establish a link with a core, or to switch to a periphery from a core unilaterally, but to establish a new link with a periphery or to switch to a core from a periphery must require mutual consent. We also use a simulation model that contains the following real world features: in urban areas, workers suffer nontrivial urban unemployment and incur a rent that may represent the extra cost of city living and congestion. The consideration of these features enables us to produce various spatial configurations with population distributions consistent with real world observations.

The main findings of the paper are as follows. A spatial equilibrium may feature monocentric, multicentric, core-periphery, urban-rural, or multiple urban areas, or the coexistence of two or more of these configurations (multiple equilibria). We establish various sets of sufficient conditions that require strong knowledge spillovers, low link costs, and a limited number of locations, under which the

2 Some host key industry sectors, such as headquarters and research centres, are cores.

single-core configuration is the only equilibrium outcome and under which any efficient configuration must be monocentric. When the underlying graph is complete, a shortest-distance single-core configuration (a star network) is the unique equilibrium pattern up to permutations of the core, which is also efficient. We further show that, when the link cost is not too high, for any configurations, inclusive of the frequently studied case of a monocentric city, any two locations connected to the common core must have the population density declining at a faster rate than knowledge decays as the distance away from the common core increases. With sufficiently weak knowledge spillovers and high link costs, there does not exist a spatial equilibrium featuring population agglomeration; that is, no core arises in equilibrium and the spatial economy degenerates. We further establish numerically the critical upper bounds on knowledge decays and link costs for supporting the monocentric configuration as the unique equilibrium outcome. We then illustrate that, in the intermediate ranges of knowledge decays and link costs, there may be multiple cores, featuring either multicentric (with all locations connected) or multiple urban areas (with different urban areas disconnected). Moreover, we identify various cases whereby different equilibrium configurations can become the efficient outcome.

By way of modelling methodology, previous studies of city formation and spatial configuration can be roughly divided into three streams: (i) new economic geography models, (ii) matching models, and (iii) production externality models. Our paper follows the third stream, by incorporating an explicit production externality rather than one with imperfect competition or frictional matching. The two pioneers studying urban configurations with production externality as the main agglomeration force are Ogawa and Fujita (1980) and Fujita and Ogawa (1982). Berliant, Peng, and Wang (2002) allow the extent to which knowledge, in forms of capital, spills across firms to depend on geographical factors, whereas Berliant and Wang (2008) establish conditions under which subcentres (with less population than the core) may form as population grows. Lucas and Rossi-Hansberg (2002) also consider production externality, but the spillover is in forms of labour. In contrast with this conventional urban configuration literature, our network formation approach permits not only more types of urban configurations (e.g., the urban-rural type) but more strategic behaviours concerning locational interactions such as severing and establishing links with other locations and switching roles between cores and peripheries.

Our paper is also related to the socioeconomic network literature in terms of methodology. In their pivotal works, Jackson and Wolinsky (1996) and Dutta and Mutuswami (1997) formulate the network equilibrium based on pairwise stability: a particular link structure (or graph) is stable if no linked agents want to sever the existing link to become unlinked and no unlinked agent wants to form a new link. Wang and Watts (2006) consider a modified (stronger) version of pairwise stability in that no linked agent wants simultaneously to sever the existing link and form a new link. Bala and Goyal (2000) examine the endogenous formation of networks using an even stronger equilibrium concept, where any individual players can form and sever links and an equilibrium is attained when no individual agent deviates. We combine these equilibrium concepts to capture the strategic interaction of city formation: while we allow a location to take some actions unilaterally (such as severing links and switching from core to periphery), we restrict some other actions to be reached by consensus from all locations incurring direct costs (such as establishing a new link and switching from periphery to core).[3] Perhaps most important, a major difference between our network game and those in the literature is that we allow the size (population) of each node (location) to be endogenously determined in equilibrium. These distinctive features are essential components of our concept of spatial equilibrium.

Finally, our paper is related to some very recent studies on network games with spatial considerations. In Ballester, Calvo-Armengal, and Zenou (2006), Johnson and Gilles (2006), and Carayol and Roux (2008), network games are modelled with nodes indexed by locations and with costs depending on the distance between nodes. While the latter restricts attention to a circle, the former two consider more general graphs. In Galeotti, Goyal, and Kamphorst (2006), heterogeneous link costs are allowed, depending on whether the connected nodes are in the same group, where groups can be viewed as being related to localities. While these studies develop various general setups with some locality considerations, they are not particularly designed to address urban configuration issues. In a more related work by Hojman and Szeidl (2008), a core-periphery (star) network emerges in equilibrium, which is shown to be efficient under strong decreasing returns. In contrast with all the above-mentioned studies, our paper endogenizes the population at each node and allows for active decision by players at each location in determining their locality roles in the network game, which are the main contributions of our paper to this growing literature.[4]

2. Theoretical model

All agents in our local economy are identical. We model agents at each location, simply called a *location*, as a collective player of a network formation game.[5] We construct stylized network games that are suitable for studying the patterns of urban configurations. While locations are nodes on the graph, links on this graph represent potential routes of communication, transportation, and knowledge trade between locations.

3 In the conventional network literature, whether to sever or to maintain an existing link or to create a new link is the only action. In Ballester, Calvo-Armengal, and Zenou (2006), network players may determine interaction effort. In our model, we allow for decision on the locality role.

4 The active locality role differs from the centrality measure proposed by Ballester, Calvo-Armengo, and Zenou (2006) and Hojman and Szeidl (2008) in which the core location is a result of its position in the network rather than an outcome of player's selection.

5 Implicitly, a location specifies a 'location planner' who represents the entire population of the given location. Since workers are identical, the decision by a representative locality is the same as any decision rule.

2.1. Core/periphery locations and urban areas

The *local network economy* consists of a finite set of locations denoted by I. Each location is resided with identical agents whose mass is endogenously determined. Locations are nodes on a graph g, which is connected. A location $i \in I$ chooses (i) whether to be a core (c) or a periphery (p), (ii) whether to maintain or server existing links with other locations, and (iii) whether to establish new links with others. Location i's choice of roles is denoted by $\sigma_i \in \{c, p\}$. The set of maintained links is denoted $l \in 2^g$. Let $\sigma = (\sigma_i)_{i \in I}$. A pair (σ, l) constitutes a *spatial configuration*. The maintained links among locations compose a graph. On graph l, two locations i and j are connected if there is a path that consists of maintained links from i to j.[6]

Let $C = \{i \in I \mid \sigma_i = c\}$ denote the set of core locations and $P = \{i \in I \mid \sigma_i = p\}$ denote the set of periphery locations. Let $d(i, j)$ denote the shortest distance (number of links) between connected players i and j on the economic graph l. When i and j are not connected, $d(i, j) = \infty$. Let $c(j)$ denote the set of core locations serving a periphery location j: $c(j) = \{i \in C \mid d(i, j) \leq d(k, j) \text{ for all } k \in C\}$. This set is a singleton unless there is a tie among core locations that have the same shortest distance to j. Let $s(j)$ denote the set of periphery locations that can transmit knowledge to a core location j, not blocked by another core: $s(j) = \{k \in P \mid jk \in l, \nexists i \in C \text{ s.t. } i \in v(jk)\}$, where $v(jk)$ denotes nodes on the shortest path connecting i, k on l. A location j is said to be in an *urban area U* if it is either a core itself or connected to a core; that is, either $j \in C$ or $\exists i \in C$ s.t. $d(i, j) < \infty$. A disconnected periphery is a rural location.

2.2. Knowledge transmission, aggregation, and spillovers

Each worker chooses a location to reside in and possesses exactly one unit of knowledge and one unit of time that is supplied to production.[7] Let N_i denote the population of employed workers at location i, so the amount of local knowledge at location i is also N_i. Knowledge is transmitted to another location with decay; a location receives $\delta^t (0 \leq \delta < 1)$ unit of knowledge for each unit of knowledge transmitted through t links. The knowledge decay captures the idea promoted by Fujita and Ogawa (1980), Berliant, Peng, and Wang (2002), and Cowan and Jonard (2004) in which the effectiveness of knowledge spillover or diffusion is decreasing in the distance between the players. Due to this decay, the only relevant transmission of knowledge must be via the shortest path along which the largest amount of knowledge flows arrives.

6 To avoid confusion, we call two locations 'linked' if they are connected by one link, and 'connected' if they are connected by one or more links.

7 We do not consider knowledge heterogeneity as in Berliant, Reed, and Wang (2006), Berliant and Fujita (2008), and Carayol and Roux (2008). This is because we have more than one player in each node and the size of each location (population) is endogenously determined based on free mobility. Allowing for heterogeneous knowledge would make knowledge transmission and aggregation too complex to be tractable.

The amount of raw knowledge at a core location i is $N_i + \sum_{j \in s(i)} \delta^{d(i,j)} N_j$, which is the sum of knowledge possessed by its own local workers and knowledge transmitted from all peripheries connected to the core. Let K_i denote the amount of knowledge available for production at location i. To allow for a more active role played by the core, the transmitted knowledge flows are transformed into better knowledge through the core's aggregation process; in addition to creation of new knowledge in the process, such betterment may simply be by means of effective filtering and improved comprehension for adaptation. Specifically, core i aggregates knowledge into a stock measured by $K_i = \kappa(N_i + \sum_{j \in s(i)} \delta^{d(i,j)} N_j)$, where parameter $\kappa \geq 1$ is the multiplier of knowledge creation by the core.

A periphery location j has its own local knowledge and receives spilled aggregated knowledge from serving cores $i \in c(j)$ with which the connection path is the shortest. Periphery j's knowledge stock available for production is $K_j = N_j + \sum_{i \in c(j)} \delta^{d(j,i)} K_i$, which adds up the periphery's own local knowledge with spilled knowledge aggregated by serving cores. Notably, we could allow information transmission to all locations, including information exchanges between cores. This, however, increases the complexity without generating any new insights toward understanding spatial networks.[8]

2.3. *Per capita income*

Links are costly to maintain, capturing communication, transportation, and trade costs between two locations. We postulate that it takes a unit cost z to maintain a link and that the total link cost paid by a location is shared equally by all workers in the location. Let $n_p(j)=\{k \mid k \in P, d(k,j)=1\}$ denote the set of periphery locations linked with j. Let $n_c(j)=\{k \mid k \in C, d(k,j)=1\}$ denote the set of cores linked with j. A core location does not pay for its links. A periphery location shares link cost with linked periphery locations and pays the entire

8 We have done some numerical analyses allowing for more general knowledge transmission/spillovers and allowing own local knowledge and transmitted knowledge from other locations to be aggregated using a more general constant-returns-to-scale function. We have found that our main conclusions about equilibrium and efficient configurations remain qualitatively unchanged. We have also considered a general distance-based function to capture knowledge aggregation and transmission. Specifically, core i aggregates knowledge into a stock according to $K_i = \kappa \Phi_i(N_i, (d(i,j), N_j)_{j \in s(i)})$, while periphery j has knowledge stock given by $K_j = \Psi_j(N_j, (d(i,j), K_i)_{i \in c(j)})$. Thus, the per capita income at the core and the periphery are, respectively, $(\kappa \Phi_i(N_i, (d(i,j), N_j)_{j \in s(i)})/N_i)^{\alpha}$ and $(\Psi_j(N_j, (d(i,j), K_i)_{i \in c(j)})/N_j)^{\alpha}$. These knowledge functions have the following properties: $\partial \Phi_i/\partial N_i > 0$, $\partial \Phi_i/\partial N_j > 0$, $\partial \Phi_i/\partial d(i,j) < 0$, $\partial \Psi_j/\partial N_j > 0$, $\partial \Psi_j/\partial K_j > 0$, $\partial \Psi_j/\partial d(i,j) < 0$. Define the decay factor from location $j \in s(i)$ to core i as $\delta_c(i,j) = \Phi_i((d(i,j), N_j)_{j \in x(i)})/\Phi_i(N_i, (d(j,j), N_j), (d(i,k), N_k)_{k \in s(i) \setminus j})$ and the decay factor from core $k \in c(j)$ to location j as $\delta_p(i,j) = \Psi_j(N_j, (d(k,j), K_k)_{k \in c(j)})/\Psi_j(N_j, (d(i,i), K_i), (d(k,j), K_k)_{k \in c(j) \setminus i})$. The qualification of our main theorems (see theorems 1 and 2 below) applies to $\delta_c(i,j)$ and $\delta_p(i,j)$ being large or small enough, though the quantitative analysis cannot be carried out without the explicit knowledge aggregation and transmission functional forms.

cost for links with cores.[9] Thus, a periphery location j pays total link costs $Z_j(\sigma, l) = z|n_p(j)|/2 + z|n_c(j)|$, where $|.|$ denotes the cardinality of a set.

Workers produce output with knowledge and time using a Cobb-Douglas production technology given by $Y_i = K_i^\alpha N_i^{1-\alpha}$, where $0 < \alpha < 1$ and the scaling factor is normalized to one by choice of units. An employed worker at location i receives per capita output $Y_i/N_i = (K_i/N_i)^\alpha$. Note that if locations i is disconnected from all other locations, its per capita income is $(N_i/N_i)^\alpha = 1$. Since total worker population in the economy is 1, the population identity requires $\sum_{i \in U} N_i + \sum_{j \in I \setminus U} N_j = 1$.

For theoretical results, we assume away unemployment and rent to ease illustration, though our results can be readily extended to such inclusions. Workers at a core location $i \in C$ earn a net income that is equal to per capita output, $y_i = (K_j/N_j)^\alpha$. Workers at a periphery location $j \in P$ earn a net income that is equal to per capita output net of per capita link costs, $y_j = (K_j/N_j)^\alpha - Z_j(\sigma, l)/N_j$. We denote by $y_h(\sigma, l, (N_i)_{i \in I})$ per capita income at location h under a given configuration (σ, l) and a population distribution $(N_i)_{i \in I}$ satisfying the population identity.[10]

2.4. *Equilibrium*

Given a configuration (σ, l), workers are freely mobile to maximize per capita income. Workers reach locational no-arbitrage if their per capita incomes are equalized among locations. Given a spatial configuration (σ, l), the economy is said to attain *locational no-arbitrage* if there exists a solution $\{(N_i)_{i \in I}, \bar{y}(\sigma, l)\}$ such that (i) $\bar{y}(\sigma, l) = (K_i/N_i)^\alpha, \forall i \in C$, (ii) $\bar{y}(\sigma, l) = (K_j/N_j)^\alpha - Z_j(\sigma, l)/N_j, \forall j \in P$, and (iii) $\sum_{i \in U} N_i + \sum_{j \in I \setminus U} N_j = 1$.

Given a population distribution, location i can choose a strategy $(\sigma_i, l_i) \in \{c, p\} \times 2^{\{ij | ij \in g\}}$ seeking maximal per capita income. We employ an equilibrium concept that is a combination of standard Nash equilibrium and pairwise stability commonly seen in network games. This is to capture more naturally the behaviour of localities to suit our study of the formation of the spatial configuration of a local economy. The main features of our equilibrium concept are as follows: (i) any location can deviate by changing the configuration in any possible ways in A_i via switching roles, sever or establish links, or combinations of all of them; (ii) any location can take action unilaterally if not imposing link costs on other locations (i.e., a location can sever links and switch from core to periphery unilaterally; and, (iii) when link costs to other locations are imposed, consent from all locations is required (i.e., when a new link is established, both locations involved must have higher per capita incomes; when a location switches from a

9 The main findings are robust to more general link cost technologies as long as the periphery pays more than half of the cost when linking with a core. Owing to locational mobility and endogenous population, however, it is very difficult, if not impossible, to allow the cost to depend on individual actions by means of transfers (e.g., as in Hojman and Szeidl 2008).

10 Notice that at any urban location $j \in U$, $\lim_{N_j \to 0} y_j = \infty$; hence, its population will not approach zero. Moreover, we rule out degenerate configurations with zero population at any rural location.

periphery to a core, its immediately connected locations must have as high per capita incomes as before). Accordingly, a spatial configuration (σ, l) is called *network-stable* under population distribution $(N_i)_{i\in I}$ if there is no location $i \in I$ and no $(\sigma_i', l_i') \in \{c, p\} \times 2^{\{\bar{ij} \mid \bar{ij} \in g\}}$ and $\sigma_j' = \sigma_j$, for all $j \neq i$ and $l' = l \backslash \{\bar{ij} \mid \bar{ij} \in g\} \cup l_i'$, such that (i) $y_i'(\sigma', l', (N_i)_{i\in I}) > \bar{y}(\sigma, l)$ and (ii) $y_j'(\sigma', l', (N_i)_{i\in I}) > \bar{y}(\sigma, l)$, $\forall j \in J_1 \cup J_2$, where $J_1 = \{j \in I \mid \bar{ij} \in l_i' \backslash l\}$ and $J_2 = \{j \in I \mid \bar{ij} \in l_i', \sigma_j = p\}$ if $\sigma_i' \neq \sigma_i = p$ and $J_2 = \emptyset$ otherwise. While J_1 is the set of new links established in l', J_2 is the set of linked peripheries if i switches to play core. A deviation initiated by location i is successful if all of the following three scenarios arrives (i) location i can reach a higher per capita income (ii) when a new link is established, both locations have higher per capita income, and (iii) when a periphery location switches role to be a core, all immediately connected periphery locations are better off. Thus, a *spatial equilibrium* is a tuple of spatial configuration, population distribution, and per capita income $((\sigma, l), (N_i)_{i\in I}, \bar{y}(\sigma, l))$, such that (σ, l) is network-stable under $(N_i)_{i\in I}$ and $\{(N_i)_{i\in I}, \bar{y}(\sigma, l)\}$ attains locational no-arbitrage under (σ, l). In contrast to pairwise-stable networks, our equilibrium is independent of the initial graph.

2.5. *General propositions*

We are now ready to present three fundamental results of spatial equilibrium in our network formation game. The first theorem focuses on the existence of spatial equilibrium with population agglomeration in the sense that the economy has at least one core emerging with links to some periphery locations. It shows that when knowledge decays not too fast and link costs are not too large, there is a spatial equilibrium with only one core that is connected to all other locations, where any location can be such a core.

Consider a general local network economy with a finite number of locations on graph g. Let (σ^*, l^*) denote the following configuration: (i) pick location c as the single core, and (ii) link every location to the core by a path with the shortest distance. Thus, (σ^*, l^*) is a tree graph where the distance from the core to location i is $d(c, i)$. For population distribution $(N_i)_{i\in I}$, the stock of aggregated knowledge at the core is $K_c = \kappa(N_c + \sum_{i\in P} \delta^{d(c,i)} N_i)$. Therefore, the core has per capita income, $y_c((\sigma^*, l^*), (N_i)_{i\in I}) = (K_c/N_c)^\alpha$, while periphery location $i \in I\backslash c$ has per capita income, $y_i((\sigma^*, l^*), (N_i)_{i\in I}) = (1 + \delta^{d(c,i)} K_c/N_i)^\alpha - Z_j(\sigma^*, l^*)/N_i$. Define the average income as follows: $\hat{y}((N_i)_{i\in I}) = \sum_{i\in I} N_i y_i((\sigma^*, l^*), (N_i)_{i\in I})$. By construction, we have $\sum_{i\in I} N_i(y_i((\sigma^*, l^*), (N_i)_{i\in I}) - \hat{y}((N_i)_{i\in I})) = 0$.

Before proving the existence of such a population distribution featuring a single core with which all periphery locations link, we establish in the next two lemmas that (i) when N_i is small enough but not approaching zero, per capita income y_i is always larger than the average income $\hat{y}$; (ii) when δ is sufficiently large and z is sufficiently small, the population at an urban location is bounded below by $\bar{\varepsilon} > 0$.

LEMMA 1. *When z is sufficiently small, in the above configuration* (σ^*, l^*) *, for each population distribution* $(N_i)_{i \in I}$ *, for each* $i \in I$ *, there is a small* $\varepsilon_i > 0$ *such that, if* $N_i = \varepsilon_i$, $y_i((\sigma^*, l^*), (N_i)_{i \in I}) > \hat{y}((\sigma^*, l^*), (N_i)_{i \in I})$.

Proof. All proofs are relegated to the appendix.

LEMMA 2. *When* δ *is sufficiently large and z is sufficiently small, in the above configuration* (σ^*, l^*) *, for each non-rural location* $i \in I$*, there is a small* $\bar{\varepsilon} > 0$ *such that,* $N_i \geq \bar{\varepsilon}$.

Define $\Omega(\beta_0, \beta_1, \beta_2, \beta_3, \beta_4) = \min_{N \in [\bar{\varepsilon}, 1]}[(\beta_0 + \beta_1 \delta\kappa/N)^\alpha - (\beta_2 + \beta_3 \delta\kappa/N)^\alpha - \beta_4 z/N]$, which can be viewed as a minimal 'net gain' function over the compact domain of population $[\bar{\varepsilon}, 1]$. It is clear that Ω is increasing in β_0 and β_1, but decreasing in β_2, β_3 and β_4. Moreover, it is unambiguous that Ω is lower with rising link maintenance costs z. This minimal net gain is positive provided that $\beta_0 \geq \beta_2$ and z is sufficiently small. We next consider the following set of (sufficient) conditions that all impose upper bounds on the severity of knowledge decays $(1 - \delta)$ and the magnitude of link maintenance costs.

Condition S (Small knowledge decays and link maintenance costs).
(S-1) $\Omega(1, \delta^{|I|-2}(\bar{\varepsilon} + \delta^{|I|-1}(1 - \bar{\varepsilon})), -\delta\kappa, 1, (|I| - 1)/2) > 0$;
(S-2) $\Omega(1, 1 + \delta\bar{\varepsilon}, 1, 1, 1/2) > 0$;
(S-3) $\min\{\Omega(1 + \delta^2\kappa, \bar{\varepsilon}, 0, 0, 1), \Omega(1, 2\bar{\varepsilon}\delta^{|I|-2}, 0, 0, 1)\} > 1$;
(S-4) $\Omega(1, 1 + \delta^{2|I|-3}\bar{\varepsilon}, 1, 1, 1/2) > 0$.

Notably, all conditions are more likely to hold when δ is higher, z is lower, and $|I|$ is smaller. It is also straightforward that (S-2) always hold true when (S-4) is met. We are now prepared to use lemmas 1 and 2 to establish the following:

THEOREM 1. *Consider a local network economy with a finite number of locations* $|I| \geq 3$ *on an arbitrary connected graph. Then,*

(i) (Existence & Indeterminacy) *under conditions (S-1) and (S-2), a spatial equilibrium with any location as the single core connected to all other locations exists;*
(ii) (Unique Pattern) *under conditions (S-1), (S-3), and (S-4), any spatial equilibrium consists of a single core connected to all other locations;*
(iii) (Efficient Pattern) *under conditions (S-1), (S-3), and (S-4), any efficient configuration is monocentric.*

Theorem 1 shows generic indeterminacy of spatial equilibria when knowledge decays and link maintenance costs are sufficiently small. The presence of multiple equilibria is nonetheless the result of indeterminacy of the core location. This finding resembles that in Wang (1990), where there can be a continuum of equilibria with the locations of competitively formed trading centres indeterminate. Interestingly, all such equilibria in our network games always feature a single core connected to all other locations, regardless of the initial graph. The network approach established in the present paper offers additional insights toward understanding the likelihood for a single-core configuration to arise in equilibrium: small knowledge decays and link maintenance costs with low number of locations.

When the underlying graph is complete, a shortest-distance single-core configuration defined above is the star network. Therefore, this is the unique equilibrium pattern up to permutations of the core and is also efficient.

COROLLARY 1 (The Star Network). *Suppose knowledge decays and link maintenance costs are sufficiently small such that conditions (S-1), (S-3), and (S-4) are met. Then, when the underlying graph g is a complete network, the start network is the only equilibrium and efficient configuration up to permutations of the core.*

The second theorem concerns the impossibility of agglomeration in spatial equilibrium due to high knowledge decays or high link costs.

THEOREM 2 (Impossibility on Spatial Networks). *Consider a local network economy with a finite number of locations on an arbitrary graph. With sufficiently large knowledge decays and link maintenance costs such that* $z > 2\delta\kappa$*,the spatial equilibrium degenerates with all locations disconnected and no population agglomeration.*

The impossibility of any forms of spatial agglomeration is a consequence of large knowledge decays and link maintenance costs. When the agglomeration force–knowledge aggregation and transmission measured by $\delta\kappa$–becomes trivial, any small link maintenance cost will lead to completely degenerate equilibrium with isolated nodes.

The third theorem characterizes patterns of population and per capita income at various locations on an arbitrary graph.

THEOREM 3. *Consider a local network economy with a finite number of locations* $|I| \geq 3$ *on an arbitrary graph. When knowledge decays and link maintenance costs are sufficiently small such that* $z < 2\alpha\delta^{|I|-1}/(|I|+1)$*, a spatial equilibrium features the following: if periphery locations i and j are served only by the same core, pays the same link cost, and* $d(c,j) > d(c,i) > 1$*, then the population declines with distance at a rate faster than the decay rate of knowledge, that is,* $N_j/N_i < \delta^{d(c,j)-d(c,i)}$.

Theorem 3 applies generally to all configurations inclusive of the most interesting case of a monocentric city with all locations connected to a single core and with declining population from the core. The results imply that under a monocentric configuration with sufficiently small knowledge decays and link maintenance costs and sufficiently low number of locations, the population density away from the centre declines at a faster rate than knowledge decays.

Denote the total population in the urban area as $\hat{N} = \sum_{i \in U} N_i$ and let $\hat{N}_c(\delta, z, |I|) = (1-\alpha)\delta^{(|I|-1)}\kappa\bar{\varepsilon}^{\alpha} z/(\alpha\delta^{(|I|-1)}\kappa^{1+\alpha}\bar{\varepsilon} - z\bar{\varepsilon}^{\alpha-1})$. Further define

$$\Upsilon\left(\hat{N};\delta,z,|I|\right)=\frac{\alpha\delta^{(|I|-1)}\kappa\bar{\varepsilon}}{\delta^{(|I|-1)}\kappa\bar{\varepsilon}+\hat{N}}\left[\kappa\bar{\varepsilon}+\left(1+\frac{1}{\bar{\varepsilon}\hat{N}}\right)z\right]-\left(1+\frac{1}{\bar{\varepsilon}\hat{N}}\right)z$$
$$-\frac{\alpha}{\bar{\varepsilon}}\frac{\delta\kappa}{\delta\kappa+\bar{\varepsilon}\hat{N}}\left(z+\frac{\kappa}{\bar{\varepsilon}}\hat{N}\right).$$

Thus, the critical value $\hat{N}_c$ is decreasing in δ and increasing in z and $|I|$. Moreover, while it is clear that Υ is always decreasing in $|I|$, it is not difficult to show that Υ is also decreasing in z because $\alpha\delta^{(|I|-1)}\kappa\bar{\varepsilon}/(\delta^{(|I|-1)}\kappa\bar{\varepsilon}+\hat{N})<1$.[11] In the case with a monocentric configuration, we can characterize equilibrium per capita income in response to population:

COROLLARY 2. *In a local network economy with a finite number of locations on an arbitrary graph, a spatial equilibrium with all locations connected to a single core and with declining population from the core features the following:*

i) *when* $\hat{N}<\hat{N}_c$, *equilibrium per capita income rises with the total population of the local network economy;*
ii) *when* $\Upsilon(\hat{N})>0$, *equilibrium per capita income declines with population.*

Corollary 2 provides an interesting comparative-static result: as the total population of the local network economy increases, equilibrium per capita income may rise or fall, depending crucially on the efficacy of knowledge creation and transmission (captured by δ and κ) and the link cost (z). A larger overall population, on the one hand, contributes to a larger knowledge stock and a lower per capita link cost. On the other hand, it lowers per capita output for a given stock of local knowledge. When the link cost is small and knowledge creation and transmission are more effective, the marginal benefit of higher overall population from creating knowledge and from reducing per capita link cost cannot outweigh the direct marginal cost (with the local knowledge stock rising proportionally less than the local population). Thus, rising population lowers equilibrium per capita income.

3. Characterization of equilibrium configurations

We now add real world features such as urban unemployment and land rent to the basic model. While the former generalization allows a gap between urban and rural realized income despite the equalization of expected income ex ante, the latter incorporates the fact that urban living incurs costs from higher land rents or congestion. These added features enable us to generate numerically

11 Yet the effect of δ on Υ remains ambiguous.

realistic population distribution over various locations (namely, core, subcentres, peripheries, and rural areas).

We consider full employment in rural areas and non-trivial unemployment in urban areas.[12] Since workers are identical, they get employed with equal probability. Given an urban employment rate $e \in (0, 1)$, the expected income for a worker residing in an urban area location $i \in U$ is ey_i. Note that N_i is the working population at location i and N_i/e is the total population including non-workers. Total working population in the economy is one. Therefore, the population identity implies $\sum_{i \in U} N_i/e + \sum_{j \in I \setminus U} N_j = 1$. We denote by $Q_i(\sigma, l)$ the exogenous per capita land rent paid by employed workers in i under a spatial configuration (σ, l). The per capita land rent in a core is simply q, and in a periphery location $j \in P$ served by core i it takes the following form: $Q_j(\sigma, l) = b^{d(j,i)}q$, where b is the rent gradient. That is, the rent schedule is downward sloped and decreasing geometrically in the distance away from the core by which a peripheral location is served. This captures the pattern in the urban land use literature as illustrated by Fujita (1989). Workers at a core location $i \in C$, a periphery location $j \in P \cap U$ linked with a core i, and a rural location $r \in I \setminus U$ are expected to earn a net income at, respectively,

$$y_i = e[(K_i/N_i)^\alpha - q], \forall i \in C$$

$$y_j = e\left[(K_j/N_j)^\alpha - \frac{z|n_p(j)|}{2N_j} - \frac{z|n_c(j)|}{N_j} - b^{d(j,i)}q\right], \forall j \in P \cap U$$

$$y_r = 1.$$

In spatial equilibrium, the population identity must be met and the net income given above must be equalized. More specifically, for any spatial configurations, we must have $y_i = y_j = \bar{y} \geq 1$ for all $i \in C$ and $j \in P \cap U$; when the spatial equilibrium features any rural locations, the net per capita income must be equal to one (i.e., $\bar{y} = 1$ if $I \setminus U \neq \emptyset$).

3.1. Benchmark setup

For simplicity, we restrict our attention to a linear spatial structure with five locations, indexed from left to right by $i = 1, 2, 3, 4, 5$, where the five locations as a whole constitute a closed local economy. This simple structure is sufficient for examining at least the following four symmetric spatial configurations that are frequently studied in the urban economics literature (refer to figure 1 for a graphic illustration of these configurations).

i) Monocentric configuration (m). Location 3 serves as the unique core, while locations 1, 2, 4, and 5 all are peripheries connected to it.

12 This captures the phenomenon that the realized income in the urban area is higher than in the rural area, but the expected income is equalized owing to urban unemployment (Harris and Todaro 1970).

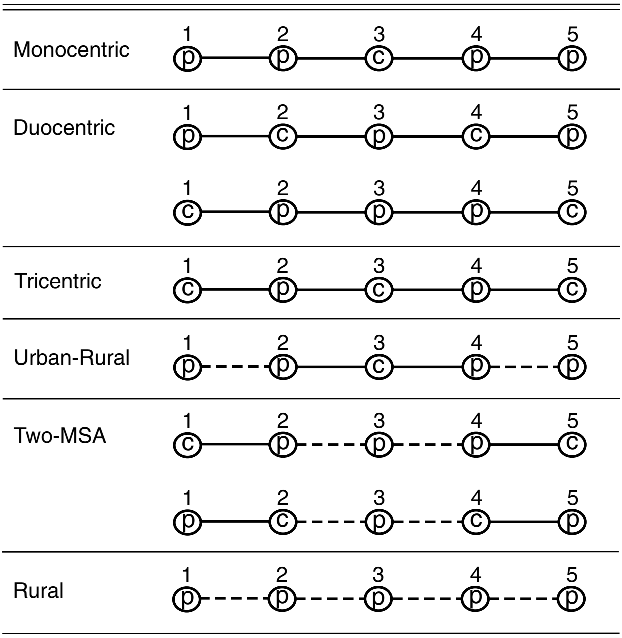


FIGURE 1 Network configurations

ii) Multicentric configuration, particularly duocentric configuration (d). There are two patterns of duocentric configuration. In the first, studied in this subsection, locations 2 and 4 serve as cores and locations 1 and 5 as peripheries served by cores 2 and 4, respectively. Location 3 is a periphery served by both cores. In the second, studied in our working paper, locations 1 and 5 serve as cores and 2, 3, and 4 as peripheries. With five locations, there is only one symmetric tricentric configuration where locations 1, 3, and 5 are cores and locations 2 and 4 are peripheries, which is also examined in our working paper.[13]

iii) Urban-rural configuration (u). Location 3 is the unique core, serving periphery locations 2 and 4; locations 1 and 5 are rural peripheries, disconnected from the urban area $U = \{2, 3, 4\}$.

iv) Two-MSA configuration (s). There are two patterns. In the first, which is our focus, location 1 (resp. 5) is a core serving peripheral cities 2 (resp. 4). In the second, examined in our working paper, location 2 (resp. 4) is a core serving peripheral cities 1 (resp. 5). In both cases, location 3 is completely disconnected; there are two separate urban areas (MSAs), $U = \{1, 2\} \cup \{4, 5\}$.

13 Our working paper can be found at: http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1420625 and http://pingwang.wustl.edu/. Notably, the range of existence and the comparative statics in the tricentric configuration are similar to those of the duocentric case.

There is a degenerate case of rural configuration where all peripheries are disconnected, which is uninteresting and hence ignored.

In this subsection, we focus exclusively on the benchmark setup as described in section 2: periphery locations transmit knowledge to core locations, whereas periphery locations receive spillovers only from the closest core. The following observations are therefore immediate: (i) A location cannot play core alone; there must be a link to a periphery location. (ii) Two core locations will not link because there is no benefit. (iii) No periphery locations will remain linked without a core connected to them, since there is no gain but only cost. (iv) A core will not sever any link to a periphery while remaining a core, since link costs are entirely paid by peripheries. (We will relegate the discussion of alternative setups in knowledge transmission and spillovers to section 3.2.)

It should be noted that, since we examine only symmetric configurations, locations at symmetric positions, 1 vs. 5 and 2 vs. 4, have similar deviation conditions to check. For illustrative purposes, we use letters and symbols to describe configurations: letters ‘c’ and ‘p’ indicate the choice of being a core or a periphery; ‘-’ indicates a maintained link between two locations and ‘..’ indicates a severed link.

3.1.1. Monocentric configuration (p-p-c-p-p)

In this configuration, location 3 receives knowledge transmitted from all periphery locations, and, upon aggregating them, serves to all periphery locations. The per capita income at each location is equal to $\bar{y}$ in equilibrium. Since all of the worker population live in urban areas, $(\sum_{i=1}^{5} N_i)/e = 1$.

We check for deviations from network stability in the following, (i) A location can sever all links and stay alone. It then yields the rural per capita income, which is 1. Equilibrium requires $\bar{y} - 1 \geq 0$. (ii) Location 2 can sever the link with location 1 (p..p-c-p-p). It pays for only one link but does not receive knowledge from location 1. Equilibrium requires $\bar{y} - e\{[(\delta^2\kappa + 1) + \delta\kappa(N_3 + \delta N_4 + \delta^2 N_5)/N_2]^\alpha - z/N_2 - bq\} \geq 0$. (iii) Location 2 can also play core and link only with location 1 (p-c..c-p-p), which must be mutually beneficial to location 1. Equilibrium requires $\bar{y} - e[(\kappa + \delta\kappa N_1/N_2)^\alpha\text{-}q] \geq 0$ or $\bar{y} - e\{[(\delta^2\kappa + 1) + \delta\kappa N_2/N_1]^\alpha - z/N_1 - bq\} \geq 0$. (iv) Location 1 can play core (c-p-c-p-p) without paying link cost but receiving knowledge from location 2, which must be mutually beneficial to location 2. Equilibrium requires $\bar{y} - e[(\kappa + \delta\kappa N_2/N_1)^\alpha - q] \geq 0$ or $\bar{y} - e\{[(2\delta^2\kappa + 1) + \delta\kappa(N_1 + N_3 + \delta N_4 + \delta^2 N_5)/N_2]^\alpha - 2z/N_2 - bq\} \geq 0$.

3.1.2. Multicentric configuration (p-c-p-c-p)

There are two types of multicentric configurations: (i) duocentric (p-c-p-c-p and c-p-p-p-c) and (ii) tricentric (c-p-c-p-c). In this subsection, for brevity we discuss only the first pattern of the duocentric configuration, p-c-p-c-p. We will relegate the second pattern of the duocentric configuration and the tricentric configuration to appendices A and B of our working paper (see fn 13, above).

We illustrate the case with locations 2 and 4 as cores. Location 2 (resp. location 4) as a core receives transmitted knowledge from locations 1 and 3 (resp. locations 3 and 5) and serves back the aggregated knowledge with decays. Again, all incomes are equalized in equilibrium $y_i=\bar{y}$, whereas population identity requires $(\sum_{i=1}^{5} N_i)/e = 1$. The following deviations by location must be ruled out for this configuration to be an equilibrium outcome. (i) Every location can sever links and stay alone. (ii) Location 2 can play periphery (p-p-p-c-p). It then receives knowledge from all locations but pays for one link cost. (iii) Location 2 can play periphery and sever the link with location 1 (p..p-p-c-p). (iv) Locations 3 can sever one link (p-c-p..c-p). It saves link cost but loses knowledge from locations 4 and 5.[14]

3.1.3. Urban-rural configuration (p..p-c-p..p)

Rural locations 1 and 5 yield unit per capita income. Location 3 receives transmitted knowledge from locations 2 and 4 and serves back with the aggregated knowledge with decays. In the presence of rural areas, per capita incomes at all locations are given by $y_i = \bar{y} = 1$. Population feasibility requires $N_1 + (N_2 + N_3 + N_4)/e + N_5 = 1$, and $N_1 = N_5$. No location will deviate to stay alone, since the equilibrium per capita income is 1. The following deviations by locations must be ruled out for this configuration to be an equilibrium outcome. (i) Location 2 can link with location 1 and form (p-p-c-p..p). This move needs to be mutually beneficial. They both pay more link costs but also receive more knowledge. (ii) Location 2 can play core and link with location 1 (p-c..c-p..p). It saves link cost by being a core. This move needs to be mutually beneficial to location 1. (iii) Location 1 can play core and link with location 2 and form (c-p-c-p..p). This move needs to be mutually beneficial.[15]

14 To rule out deviations (i)–(iv), it is required that $\bar{y} - 1 \geq 0$ and

$$\bar{y} - e\left[\left(\left(\delta^4\kappa + 1\right) + \delta^2\kappa\left(\delta^3 N_1 + \delta N_3 + N_4 + \delta N_5\right)/N_2\right)^{\alpha} - z/N_2 - b^2 q\right] \geq 0$$
$$\bar{y} - e\left[\left(\left(\delta^4\kappa + 1\right) + \delta^2\kappa\left(\delta N_3 + N_4 + \delta N_5\right)/N_2\right)^{\alpha} - z/(2N_2) - b^2 q\right] \geq 0$$
$$\bar{y} - e\left[\left(\left(\delta^2\kappa + 1\right) + \delta\kappa\left(\delta N_1 + N_2\right)/N_3\right)^{\alpha} - z/N_3 - bq\right] \geq 0.$$

15 To rule out deviations (i)–(iii), it is required that

$$\bar{y} - e\left[\left(\left(\delta^2\kappa + 1\right) + \delta\kappa\left(\delta^2 e N_1 + N_3 + \delta N_4\right)/N_2\right)^{\alpha} - 3z/(2N_2) - bq\right] \geq 0 \text{ or}$$
$$\bar{y} - e\left[\left(\left(\delta^4\kappa + 1\right) + \delta^2\kappa\left(\delta N_2 + N_3 + \delta N_4\right)/(eN_1)\right)^{\alpha} - z/(2eN_1) - b^2 q\right] \geq 0$$

$$\bar{y} - e\left[\left(\kappa + \delta\kappa e N_1/N_2\right)^{\alpha} - q\right] \geq 0 \text{ or}$$
$$\bar{y} - e\left[\left(\left(\delta^2\kappa + 1\right) + \delta\kappa N_2/(eN_1)\right)^{\alpha} - z/(eN_1) - bq\right]$$

$$\bar{y} - e\left[\left(\kappa + \delta\kappa N_2/(eN_1)\right)^{\alpha} - q\right] \geq 0 \text{ or}$$
$$\bar{y} - e\left[\left(\left(2\delta^2\kappa + 1\right) + \delta\left(\kappa e N_1 + N_3\right)/N_2\right)^{\alpha} - 2z/N_2 - bq\right].$$

3.1.4. Two-MSA configuration (c-p..p..p-c)

There are two cases of two-MSA systems, with locations 1 and 5 as cores (c-p..p..p-c) and with locations 2 and 4 as cores (p-c..p..c-p); location 3 is the rural place separating the two MSAs. For brevity, we examine only the first case here. Locations 1 and 5 receive transmitted knowledge from locations 2 and 4, respectively and serve back the aggregated knowledge with decays. Rural location 3 yields unit per capita income. While incomes are equalized in equilibrium, population identity requires $(N_1 + N_2 + N_4 + N_5/e) + N_3 = 1$. The following deviations by location must be ruled out for this configuration to be an equilibrium outcome. (i) Locations 2 and 3 can link (c-p-p..p-c). They both pay more link costs but receive more knowledge. This move should be mutually beneficial. (ii) Location 3 can link with locations 2 and 4 (c-p-p-p-c) and forms a duocentric configuration. This move should be beneficial to locations 2 and 4.[16]

3.2. Alternative setups

There are alternative ways to model the transition of knowledge. First, a periphery location may transmit knowledge to other peripheries. The available knowledge at periphery j becomes $K_j = N_j + \sum_{k\in s(j)} \delta^{d(j,k)} N_k + \sum_{i\in c(j)} \delta^{d(j,i)} K_i$. We will examine only the monocentric configuration for this alternative setup. Second, a periphery location may receive knowledge spillovers from all cores in addition to the closest one. We will examine only the duocentric configuration for this setup. The equilibria in these alternative setups are contrasted with the benchmarks in the numerical analysis in section 4.1.

3.2.1. Alternative monocentric (p-p-c-p-p)

The monocentric configuration is now re-examined when knowledge from a periphery location can spill over to another periphery location until reaching another serving core. For example, the knowledge from location 1 shows up at location 2 once without the multiplier and once with the multiplier. One can easily see that the equilibrium conditions for network stability are the same as for the benchmark model in section 3.1.1.

3.2.2. Alternative duocentric (p-c-p-c-p)

The duocentric configuration, with locations 2 and 4 as cores, is now re-examined when the cores serve all periphery locations. The equilibrium conditions for

16 To rule out deviations (i)–(ii), it is required that

$$\bar{y} - e\left[\left((\delta^2\kappa + 1) + \delta\kappa\left(N_1 + \delta^2 e N_3\right)/N_2\right)^{\alpha} - 3z/(2N_2) - bq\right] \geq 0 \text{ or}$$
$$\bar{y} - e\left[\left((\delta^4\kappa + 1) + \delta^2\kappa\left(N_1 + \delta N_2\right)/(eN_3)\right)^{\alpha} - z/(2eN_3) - b^2 q\right] \geq 0$$

$$\bar{y} - e\left[\left((2\delta^4\kappa + 1) + \delta^2\kappa\left(N_1 + (\delta + \delta^3)N_2 + (\delta + \delta^3)N_4 + N_5\right)/(eN_3)\right)^{\alpha} \text{-} z/(eN_3) \text{ -} b^2 q\right] \geq 0 \text{ or}$$
$$\bar{y} - e\left[\left((\delta^2\kappa + 1) + \delta\kappa\left(N_1 + \delta^2 e N_3 + \delta^3 N_4\right)/N_2\right)^{\alpha} - 3z/(2N_2) - bq\right] \geq 0 \text{ or}$$
$$\bar{y} - e\left[\left((\delta^2\kappa + 1) + \delta\kappa\left(\delta^3 N_2 + \delta^2 e N_3 + N_5\right)/N_4\right)^{\alpha} - 3z/(2N_4) - bq\right] \geq 0.$$

network stability are the same as for the benchmark model in section 3.1.2. With this alternative mode of service by the cores, our theorem 3 can be generalized for cores with the same amount of aggregated knowledge. In this case, the relative size of local populations is determined by the sum of decay factors. This is particularly useful for characterizing equilibria with symmetric cores, for example, the duocentric configuration or a ring configuration with multiple cores.

PROPOSITION. *Consider a local network economy with a finite number of locations* $|I| \geq 3$ *on an arbitrary graph. When knowledge decays and link maintenance costs are sufficiently small that* $z < 2\alpha\delta^{|I|-1}/(|I|+1)$*, a spatial equilibrium features the following: for two periphery locations i and j that are served by cores with the same amount of aggregated knowledge and pays the same link cost,*

$$\frac{N_j}{N_i} < \frac{\sum_{k=1}^{K} \delta^{d(c_k,j)}}{\sum_{k=1}^{K} \delta^{d(c_k,i)}} < 1 \text{ or } \frac{N_j}{N_i} > \frac{\sum_{k=1}^{K} \delta^{d(c_k,j)}}{\sum_{k=1}^{K} \delta^{d(c_k,i)}} > 1.$$

That is, the population changes faster proportionally to the sum of knowledge decay factors.

4. Numerical analysis

In this section, we establish the range of parameters to support a particular configuration in spatial equilibrium, and the range a configuration brings the maximum (equal treatment) welfare. We then perform comparative-static analysis with respect to changes in the land rent, the link cost, and the decay rate of knowledge transmission/spillovers.[17]

4.1. Equilibrium configuration

We begin by examining the benchmark setup in which only cores create knowledge spillovers and peripheries are only served by the closest core(s). We focus on two key parameters, z and δ, that are crucial for determining the equilibrium configuration. For the time being, we keep other parameters fixed at $\alpha = 0.5$, $\kappa = 1.2$, $e = 0.9$, $b = 0.8$, and $q = 0.002$. In section 4.2 below, we will return to examining the effects of changes in these other parameters.

First, we present a representative equilibrium for each of these four spatial configurations. The benchmark values of (z, δ) supporting each case of the three configurations in equilibrium, as well as the equilibrium distribution of working populations and expected per capita net income, are reported in the following table:

17 All numerical and graphical details are relegated to the working paper version of this paper.

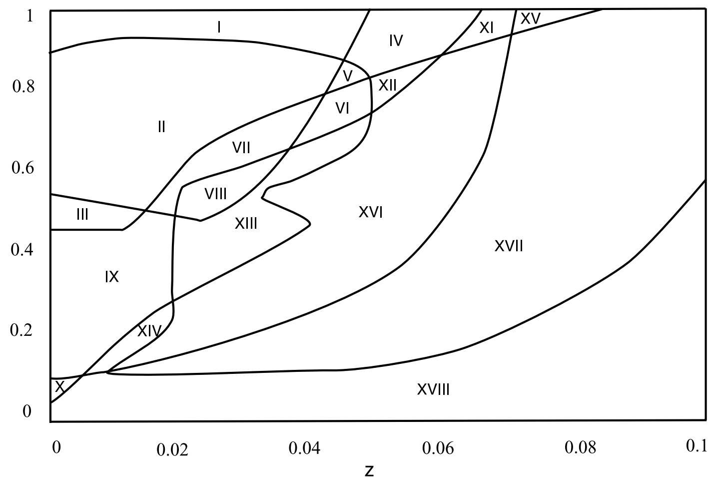
		z	δ	N_1	N_2	N_3	N_4	N_5	$\bar{y}$
i)	Monocentric	0.02	0.7	0.1648	0.2057	0.2591	0.2057	0.1648	1.6286
ii)	Multicentric	0.008	0.24	0.1227	0.2546	0.2454	0.2546	0.1227	1.1425
iii)	Urban-rural	0.04	0.2	0.1466	0.0494	0.6079	0.0494	0.1466	1.000
iv)	Two-MSA	0.05	0.5	0.0261	0.4008	0.1626	0.4008	0.0261	1.000

In the following, we investigate the range of existence for each configuration in the parameter space. We set the ranges for the two key parameters as $z \in [0, 0.1]$ and $\delta \in [0, 1]$. Over this parameter space, we can pin down equilibrium configurations, based on different combinations of the values of (z, δ). For illustrative purposes, we summarize only the most representative cases in the table below, where we use 'none' to represent cases with no agglomerative symmetric equilibrium (p..p..p..p..p). In our terminology, we refer to this no-agglomeration case as 'purely rural,' although this case is comparable to the completely mixed configuration studied in Ogawa and Fujita (1980), Fujita and Ogawa (1982), and Berliant, Peng, and Wang (2002), where workers and firms reside uniformly in every location. The complete characterization of all symmetric equilibrium configurations, including tricentric (with three cores) and pure rural (with no population agglomeration), are depicted in figure 2. The zones represent the union of existence for each configuration; for example, the zones for *d* includes all symmetric duocentric and also tricentric equilibria.

δ z	Low	Intermediate-low	Intermediate	Intermediate-high	High
High	m	m	m, s	m, u, s	m, u
Intermediate-high	m, d	m, d	m, s	u, s	u
Intermediate	m, d, s	d	s	u, s	u
Intermediate-low	d, s	d, u, s	u, s	u	None
Low	d, s	s	u	None	None

(m = monocentric, d = multicentric, u = urban-rural, s = two-MSA, None = rural)

Depending on the values of the two key parameters, an equilibrium with population agglomeration may or may not exist. With sufficiently low knowledge spillovers (δ sufficiently low) and sufficiently high link costs (z sufficiently high), the environment approaches to one described by Starrett's Spatial Impossibility Theorem under which there does not exist a spatial equilibrium featuring population agglomeration. Thus, no core is ever formed and every location of the entire local economy has exactly the same measure of population (i.e., purely rural). When knowledge spillovers are strong enough or inter-location links are not too costly, an agglomerative symmetric equilibrium exists. Under these circumstances, one may have a unique spatial configuration, or coexistence of more than one spatial configurations (multiple equilibria), which we now characterize.



Unique configuration	Zone	Two coexistent configurations	Zone	Three coexistent configurations	Zone
m	I	m, d	II	m, d, s	III, V
d	VII, X	m, u	XV	m, u, s	XI
u	XVII	m, s	IV	d, u, s	XIII
s	XII, XIV	d, u	VIII		
		d, s	VI, IX		
		u, s	XVI		

FIGURE 2 Equilibrium configurations

First, when δ is high, knowledge spillovers are strong enough for a single core to serve the entire local economy, provided that the link cost z is not too large. This gives rise to the monocentric configuration. As the link cost increases, some locations may sever links with this single core, thereby yielding the urban-rural configuration; alternatively, the two-MSA configuration may arise with two cores each serving one periphery. These three configurations can coexist when the link costs take an intermediate range (between 0.050 and 0.072).

Second, when δ is moderately high, the equilibrium configuration need not be concentrated, even when the link cost is not too large. With a sufficiently low link cost, monocentric and duocentric configurations can coexist. As the link cost rises, those residing at location 3 are not willing to pay both link costs and hence the monocentric configuration becomes the sole equilibrium outcome. As the link cost continues to increase, those residing in the outskirts (locations 1 and 5) are not willing to maintain links with the urban area. As a consequence, the equilibrium features either the urban-rural configuration, where the outskirt locations are disconnected, or the two-MSA configuration, where the outskirts themselves become cores serving in respective urban areas.

Third, when δ is moderately low, a single core cannot serve the entire local economy. The duocentric configuration emerges if the link cost is low. As the link cost rises (falling in the range between 0.046 and 0.056), the equilibrium features two MSAs. As the link cost continues to increase, it is too costly for the entire local economy to be linked; as a result, the outskirt locations sever links with the geographically centred urban area and the urban-rural configuration arises in equilibrium.

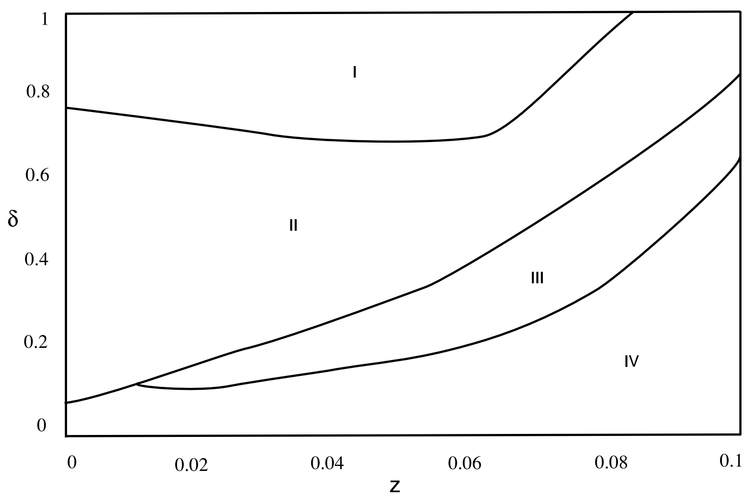
Fourth, when δ is sufficiently low, the area that a single core can serve becomes more limited. With a sufficiently low link cost, duocentric and urban-rural configurations can coexist. As the link cost rises, the urban-rural configuration becomes the sole equilibrium outcome. As the link cost continues to increase, there does not exist an equilibrium configuration with population agglomeration.

Remark. Should peripheries also generate knowledge spillovers, the overall gain from knowledge spillovers increases, making those residing in locations 2 and 4 more willing to maintain the link with the outskirts and the monocentric configuration more likely to emerge. Should peripheries be served by both cores in the duocentric configuration at locations 2 and 4, the advantage of the middle location reduces relative to the outskirt locations, making those residing at location 3 less willing to pay link costs to both cores and the duocentric configuration less likely to arise.

4.2. Efficient configuration

We examine equal-treatment welfare where utility levels are equalized at all locations. This implies locational no-arbitrage. It turns out that only monocentric, duocentric, urban-rural, and rural configurations can be efficient in different range of parameter values; others are not efficient. Specifically, for high δ and low z, knowledge does not decay much, it is more efficient to have one core processing all the spillover knowledge. When δ and z are in the intermediate range, transmitting to a single core causes high decay and it is more efficient to have two cores serving local peripheries. Also, the link cost will reduce income at peripheries linked to the core, they will need stronger core knowledge coming from shorter distances. When decay and link cost are large, the benefits from connecting to the core are not strong enough to reach the extreme locations on the network and some will be disconnected. Therefore, only urban-rural and rural configurations are possible to bring the highest income. The following table summarizes the efficient range (see also graphics in figure 3).

δ z	Low	Intermediate	High
High	m	m	d
Intermediate-high	d	d	u,r
Intermediate-low	d	u, r	u, r
Low	r	r	r



Unique configuration	Zone	Two coexistent configurations	Zone
m	I	r, u	III
d	II		
r	IV		

FIGURE 3 Efficient configurations

We examine the following cases of welfare comparison among different configurations.[18]

(i) When $z = 0.004$ and $\delta = 0.2$, the link cost is very small and the knowledge decay is very severe. The two-MSA (p-c..p..c-p) configuration is an equilibrium. It can be improved by linking the middle periphery into a duocentric network (p-c-p-c-p), which is an equilibrium. (ii) When $z = 0.008$ and $\delta = 0.6$, the link cost is small and the knowledge decay is severe. The monocentric configuration is an equilibrium but owing to knowledge decay, yields an income level lower than the duocentric (p-c-p-c-p) configuration, which is an equilibrium. (iii) When

18 The numerical results of cases (i)–(iv), respectively, are given as follows:

		N_1	N_2	N_3	N_4	N_5	$\bar{y}$
i)	Two-MSA	0.0140	0.0861	0.8886	0.0861	0.0140	1.0000
	Duocentric	0.1194	0.2612	0.2388	0.2612	0.1194	1.1111
ii)	Monocentric	0.1412	0.2242	0.2692	0.2242	0.1412	1.5182
	Duocentric	0.1554	0.1892	0.3108	0.1892	0.1554	1.5502
iii)	Urban-Rural	0.2513	0.0152	0.4671	0.0152	0.2513	1.0000
	Duocentri	0.1109	0.2782	0.2218	0.2782	0.1109	1.2445
iv)	Urban-Rural	0.1903	0.0127	0.5940	0.0127	0.1903	1.0000
	Monocentric	0.2130	0.1288	0.3164	0.1288	0.2130	1.5245

$z = 0.046$ and $\delta = 0.5$, the link cost is moderate and the knowledge decay is severe. The urban-rural configuration is an equilibrium, but it does not utilize all available knowledge and can be improved into the duocentric configuration (p-c-p-c-p). The duocentric network, however, is not an equilibrium, since the middle periphery has an incentive to sever the link with one of the cores. Apparently, the benefit of linking all locations is large enough for the link cost, yet location 3 can save the link cost that is larger than the spillover benefit from one of the cores. Thus, a subsidy is needed in order to maintain this duocentric network.[19] (iv) When $z = 0.064$ and $\delta = 0.76$, the link cost is relatively large and the knowledge decay is moderate. The urban-rural is an equilibrium, but it does not utilize knowledge from two locations at the ends of the network. Linking these two can raise income, since the added knowledge outweighs the additional link costs. However, assuming other locations not to deviate, periphery location 2 has an incentive to sever the link with location 1, because it can save one the link cost yet still enjoy knowledge spillover from location 5. This example illustrates a free-rider problem, owing to the public good nature of spillover. To maintain a monocentric network, location 2 (and 4) needs to be subsidized.

4.3. Comparative statics

The per capita income of a location is determined by three factors: (i) its own local knowledge and spilled aggregated knowledge, (ii) linked costs and land rents incurred, and (iii) its own population. Before turning to quantitative analysis of the effects of parameter shifts on population distribution, we can gain insight from characterizing the effects of equilibrium population analytically.

Given equilibrium configuration (σ, l) and population $(N_i^*)_{i\in I}$, let $S_i(N_i^*)$ denote the knowledge spillovers received by location i in equilibrium. Let λ_i denote local knowledge. Local knowledge and spilled knowledge show up in the per capita income function: $y_i = e[(\lambda_i + S_i(N_i^*)/N_i^*)^\alpha - Z_i(\sigma, l)/N_i^* - Q_i(\sigma, l)]$, where $\lambda_i = 1$ if $i \in P$ and $\lambda_i = \kappa$ if $i \in C$ (also, $Z_i(\sigma, l) = 0$ if $i \in C$).

Suppose the change for location i is small, so that the equilibrium income can stay constant. We focus on a local perturbation in one of the parameters ($\theta = z, \delta, b, q$ or e), holding populations in other locations unchanged. Then, we can express per capita income at location i as a function $\hat{y}_i(N_i^*, \theta)$ and denote $\varepsilon^S = N_i^*(dS_i/dN_i^*)/S_i^*$ as the spillover elasticity. We obtain for each θ: $dN_i^*/d\theta = -(\partial\hat{y}_i/\partial\theta)/(\partial\hat{y}_i/\partial N_i^*)$, where $\partial\hat{y}_i/\partial N_i^* = (e/(N_i^*)^2)[\alpha(\varepsilon^S - 1)S_i/(\lambda_i + S_i/N_i^*)^{1-\alpha} + Z_i(\sigma, l)]$, $\partial\hat{y}_i/\partial\delta > 0$, $\partial\hat{y}_i/\partial z < 0$, $\partial\hat{y}_i/\partial b < 0$, $\partial\hat{y}_i/\partial q < 0$, and $\partial\hat{y}_i/\partial e > 0$. Thus, the sign of $dN_i^*/d\theta$ depends on the sign of $\partial\hat{y}_i/\partial N_i^*$, which relies on the interaction of local as well as spilled knowledge. Under normal circumstances, one may consider

19 For example, the local government may levy a uniform per capita tax on all involved locations (locations 2–4 in this case) to finance the subsidy to the link cost incurred by location 3. When the spillover benefit is sufficiently large and the population at locations 2 and 4 is sufficiently higher than that at location 3 (so that locations 2 and 4 would not sever the links), such a subsidy policy can lead to an efficiency gain.

TABLE 1
Numerical comparative-static results

		N_1	N_2	N_3	N_4	N_5	$\bar{y}$
i) Monocentric configuration	b	−	−	+	−	−	−
	q	+	+	−	+	+	−
	z	+	−	+	−	+	−
	δ	+	−	−	−	+	+
ii) Duocentric configuration	b	−	+	−	+	−	−
	q	+	−	+	−	+	−
	z	−	+	−	+	−	−
	δ	+	−	+	−	+	+
iii) Urban-rural configuration	b	−	−	+	−	−	0
	q	+	+	−	+	+	0
	z	−	+	+	+	−	0
	δ	+	−	−	−	+	0
	e	−	−	+	−	−	0
iv) Two-MSA configuration	b	−	+	−	+	−	0
	q	+	−	+	−	+	0
	z	+	+	−	+	+	0
	δ	−	−	+	−	−	0

$\varepsilon^S < 1$ and $Z_i(\sigma, l)/S_i \to 0$, and therefore $\partial\hat{y}_i/\partial N_i^* < 0$. In this case, we have $\partial N_i^*/\partial\delta > 0, \partial N_i^*/\partial z < 0, \partial N_i^*/\partial b < 0, \partial N_i^*/\partial q < 0,$ and $\partial N_i^*/\partial e > 0$. That is, other things being equal, population in a location rises when knowledge transmission is more effective, the unit link cost is lower, the land rent schedule is less significant and steeper, and urban unemployment is less severe.

Tooled with these analytic insights, we examine comparative statics for each equilibrium configuration in the following four subsections, respectively. Throughout all configurations we take the ranges of the land rent and rent gradient parameters as $b \in [0.6, 1]$ and $q \in [0, 0.04]$. In all but the rural-urban configuration, we set the employment rate in the urban area at 90%; that is, $e = 0.9$. For the urban-rural configuration, we will study the Harris-Todaro proposition of rural-urban migration by perturbing this employment rate parameter in the range of $e \in [0.88, 0.92]$. With regard to the two key parameters (z, δ), we will perturb them by $(\pm 0.01, \pm 0.1)$ respectively around their benchmark values (which vary with different spatial configurations). We summarize all such numerical results qualitatively in table 1.

4.3.1. Monocentric

Under the benchmark parametrization, the spillover elasticity is less than one and the knowledge spillover effect outweighs the link cost effect. Thus, a flatter land rent gradient (higher b) reduces the relative disadvantage for residing in the core. As a result, the population distribution becomes more concentrated. An increase in the land rent (q), on the contrary, makes it more costly to reside in the core and hence leads to a flatter population distribution. With a higher link cost (z), the two locations connected to the core (locations 2 and 4) become most disadvantageous. Thus, the working populations of locations 2 and 4 fall, whereas those of locations 1 and 5 rise. Finally, in response to a stronger knowledge spillover (lower decay rate δ), the disadvantage of outskirt locations reduces, so the working population distribution becomes flatter.

4.3.2. Multicentric

The intuition with regard to changes in the land rent gradient and the level of land rent is identical to that in the case of monocentric configuration. For an increase in the link cost, however, we note that location 3 is most disadvantageous (as it must pay both link costs with the two cores). Therefore, the working population in location 3 shrinks more than the two outskirt peripheries. On the contrary, as knowledge spillovers become stronger, location 3 benefits most (as it is served by both cores); its working population thus rises by more than that of the two outskirt peripheries. When the link cost becomes too high or the knowledge transmission becomes too weak, the two cores are not sufficient to serve the entire local economy and the two-MSA configuration arises in equilibrium. When the link cost continues to increase, an urban-rural configuration emerges as an equilibrium outcome.

4.3.3. Urban-rural (p..p-c-p..p)

Under the benchmark parametrization, the spillover elasticity need not outweigh the link cost effect. In response to a decrease in the land rent gradient, the core attracts more working population from the rural areas. When the level of land rent rises, the changes in working population are not monotone: the core shrinks, whereas the peripheries in the urban area expand more proportionately than in the rural areas. Thus, the overall population in the urban area falls and the working population distribution within the urban area becomes flatter. As the link cost increases, the urban area gains more working population, indicating that the spillover elasticity effect is outweighed by the link cost effect. By similar arguments, a stronger knowledge spillover leads to a population reduction in the urban area. In this case, the working population distribution within the urban area is steeper. When knowledge transmissions become too weak, an urban-rural configuration is no longer stable. In this case, the only configuration that may arise is the non-agglomerative equilibrium, where all locations are disconnected. Finally, in response to a higher employment opportunity in the urban area, the overall urban working population rises and the population distribution becomes more concentrated within the urban area. When urban employment opportunities continue to rise, the urban-rural configuration also collapses, as all workers desire to migrate to the core. To compare the degree of population concentration between the monocentric and the urban-rural configurations, we select parameters in regions XI and XV in figure 2. We find that, given the same set of parameters, the core always has more concentrated population in the urban-rural configuration than the monocentric case. Thus, the presence of disconnected rural area serves as a potential threat under which the size of the monocentric city must be sufficiently large to generate strong knowledge spillovers in order to prevent the periphery locations from severing their links.

4.3.4. Two-MSA (c-p..p..p-c)

As one can see, an increase in the land rent gradient encourages more working population to move into the disconnected rural area (location 3). Since a higher level of land rent hurts connected peripheral locations (locations 2 and 4) more than proportionately, workers emigrate from these locations either to become disconnected (residing in location 3) or to join the cores (locations 1 and 5). In response to an increase in the link cost or a reduction in the knowledge transmission, the urban areas (both cores and connected peripheries) gain more working population from the disconnected rural area, again owing to a strong link cost effect. When maintaining the link becomes too costly, this configuration can no longer arise in equilibrium, because the population in the economy is not large enough for the two MSAs to sustain to generate income no less than the rural income ($\bar{y} = 1$). In this case, the only equilibrium outcome is the urban-rural configuration.

5. Concluding remarks

We have developed a network formation approach to knowledge-based city formation and spatial agglomeration. This framework allows a thorough analysis of the transmission, aggregation, and spillover of knowledge on a spatial network. We have shown that spatial equilibrium may feature monocentric, multicentric, urban-rural, or multiple urban areas, or coexistence of more than one spatial configuration. The stronger knowledge spillovers and the lower link costs are, the more likely it is that the local economy features a single core and the monocentric configuration becomes the unique and efficient equilibrium outcome. As the strength of knowledge spillovers declines, more than one core may form. In response to an increase in the link cost, some locations may become disconnected, thereby generating multiple-urban-area or urban-rural configurations.

A natural extension would be to examine how the spatial network configuration changes over time in response to a continual increase in population. One may adopt a modelling strategy similar to the one proposed by Berliant and Wang (2008), where, with discrete locational choice and full capital depreciation, the dynamic optimization problem with population change reduces to period-by-period optimization. One may further assume location players are myopic in the network formation sense, as defined by Jackson and Watts (2002).

Appendix

Proof of lemma 1

Let $N_i^1(\varepsilon_i) = \{(N_j)_{j\in I} \in \Re_+^{|I|} \mid N_i \geq \varepsilon_i, \sum_{j\in I\setminus i} N_j = 1 - \varepsilon_i\}$ and $N_i^2(\varepsilon_i) = \{(N_j)_{j\in I} \in \Re_+^{|I|} \mid N_i = \varepsilon_i, \sum_{j\in I\setminus i} N_j = 1 - \varepsilon_i\}$. Since $\lim_{N_c\to 0} y_c((\sigma^*, l^*), (N_i)_{i\in I}) =$

∞, we can find a small enough $\varepsilon_c > 0$, a large enough M_c, and a small $r_c > 0$ such that $(1\text{-}r_c)M_c < \min_{(N_i)_{i\in I}\in N_c^2(\varepsilon_c)} y_c((\sigma^*, l^*), (N_i)_{i\in I}) \leq \max_{(N_i)_{i\in I}\in N_c^1(\varepsilon_c)} y_c((\sigma^*, l^*), (N_i)_{i\in I}) < (1 + r_c)M_c$. For all $i \in I\backslash c, \lim_{N_i\to 0} y_i((\sigma^*, l^*), (N_i)_{i\in I}) = \lim_{N_i\to 0}[(1 + \delta^{d(c,i)}K_c/N_i)^\alpha - Z_i(\sigma^*, l^*)/N_i] = \infty$, we can find a small enough $\varepsilon_i > 0$, a large enough M_i, and a small $r_i > 0$ such that $(1\text{-}r_i)M_i < \min_{(N_i)_{i\in I}\in N_i^2(\varepsilon_i)} y_i((\sigma^*,l^*),(N_i)_{i\in I}) \leq \max_{(N_i)_{i\in I}\in N_i^1(\varepsilon_i)} y_i((\sigma^*,l^*),(N_i)_{i\in I}) < (1 + r_i)M_i$. By taking ε_i sufficiently small, we can find arbitrarily large M_i and small r_i. We can manipulating all ε_i small enough for all $i \in I$, and obtain the same value $M_i = M, r_i = r$, for all $i \in I$ in the above expressions. There is at least one $h \in I$ with population $N_h \geq 1/|I|$ since $\sum_{i\in I} N_i = 1$. Let $W = \max_{i\in I} \max_{(N_i)_{i\in I}\in N_i^1(1/|I|)} y_i((\sigma^*, l^*), (N_i)_{i\in I})$. By taking $M \gg W$ and $N_i = \varepsilon_i$, we have $y_i((\sigma^*, l^*), (N_i)_{i\in I}) - \hat{y}((\sigma^*, l^*), (N_i)_{i\in I}) > N_h(1 - r)M - (1 - N_i - N_h)2rM - N_h W \geq (1 - r - 2r|I|)M/|I| - W > 0$, for large enough M and small enough r. ■

Proof of lemma 2

We begin by examining an extreme case with $\delta \to 1$ and $z < 1/|I|^3$. We then take $\bar{\varepsilon} = 1/|I|^2$. If the core location has population $\bar{\varepsilon}$, its income is close to $(\kappa|I|^2)^\alpha$; if a periphery location has population $\bar{\varepsilon}$, its income approaches $(1 + \kappa|I|^2)^\alpha - Z_j(\sigma^m, l^m)|I|^2 > (1 + \kappa|I|^2)^\alpha - 1/|I|$. By construction, there must be a location with population more than $1/|I|$. If this location is the core, its income is less than $(\kappa Z_j(\sigma^m, l^m)|I|)^\alpha$; if it is a periphery, its income is less than $(1 + \kappa|I|)^\alpha - Z_j(\sigma^m, l^m)|I| < (1 + \kappa|I|)^\alpha$. When $|I|$ is not too small, income at location with population less than $\bar{\varepsilon}$ is larger than income at location with population more than $1/|I|$. Thus, $\bar{\varepsilon}$ is a valid lower bound for equilibrium population in the monocentric configuration. By continuity, a proper lower bound $\bar{\varepsilon}$ on population can always be constructed for δ not too far below 1 and z not too much above $z < 1/|I|^3$. ■

Proof of theorem 1

i) The proof contains three parts.

a) First, we know from lemma 1 that if there is a solution of population distribution that equalizes per capita income, then $y_i((\sigma^*, l^*), (N_i)_{i\in I}) > \hat{y}((\sigma^*, l^*), (N_i)_{i\in I})$ for sufficiently small $N_i = \varepsilon_i > 0$. Next, we construct a fixed point map. Define $\phi_i((\sigma^*, l^*), (N_i)_{i\in I}) = \max[0, y_i((\sigma^*, l^*), (N_i)_{i\in I}) - \hat{y}((\sigma^*, l^*), (N_i)_{i\in I})]$ for $i \in I$. Construct for $i \in I, F_i((N_i)_{i\in I}) = \varepsilon_i + (1 - \sum_{i\in I} \varepsilon_i)(N_i - \varepsilon_i + \phi_i((\sigma^*, l^*), (N_i)_{i\in I}))/ \sum_{i\in I}(N_i\text{-}\varepsilon_i + \phi_i((\sigma^*, l^*), (N_i)_{i\in I}))$, which maps continuously a population distribution from $\{(N_i)_{i\in I} \in [\varepsilon_i, 1 - \sum_{i\in I} \varepsilon_i]^{|I|} \mid \sum_{i\in I} N_i{=}1\}$ to itself. By Brower's Fixed Point Theorem, there is a population distribution $(N_i^*)_{i\in I}$ such that $F_i((N_i^*)_{i\in I}) = N_i^*$ for all $i \in I$, or, rearranging, $(N_i^* - \varepsilon_i) \sum_{i\in I} \phi_i((\sigma^*, l^*), (N_i^*)_{i\in I}) = (1 - \sum_{i\in I} \varepsilon_i)\phi_i((\sigma^*, l^*), (N_i^*)_{i\in I})$, for $i \in I$. Suppose $\sum_{i\in I} \phi_i((\sigma^*, l^*), (N_i^*)_{i\in I}) >$

0. There is an $j \in I$ such that $\phi_j((\sigma^*, l^*), (N_i^*)_{i\in I}) > 0$. From the above equation, $N_j^* - \varepsilon_j > 0$ if and only if $\phi_j((\sigma^*, l^*), (N_i^*)_{i\in I}) > 0$, implying $y_j((\sigma^*, l^*), (N_i^*)_{i\in I}) - \hat{y}((\sigma^*, l^*), (N_i^*)_{i\in I}) > 0$. But when $N_i = \varepsilon_i, y_i((\sigma^*, l^*), (N_i^*)_{i\in I}) - \hat{y}((\sigma^*, l^*), (N_i^*)_{i\in I}) > 0$, contradicting the average income equation. We can conclude that $\sum_{i\in I} \phi_i((\sigma^*, l^*), (N_i^*)_{i\in I}) = 0$, and this means for all $i \in I, y_i((\sigma^*, l^*), (N_i^*)_{i\in I}) - \hat{y}((\sigma^*, l^*), (N_i^*)_{i\in I}) \leq 0$. Therefore, $N_i > \varepsilon_i$ for all $i \in I$ and we have $y_i((\sigma^*, l^*), (N_i^*)_{i\in I}) - \hat{y}((\sigma^*, l^*), (N_i^*)_{i\in I}) = 0$ for all i. So, $(N_i^*)_{i\in I}$ equalizes the per capita income of locations with positive populations. Since the population at each non-rural location never approaches zero, we can utilize lemma 2, imposing a lower bound $\bar{\varepsilon}$ on population from here on.

b) No periphery location wants to switch to be a core. Suppose location $j \in I\backslash c$ has income $[1 + \delta^{d(c,j)}\kappa(N_c + \sum_{k\in I\backslash c} \delta^{d(c,k)} N_k^*)/N_j^*]^\alpha - Z_j(\sigma^*, l^*)/N_j^*$, and switches to be a core. The set of peripheries connected to j and not blocked by c is $s(j)$. The per capita income at j is $\kappa \sum_{k\in s(j)} \delta^{d(j,k)} N_k^*/N_j^*)^\alpha$. Comparing the above two expressions and manipulating establish the sufficient condition given in (S-1).

c) No location wants to change links, either by severing old links or by stablishing new links. By changing links, a location will not increase the knowledge received from the core, since it is already connected to the core with the shortest distance. A location may sever a link (but still be connected to the core) to reduce the cost. Yet any link severed will reduce K_c, since (σ^*, l^*) is a tree. However, if δ is large enough and z is small enough, the loss in knowledge dominates the saving in link costs. Let $L(j) = \{i \in I \mid ij \in l^*, d(c, j) = d(c, i) + 1\}$ be the set of locations that are linked with j and are one link away from the core. If location j severs a link to $h \in L(j)$, all peripheries whose paths to the core contain link jh are cut off; denote this set $T(jh)$. A sufficient condition for δ, z is (S-2).

ii) When δ is sufficiently large and z sufficiently small, there is at least one core linked with a periphery in an equilibrium. Otherwise, a periphery location j can switch to being a core and link with adjacent periphery h (notice that the core does not pay link cost). This link is mutually beneficial, since the aggregated knowledge can outweigh the link cost paid by the periphery. A sufficient condition is $\min_{N\in[\bar{\varepsilon},1]}[1 + \delta\kappa(\bar{\varepsilon} + \delta N)/N]^\alpha - z/N - 1 > 0$. With one core in equilibrium, any periphery location will be connected to the core under sufficiently large δ and sufficiently small z. Suppose, otherwise, some periphery locations are not connected to the core. We can find a periphery p_h that is adjacent to another p_j that is connected to the core. Peripheries p_h and p_j can deviate by linking together. A sufficient condition is $\min_{N\in[\bar{\varepsilon},1]}(1 + \delta^{|I|-1}\kappa 2\bar{\varepsilon}/N)^\alpha - z/N - 1 > 0$. The above two conditions can be combined to (S-3). Applying similar arguments to the derivation of (S-2), we can obtain a sufficient condition for the link to be beneficial to peripheries p_j, thereby yielding (S-4), which is stronger than (S-2). When δ is sufficiently large and z sufficiently small to meet (S-1), (S-3), and (S-4), there is only one core in (σ^*, l^*). From (i(b)), the per capita income is higher as a periphery connected with another core than as a core.

iii) Suppose there is a population distribution $\{\bar{N}_i\}$ that equalizes utility in $(\bar{\sigma}, \bar{l})$; the income of a periphery j served by one of the cores c is $\bar{y} = [1 + \delta^{d(i,j)}\kappa(\bar{N}_c + \sum_{k\in s(i)} \delta^{d(c,k)}\bar{N}_k)/\bar{N}_j]^\alpha - Z_j(\bar{\sigma}, \bar{l})/\bar{N}_j$. Suppose there is a monocentric configuration (σ^m, l^m) with c as the single core. Keep the same population distribution $(\bar{N}_i)_{i\in I}$. In this monocentric configuration when δ is large and z is small such that conditions (S-1), (S-3) and (S-4) are met, the core income at c is $[(\bar{N}_c + \sum_{k\in I\backslash c} \delta^{d(c,k)}\bar{N}_k)/\bar{N}_c]^\alpha > \bar{y}$ and the periphery income is $[1 + \delta^{d(i,j)}\kappa(\bar{N}_c + \sum_{k\in I\backslash c} \delta^{d(c,k)}\bar{N}_k)/\bar{N}_j]^\alpha - Z_j(\sigma^m, l^m)/\bar{N}_j > \bar{y}$, for all peripheries j. The monocentric configuration brings higher income at all locations. This holds because the population is kept the same while the aggregate knowledge is approaching the highest value (which is 1) and link cost is small. Thus, we have a system of income functions $y_i(\sigma^m, l^m, (N_i)_{i\in I})$ having values all larger than $\bar{y}$ at $(N_i)_{i\in I} = (\bar{N}_i)_{i\in I}$. When z is sufficiently small, these y_i are monotonic in N_i. By continuity, a solution to $y_i(\sigma^m, l^m, (N_i)_{i\in I}) = \bar{y}^m$ exists with $\bar{y}^m \geq \bar{y}$. Therefore, any configuration will be weakly dominated by a monocentric configuration. ■

Proof of theorem 2

Suppose c is a core in any configuration. Any periphery location j connected to core c has income $[1 + \delta^{d(c,j)}\kappa(N_c + \sum_{i\in s(c)} \delta^{d(c,i)}N_i)/N_j]^\alpha - Z_j(\sigma^*, l^*)/N_j$. As δ decreases or z increases, this value decreases below 1, which is the per capita income in isolation. In this case, no locations will link. This income will be no larger than $(1 + \delta\kappa/N)^\alpha - z/(2N)$ (getting all knowledge from the core and having one link). When $z > 2\delta\kappa$, $(1 + \delta\kappa/N)^\alpha < 1 + z/(2N)$ for all values of α. ■

Proof of theorem 3

Suppose core c transmits knowledge K_c and both i and j pay link cost Z. In equilibrium, $y_h = (1 + \delta^{d(c,h)}K_c/N_h)^\alpha - Z/N_h = \bar{y}$, for h=i,j. It is clear that the population decreases with distance when the link cost z (and hence Z) is small. Thus, $N_j > N_i$ if and only if $\delta^{d(c,i)} > \delta^{d(c,j)}$. Equalizing income, we obtain $(1 + \delta^{d(c,j)}K_c/N_j)^\alpha - (1 + \delta^{d(c,i)}K_c/N_i)^\alpha = Z/N_j - Z/N_i > 0$. This means $\delta^{d(c,j)}K_c/N_j > \delta^{d(c,i)}K_c/N_i$, or, $N_j/N_i < \delta^{d(c,j)-d(c,i)}$, which holds under the sufficient condition. ■

Proof of corollary 2

i) Let total population be $\hat{N}$. Then the population at location i can be written as $N_i\hat{N}$, where N_i is the share in total population. Let $d(c, i)$ denote the distance from i to the core and Z denote the link cost location i pays. Straightforward differentiation implies $d\bar{y}/d\hat{N} = \{\Lambda_i(K_c, N_i)[\bar{y} + Z/(N_i\hat{N})](d\ln K_c/d\ln\hat{N}) + [(1\text{-}\Lambda_i(K_c, N_i))Z/(N_i\hat{N}) - \Lambda_i(K_c, N_i)\bar{y}][1 + (d\ln K_i/d\ln\hat{N})]\}/\hat{N}$, where $\Lambda_i(K_c, N_i) \equiv \alpha/[1 + N_i\hat{N}(\delta^{d(c,i)}K_c)] \in (0, 1)$.

When δ and κ are sufficiently small, Λ_i becomes so low that $(1 - \Lambda_i(K_c, N_i))Z/(N_i\hat{N}) - \Lambda_i(K_c, N_i)\bar{y} > 0$. So if this single-core, all-connected configuration is still a spatial equilibrium, then we have $d\bar{y}/d\hat{N} > 0$. The above expression will be no smaller than $[1 - \alpha\delta^{(|I|-1)}\kappa\bar{\varepsilon}/(\delta^{(|I|-1)}\kappa\bar{\varepsilon} + \hat{N})]z/(\bar{\varepsilon}\hat{N}) - \alpha\delta^{(|I|-1)}\kappa\bar{\varepsilon}/(\delta^{(|I|-1)}\kappa\bar{\varepsilon} + \hat{N})(\kappa/\bar{\varepsilon})^{\alpha}$, which is positive under the sufficient condition.

ii) When z (and hence Z) is sufficiently small, $d\bar{y}/d\hat{N} \to -(\Lambda_i(K_c, N_i)\bar{y}/\hat{N})\{1 + [1 - \delta^{d(c,i)}N_i/(N_c + \sum_{j\in s(c)} \delta^{d(c,j)}N_j)](d\ln K_c/d\ln\hat{N})\} < 0$. We need the following for a sufficient condition. The value of Λ_i will be no larger than $\alpha/[1 + \bar{\varepsilon}\hat{N}/(\delta\kappa)] = \alpha\delta\kappa/(\delta\kappa + \bar{\varepsilon}\hat{N})$, so $d\bar{y}/d\hat{N} \leq \{[\alpha\delta\kappa/(\delta\kappa + \bar{\varepsilon}\hat{N})](\kappa/\bar{\varepsilon} + z/\hat{N})/\bar{\varepsilon} + [1 - \alpha\delta^{(|I|-1)}\kappa\bar{\varepsilon}/(\delta^{(|I|-1)}\kappa\bar{\varepsilon} + \hat{N})](z/\hat{N})(1 + 1/(\bar{\varepsilon}\hat{N})) - [\alpha\delta^{(|I|-1)}\kappa\bar{\varepsilon}/(\delta^{(|I|-1)}\kappa\bar{\varepsilon} + \hat{N})]\kappa\bar{\varepsilon}/\hat{N}\}/\hat{N}$, which is negative under the sufficient condition. ■

Proof of the proposition

Suppose two locations i and j are served by cores $c_1, \ldots, c_K$. Let K_c denote the knowledge aggregated at each of the core c_k (they have the same amount of knowledge). Suppose both i and j pay the same link cost Z. Then, we can follow the same argument as in the proof of theorem 3 and conclude that $N_i > N_j$ if and only if $\sum_{k=1}^{K} \delta^{d(c_k,i)} > \sum_{k=1}^{K} \delta^{d(c_k,j)}$. Equalizing income, we obtain $(1 + \sum_{k=1}^{K} \delta^{d(c_k,j)}K_c/N_j)^{\alpha} - (1 + \sum_{k=1}^{K} \delta^{d(c_k,i)}K_c/N_i)^{\alpha} = Z/N_j - Z/N_i > 0$. Thus, if $\sum_{k=1}^{K} \delta^{d(c_k,i)} > \sum_{k=1}^{K} \delta^{d(c_k,j)}$, we have $N_i > N_j$ and $N_j/N_i < \sum_{k=1}^{K} \delta^{d(c_k,j)}/\sum_{k=1}^{K} \delta^{d(c_k,i)} < 1$. If $\sum_{k=1}^{K} \delta^{d(c_k,i)} < \sum_{k=1}^{K} \delta^{d(c_k,j)}$, we have $N_i < N_j$ and $N_j/N_i > \sum_{k=1}^{K} \delta^{d(c_k,j)}/\sum_{k=1}^{K} \delta^{d(c_k,i)} > 1$. ■

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