Economic Theory (2008) 34: 1–23
DOI 10.1007/s00199-007-0211-1

RESEARCH ARTICLE

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Equilibrium dynamics in discrete-time endogenous growth models with social constant returns

Received: 21 November 2005 / Revised: 28 December 2006 /
Published online: 22 February 2007

Abstract The existing literature establishes possibilities of local determinacy and dynamic indeterminacy in continuous-time two-sector models of endogenous growth with social constant returns. The necessary and sufficient condition for local determinacy is that the factor intensity rankings of the two sectors are consistent in the private/physical and social/value sense. The necessary and sufficient condition for dynamic indeterminacy is that the final (consumable) good sector is human (pure) capital intensive in the private sense but physical (consumable) capital intensive in the social sense. This paper re-examines the dynamic properties in a discrete-time endogenous growth framework and finds that conventional propositions obtained in continuous time need not be valid. It is shown that the established necessary and sufficient conditions on factor intensity rankings for

We have benefitted from discussion with Robert Becker, Eric Bond, Michael Kaganovich, Karl Shell and participants of the Midwest Macroeconomic Conference in Chicago and the Midwest Economic Theory and International Trade Meetings at Indiana University. The fourth author acknowledges financial support from the Institute of Economics and Business Administration of Kobe University and the Institute of Economic Research of Kyoto University to enable this international collaboration.

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local determinacy and dynamic indeterminacy are neither sufficient nor necessary, as the magnitudes of time preference and capital depreciation rates both play essential roles.

Keywords Sectoral externalities · Endogenous growth · Dynamic determinacy/indeterminacy

JEL Classification Numbers E32 · J24 · O40

1 Introduction

In the past decade, there has been a growing literature studying under what circumstances dynamic indeterminacy may occur. When dynamic indeterminacy arises, the perfect-foresight equilibrium is at least locally indeterminate in the sense that there are multiple (often a continuum of) converging transition paths. The case of indeterminacy has created vast interest due to the observations that many macro variables are volatile over time and that economic growth rates are dispersed across countries. It is worth emphasizing that most of the applied papers focusing on calibrated comparative dynamics or policy simulations use discrete-time models in which there exists a unique trajectory satisfying saddle-path stability (local determinacy). In contrast, the theoretical researches on dynamic indeterminacy has been mostly based on the continuous-time setting due to its tractability. This is particularly true for the study of equilibrium indeterminacy in endogenous growth models. In fact, to our knowledge, no one has undertaken a complete examination of the validity of continuous-time dynamics properties under discrete time in an endogenously growing economy where indeterminacy may be present.[1] This paper provides the first attempt at such an endeavor.

Within the endogenous growth framework *á la* Romer (1986) and Lucas (1988), previous studies conclude that dynamic indeterminacy may occur when the model economy involves external increasing returns or distortionary taxes. The early literature, best represented by Benhabib and Perli (1994), Boldrin and Rustichini (1994) and Xie (1994), focuses on increasing returns as a result of uncompensated positive spillovers.[2] More recent work, instead, considers overall constant-returns production technologies, which the present paper follows.[3] The analysis of the multi-sector dynamical system with perpetual growth has been viewed rather complex, because the conventional algorithm transforms the nonstationary system into stationary ones by constructing ratios of growing quantities and then studying their dynamic properties in conjunction with factor allocation variables.[4] Yet, as pointed

[1] Dynamic properties have been examined in discrete-time exogenous growth models with sector-specific externalities. The reader is referred to Benhabib et al. (2002) and papers cited therein. In a companion paper, Nishimura and Venditti (2002) revisit the issue with intersectoral externalities.

[2] See a survey by Benhabib and Rustichini (1994). It should be noted that dynamic indeterminacy may also arise in other types of growth models with increasing returns or matching externalities, such as Benhabib and Farmer (1994), Farmer and Guo (1994) and Laing et al. (1995).

[3] For considerations of sector-specific externalities with social constant returns, the reader is referred to Benhabib and Nishimura (1998, 1999), Benhabib et al. (2000) and Mino (2001).

[4] See Benhabib and Perli (1994), among many others.

out by Bond et al. (1996), the analysis is greatly simplified if the transformation takes place by separately characterizing the price and the quantity dynamics using both the primal and the dual. The transformed prices can be in forms of either relative prices (as in Bond et al. 1996; Mino 2001) or nominal prices (as in Benhabib et al. 2000). Under this transformation, price dynamics is determined by prices alone and thus the dynamical system is recursive. Moreover, the dynamic properties are found to depend exclusively on the factor intensity rankings of sectoral productions.

More specifically, under private constant returns without externalities but with distortionary factor taxes, Bond et al. (1996) establish the *polarization theorem*: when factor tax distortion is moderate such that the factor intensity rankings in the physical and value sense are consistent, the price and quantity dynamics are polarized—when one is stable, another must be unstable, yielding saddle-path stability.[5] They also show that dynamic indeterminacy can arise if the final good sector is human capital intensive in the physical sense but physical capital intensive in the value sense. While the former factor intensity ranking ensures stable quantity dynamics, the latter guarantees stable price adjustments, thus resulting in locally indeterminate transition. In another class of endogenous growth models, production exhibits social constant returns with positive externalities in the absence of distortionary factor taxation. As shown by Benhabib et al. (2000) and Mino (2001), the polarization theorem for local determinacy holds, whereas dynamic indeterminacy emerges when the final good sector is human capital intensive in the private sense but physical capital intensive in the social sense. Notably, a general property can be established in that the factor intensity ranking in the private sense corresponds to that in the physical sense whereas the factor intensity ranking in the social sense corresponds to that in the value sense. Thus, one can easily reconcile the findings regarding local determinacy and dynamic indeterminacy in Bond et al. (1996) and those in Benhabib et al. (2000) and Mino (2001).[6]

This paper considers a class of endogenous growth models with social constant returns in the presence of positive externalities. The departure from the literature summarized above is that time is *discrete*. Our main contribution is to show that conventional propositions obtained in continuous-time models are no longer valid. First, the balanced-growth path need not be locally determinate even if the factor intensity rankings in the private and social sense are consistent. Second and most interestingly, even under the conventional necessary and sufficient conditions that the final good sector is human capital intensive in the private sense and physical capital intensive in the social sense, dynamic indeterminacy need not emerge unless the rates of time preference and capital depreciation are sufficiently small and the rates of depreciations of the two capitals are sufficiently close. Finally, even if the conventional necessary and sufficient conditions fail to hold, dynamic indeterminacy may still arise when the final good sector is human capital intensive in both private and social sense. These findings are powerful because if one desires to conduct calibrated comparative dynamics or policy simulations under

[5] While Bond et al. (1996) examine both capital and labor income taxes, Mino (1996) studies the effects of capital taxation.

[6] Similar propositions can be extended to exogenous growth models (e.g., Benhabib and Nishimura 1998, 1999) and open-economy endogenous growth models (e.g., Nishimura and Shimomura 2002; Bond et al. 2003).

local determinacy, the parameters must be so chosen that dynamic indeterminacy or dynamic instability would not occur.[7]

The remainder of the paper is summarized as follows. In Sect. 2, we describe the basic environment of the model economy by specifying the economic agents, their preferences and capital endowments, the production technologies and the optimizing behavior. Section 3 defines the concepts of dynamic competitive equilibrium and balanced growth path and establishes the existence and uniqueness of the balanced growth path. In Sect. 4, we characterize the local dynamics of the system. Finally, Sect. 5 concludes the paper and provides possible avenues for future work.

2 The basic model

2.1 Production technology and capital formation

We consider a discrete-time model (indexed by $t = 0, 1, 2, \ldots$) of endogenous growth with two perpetually accumulated reproducible capitals (denoted X_i, $i = 1, 2$) and two distinctive sectors (labeled by $j = 1, 2$). Let the first reproducible capital be consumable and the second be non-consumable. For illustrative purposes, we may refer X_1 as physical capital and X_2 as human capital. Thus, sector 1 produces the final good for consumption and physical capital investment, whereas sector 2 produces "education services" that enhance human capital. The final good and education services are produced by both physical and human capitals.

Following the setup of the continuous-time framework by Benhabib et al. (2000), we consider that both capitals generate sector-specific, positive external effects and that both production technologies exhibit social constant returns. More specifically, denoting Y_{jt} as the output of sector j in time t and v_{ijt} as the fraction of factor X_{it} (measured at the beginning of period t) devoted to sector j production ($\sum_{j=1}^{2} v_{ij} = 1$), the production technologies with time-to-build is given by,

$$Y_{jt} = \alpha_j \prod_{i=1}^{2} \left(v_{ijt} X_{it}\right)^{\beta_{ij}} \left(\overline{v_{ijt} X_{it}}\right)^{b_{ij}}, \tag{1}$$

where sector-specific externalities are captured by the society's average values of sector-specific factors (labeled with bars), $\alpha_j > 0$ is a scaling factor, $0 < \beta_{ij} < 1$ and $\beta_{1j} + \beta_{2j} < 1$. Thus, b_{1j} and b_{2j}, respectively, measure the degree of positive externalities from physical and human capital in sector j. The private returns to scale of the two reproducible factors are measured by $\sum_{i=1}^{2} \beta_{ij}$, whereas the social returns are by $\sum_{i=1}^{2} (\beta_{ij} + b_{ij}) \equiv \sum_{i=1}^{2} \widehat{\beta}_{ij}$. By assumption, constant social

[7] It has been known that time frequency may play an important role for the stability of dynamical systems. For example, in the optimal exogenous growth literature that treats discrete-time dynamic systems, Mitra (1998) and Baierl et al. (1998) show that levels of capital depreciation rates as well as time discount rate affect the stability conditions, while the dynamic properties are generally independent of these parameters in continuous-time models. Similarly, in her study of calibrated, discrete-time models of real business cycles with market distortions, Schmitte-Grohé (1997) reveals that the conditions for indeterminacy of equilibrium involve the magnitude of capital depreciate rate, which plays no role in the continuous-time counterparts. As summarized above, the present study also confirms the relevance of time frequency in the context of endogenous growth models with externalities and social constant returns.

returns imply $\sum_{i=1}^{2} \widehat{\beta}_{ij} = 1$ for $j = 1, 2$. As to be formally defined later, we have $\overline{v_{ijt} X_{it}} = v_{ijt} X_{it}$ in equilibrium.

Let C_t denote period t consumption of the final good and $\delta_i > 0$ denote the (constant) depreciation rate of capital X_i). The evolution of the two capital stocks can therefore be expressed as ($t = 0, 1, 2, \ldots$):

$$X_{1,t+1} = Y_{1t} + (1 - \delta_1) X_{1t} - C_t, \tag{2}$$

$$X_{2,t+1} = Y_{2t} + (1 - \delta_2) X_{2t}, \tag{3}$$

where the initial stocks of both capitals, X_{10} and X_{20}, are historically given and positive.

2.2 Producer optimization and factor market equilibrium

The infinite-horizon problem of firms discounted by the market interest rate under our competitive equilibrium framework can be reduced to a simple period optimization problem. Specifically, each competitive firm in sector j chooses factor allocation $\{v_{ijt} X_{it}\}$ and output to maximize its period profit given output price P_j and input prices $\{W_1, W_2\}$:

$$\pi_{jt} = P_{jt} Y_{jt} - \sum_{i=1}^{2} W_{it} v_{ijt} X_{it}$$

subject to the production technologies specified as in (1). The first-order conditions are ($i, j = 1, 2$ and $t = 0, 1, 2, \ldots$):

$$P_{jt} \beta_{ijt} \frac{Y_{jt}}{v_{ijt} X_{it}} = W_{it}. \tag{4}$$

The cost minimizing solution of $v_{ijt} X_{it}$ is given by $a_{ij} Y_{jt}$, where $a_{ij} \equiv \frac{v_{ijt} X_{it}}{Y_{jt}}$ can be referred to as the input coefficients (which is time-invariant under the time-invariant production technologies specified above). From (4), we have:

$$a_{ij} = \beta_{ij} P_{jt} W_{it}^{-1}. \tag{5}$$

This can be combined with the production technologies (1) under the equilibrium condition, $\overline{v_{ijt} X_{it}} = v_{ijt} X_{it}$, to yield:

$$P_{jt} = \alpha_j^{-1} \prod_{h=1}^{2} \left(\beta_{hj}^{-1} W_{ht} \right)^{\widehat{\beta}_{hj}}. \tag{6}$$

Utilizing the property of constant social returns $\sum_{i=1}^{2} \widehat{\beta}_{ij} = 1$,we can substitute (6) into (5) to eliminate output prices:

$$a_{ij} = \alpha_j^{-1} \prod_{h=1}^{2} \left(\frac{\beta_{hj}^{-1} W_{ht}}{\beta_{ij}^{-1} W_{it}} \right)^{\widehat{\beta}_{hj}}. \tag{7}$$

It is convenient to express in matrix form. Let $A \equiv \left(a_{ij}\right)$, $Y_t \equiv (Y_{1t}, Y_{2t})'$ and $X_t \equiv (X_{1t}, X_{2t})'$, where the superscript "$'$" denotes the transpose operation. Then, full employment of both factor inputs requires $\sum_{j=1}^{2} a_{ij} Y_{jt} = X_{it}$, or, equivalently,

$$AY_t = X_t. \tag{8}$$

Next, define $\widehat{a}_{ij} \equiv a_{ij}\widehat{\beta}_{ij}/\beta_{ij}$, $\widehat{A} \equiv \left(a_{ij}\right)$, $P \equiv (P_1, P_2)'$ and $W \equiv (W_1, W_2)'$. Let $\widehat{B}$ be a matrix with elements $\widehat{\beta}_{ij}$ $(i, j = 1, 2)$ and I_w and I_p be diagonal matrices with diagonal elements (w_1, w_2) and (p_1, p_2), respectively. It can be easily seen from (5) that $P_{jt}\widehat{\beta}_{ij} = \widehat{a}_{ij} W_{it}$, or, in matrix form,

$$I_p \widehat{B} = (\widehat{A}) I_w. \tag{9}$$

Since $\sum_{i=1}^{2} \widehat{\beta}_{ij} = 1$, we obtain: $P_j = \sum_{i=1}^{2} W_i \widehat{a}_{ij}$, or, in matrix form,

$$P_t = (\widehat{A})' W_t, \tag{10}$$

which can be conveniently referred to as the competitive profit condition. From (6), we have a pair of Samuelsonian relationships that relate factor prices to output prices under factor price equalization:

$$\begin{aligned}
\log P_1\alpha_1 &= \widehat{\beta}_{11} \log W_1 + \widehat{\beta}_{21} \log W_2 - \left(\widehat{\beta}_{11} \log \beta_{11} + \widehat{\beta}_{21} \log \beta_{21}\right) \\
\log P_2\alpha_2 &= \widehat{\beta}_{12} \log W_1 + \widehat{\beta}_{22} \log W_2 - \left(\widehat{\beta}_{12} \log \beta_{12} + \widehat{\beta}_{22} \log \beta_{22}\right).
\end{aligned}$$

Define $\Delta \equiv \beta_{11}\beta_{22} - \beta_{12}\beta_{21}$ as the private factor share determinant and $\widehat{\Delta} \equiv \widehat{\beta}_{11}\widehat{\beta}_{22} - \widehat{\beta}_{12}\widehat{\beta}_{21}$ as the social factor share determinant. While the sign of Δ indicates the factor intensity ranking in the private (physical) sense, the sign of $\widehat{\Delta}$ gives such a ranking in the social (value) sense. Consider that the two sectors have different factor intensity rankings:

Assumption 1 $\Delta \neq 0$ and $\widehat{\Delta} \neq 0$.

Under this assumption, we can invert the Samuelsonian relationships to solve W_i $(i = 1, 2)$ as a function of $\{P_1, P_2\}$:

$$W_{1t} = \overline{w}_1 P_{1t}^{\widehat{\beta}_{22}/\widehat{\Delta}} P_{2t}^{-\widehat{\beta}_{21}/\widehat{\Delta}}; \quad W_{2t} = \overline{w}_2 P_{1t}^{-\widehat{\beta}_{12}/\widehat{\Delta}} P_2^{\widehat{\beta}_{11}/\widehat{\Delta}}, \tag{11}$$

where

$$\begin{aligned}
\overline{w}_1 &= \left(\alpha_1 \beta_{11}^{\widehat{\beta}_{11}} \beta_{21}^{\widehat{\beta}_{21}}\right)^{\widehat{\beta}_{22}/\widehat{\Delta}} \left(\alpha_2 \beta_{12}^{\widehat{\beta}_{12}} \beta_{22}^{\widehat{\beta}_{22}}\right)^{-\widehat{\beta}_{21}/\widehat{\Delta}} \\
\overline{w}_2 &= \left(\alpha_1 \beta_{11}^{\widehat{\beta}_{11}} \beta_{21}^{\widehat{\beta}_{21}}\right)^{-\widehat{\beta}_{12}/\widehat{\Delta}} \left(\alpha_2 \beta_{12}^{\widehat{\beta}_{12}} \beta_{22}^{\widehat{\beta}_{22}}\right)^{\widehat{\beta}_{11}/\widehat{\Delta}}.
\end{aligned}$$

These expressions yield the standard Stolper–Samuelson property: how factor prices are related to output prices depends crucially on the factor intensity ranking in the value sense, i.e., the sign of the social factor share determinant $\widehat{\Delta}$.

Remark 1 Our assumption of decreasing returns in the private technology with respect to capital inputs can be consistent with the free entry and zero-excess profit conditions. Suppose that, in addition to X_{1j} and X_{2j}, each production sector employes a sector-specific input denoted by X_{3j} $(j = 1, 2)$. We assume that the supply of X_{3j} is fixed and that it has negative external effects because of congestion. Letting $\alpha_j = X_{3j}^{\beta_{3j}} \bar{X}_{3j}^{b_{3j}}$ in (1), we assume that $\Sigma_{i=1}^{3} \beta_{ij} = 1$, $\beta_{ij} > 0$ $(j = 1, 2)$ and $\Sigma_{i=1}^{3} b_{ij} = 0$. Notice that since X_1 and X_2 are assumed to have positive externalities, $\Sigma_{i=1}^{3} b_{ij} = 0$ means that b_{3j} has a negative value. Given these assumptions, both social and private technologies exhibit constant returns to scale with respect to X_{1j}, X_{2j} and X_{3j}, so that firms cannot earn excess profits under free entry. Now, suppose further that $b_3 = -\beta_3 \, (< 0)$. Then (1) satisfies constant and decreasing returns to scale in X_{1j} and X_{2j} from the social and private perspectives, respectively.

2.3 Household optimization

The representative household has a time-additive preference with a subjective discount factor $\rho \in (0, 1)$ and a constant elasticity of intertemporal substitution $\sigma^{-1} > 0$. Her lifetime utility is thus given by:

$$U = \sum_{t=0}^{\infty} \rho^t \frac{C_t^{1-\sigma} - 1}{1 - \sigma}, \quad \sigma > 0, \ \sigma \neq 1.$$

Let Π_t be the distribution of profits from both sectors to the representative agent. Denote W_{it} and P_{jt} $(i, j = 1, 2)$ as the nominal factor prices and nominal good/education service prices of the ith capital and jth output at time t where all prices are measured in utils without accounting for intertemporal variation from time-discounting. Assuming that both capitals are owned by households (of mass one), the flow budget constraint facing the presentative household at time t can be written as:

$$\begin{aligned} & W_{1,t} X_{1,t} + W_{2,t} X_{2,t} + \pi_{1t} + \pi_{2t} \\ & \quad = P_{1t} C_t + P_{1t} \left[X_{1,t+1} - (1 - \delta_1) X_{1,t} \right] + P_{2t} \left[X_{2,t+1} - (1 - \delta_2) X_{2,t} \right], \end{aligned} \tag{12}$$

where the lefthand side is the sum of factor incomes and profit redistributions and the righthand side is the total expenditure (consumption plus gross investments in physical and human capitals).[8]

Therefore, the representative household chooses $\{C_t, X_{1,t}, X_{2,t}\}$ to maximize the lifetime utility U subject to the budget constraint (12). Since input and output

[8] If we follow the implication pointed out in Remark 1 in Sect. 2.1, we may denote $\pi_j = W_3 X_{3ji} (j = 1, 2)$, where W_3 is the price of X_{3j} and it satisfies $W_3 = P_j \beta_{3j} Y_j / X_{3j}$.

prices are defined in utils without time-discounting, the Lagrangean function can be specified as:

$$\mathcal{L} = \sum_{t=0}^{\infty} \rho^t \left\{ \frac{C_t^{1-\sigma} - 1}{1-\sigma} + W_{1t}X_{1t} + W_{2t}X_{2t} + \pi_{1t} + \pi_{2t} \right.$$
$$\left. -P_{1t}C_t - P_{1t}[X_{1,t+1} - (1-\delta_1)X_{1t}] - P_{2t}[X_{2,t+1} - (1-\delta_2)X_{2t}] \right\}.$$

The first-order necessary conditions are ($i = 1, 2$):

$$\frac{\partial \mathcal{L}}{\partial C_t} = \rho^t \left(C_t^{-\sigma} - P_{1t} \right) = 0, \tag{13}$$
$$\frac{\partial \mathcal{L}}{\partial X_{it}} = \rho^t [W_{it} + (1-\delta_1)P_{it}] - \rho^{t-1} P_{i,t-1} = 0, \tag{14}$$

which are sufficient for optimization under the transversality conditions given by ($i = 1, 2$):

$$\lim_{t\to\infty} \rho^t P_{it} X_{i,t+1} = 0.$$

3 Equilibrium

Within our endogenous-growth framework with sector-specific externalities, a *dynamic competitive equilibrium* satisfies the following conditions:

(i) Each consumer maximizes her life-time utility subject to the flow budget constraint (4) under a given sequence of perfectly anticipated prices and profits, $\left\{W_{i,t}, P_{i,t}, \pi_{i,t}\right\}_{t=0}^{\infty}$ (i =1,2). Additionally, the consumer's optimal plan stisfies the transversality condition: $\lim_{t\to\infty} \rho^t P_{it} X_{i,t+1} = 0$.
(ii) Each firm maximizes its profits in each moment by taking external effects, $\overline{v_{ij,t_i} X_{i,t}}$ ($i, j = 1, 2$), as given.
(iii) The goods market clearing conditions, (2) and (3), always hold.
(iv) Both capitals are fully employed: $AY_t = X_t$ and $\Sigma_{j-1}^2 v_{ij} = 1$ ($i, j = 1, 2$) for all $t \geq 0$.
(v) The external effects satisfy the consistency conditions so that $\overline{v_{ij,t_i} X_{i,t}} = v_{ij,t_i} X_{i,t}$ ($i, j = 1, 2$) for all $t \geq 0$.

Notice that under the budget constraint, the production technology, the capital evolution equations, the factor demand conditions, and the full employment conditions, one of the goods market clearing conditions is redundant, i.e., the Walras' law holds.

Since both capitals are accumulated perpetually, the quantity and price variables stated above are nonstationary. A *balanced growth path* (BGP) is a dynamic competitive equilibrium in which quantities and nominal prices $\{X_{it}, Y_{jt}, C_t, W_{it}, P_{jt}\}$ grow at some constant rates and factor allocation shares are time-invariant. A BGP is called *nondegenerate* if the rates of growth of quantities $\{X_{it}, Y_{jt}, C_t\}$ are positive.

3.1 Transformation of the dynamical system

To characterize the dynamical system, it is important to properly transform the non-stationary system into a stationary one. Given the property of constant social returns of the production technologies and the time additive preference with constant elasticity of intertemporal substitution, it is straightforward from (1)–(3), (13) and (6) that, along a BGP, factor inputs (X_i), outputs (Y_j) and consumption (C) will all grow at a common rate, denoted g, whereas factor prices (W_i) and good/education service prices (P_j) will decrease at a common rate, $(1+g)^{\sigma}$. To transform these endogenous variables into stationary variables, we therefore discount quantities by dividing by $(1+g)^t$ and upcount prices by multiplying $(1+g)^{\sigma t}$:

$$
\begin{aligned}
&x_t \equiv X_t\,(1+g)^{-t}\,; \quad y_t \equiv Y_t\,(1+g)^{-t}\,; \quad c_t \equiv C_t\,(1+g)^{-t}\,;\\
&p_t \equiv P_t\,(1+g)^{\sigma t}\,; \quad w_t \equiv W_t\,(1+g)^{\sigma t}\,,
\end{aligned}
$$

where all variables in lower case denote the corresponding stationary values after transformation. The transversality conditions require:

$$\rho\,(1+g)^{1-\sigma} < 1. \tag{15}$$

Thus, (2), (3) and (13) imply:

$$(1+g)\,x_{t+1} = y_t + (I - I_\delta)\,x_t - z_t, \tag{16}$$

while updating (14) by one period yields:

$$(1+g)^{\sigma}\,p_t = \rho\,(I - I_\delta)\,p_{t+1} + \rho w_{t+1}, \tag{17}$$

where $z_t \equiv (c_t, 0)' = \left(p_{1t}^{-1/\sigma}, 0\right)'$ and I_δ is a diagonal matrix with diagonal elements, δ_1 and δ_2. Moreover, we can rewrite (8) and (10) in terms of transformed stationary variables:

$$A y_t = x_t \tag{18}$$
$$w_t = (\hat{A}')^{-1} p_t. \tag{19}$$

Finally, substituting (18) and (19) into (16) and (17), we obtain the two fundamental equations governing the dynamical system in $\{x_{1t}, x_{2t}, p_{1t}, p_{2t}\}$:

$$(1+g)\,x_{t+1} = [(I - I_\delta) + A^{-1}]x_t - z_t \tag{20}$$
$$\rho[(I - I_\delta) + (\hat{A}')^{-1}]p_{t+1} = (1+g)^{\sigma}\,p_t. \tag{21}$$

It is important to note that once the balanced growth rate, g, is solved, these two fundamental equations alone determine the dynamics of normalized capital inputs and normalized nominal output prices (x, p). Since there is no money illusion, this system in nominal prices must exhibit one-dimensional linear dependency, containing a unit root (to be shown later).

3.2 Balanced growth path

Along a BGP, the transformed variables are stationary and hence, (20) and (21) imply:

$$\left[A^{-1} - (gI + I_\delta)\right] x = z \tag{22}$$

$$\left[(1+g)^\sigma I - \rho\left(I - I_\delta + \widehat{A}^{-1}\right)'\right] p = 0. \tag{23}$$

Since prices are positive, (23) alone determines g by,

$$\det\left[(1+g)^\sigma I - \rho\left(I - I_\delta + \widehat{A}^{-1}\right)'\right] = 0. \tag{24}$$

Upon pinning down the balanced growth value of g, x can be determined by inverting (22), provided that $\det\left[I - A\left(gI + I_\delta\right)\right] \neq 0$ (to be shown later):

$$x = \left[I - A\left(gI + I_\delta\right)\right]^{-1} Az. \tag{25}$$

For convenience, define $s_i\left(g\right) \equiv (1+g)^\sigma - \rho\left(1 - \delta_i\right)$. We can then write (23) as:

$$I_s p = \rho \widehat{A}^{-1} p, \tag{26}$$

where I_s is a diagonal matrix with diagonal elements s_i $(i = 1, 2)$.

The remainder of this subsection is devoted to proving the existence and uniqueness of the BGP by examining normalized prices and quantities as well as the balanced growth rate, to which we now turn.

3.2.1 Existence and uniqueness of normalized prices and balanced growth rate

As in the continuous-time model, the balanced growth rate and price variables are pinned down in the system in a recursive manner prior to the determination of the quantity variables. For the BGP to be nondegenerate and to satisfy the transversality condition, it must hold that $\max\left\{1, \rho\left(1+g\right)\right\} < (1+g)^\sigma$. If $\sigma \geq 1$, this is satisfied for any $g > 0$ (because $\rho < 1$). If $0 < \sigma < 1$, then it is equivalent to the following conditions:

$$1 < (1+g)^\sigma < \rho^{-\sigma/(1-\sigma)}. \tag{27}$$

This resembles the Jones–Manuelli conditions (first inequality) and the Brock–Gale condition (second inequality). Since the balanced-growth rate is given by the root of (24), the above inequalities mean that (24) should have a solution in between 0 and $\rho^{-\sigma/(1-\sigma)}$. This holds, if α_1 and α_2 satisfy following restrictions[9]:

[9] As shown by the proof of Lemma 1 in Appendix of the paper, (24) is written as

$$\left(s_1(g)\right)^{\widehat{\beta}_{12}} \left(s_2(g)\right)^{\widehat{\beta}_{21}} - \left(\rho\alpha_1 \beta_{11}^{\widehat{\beta}_{11}} \beta_{21}^{\widehat{\beta}_{21}}\right)^{\widehat{\beta}_{12}} \left(\rho\alpha_2 \beta_{12}^{\widehat{\beta}_{12}} \beta_{22}^{\widehat{\beta}_{22}}\right)^{\widehat{\beta}_{21}} = 0.$$

It is easy to see that if α_1 and α_2 satisfy Assumption 2, the balanced growth rate, g, fulfills (27).

Assumption 2 $\frac{s_1(0)}{\rho\beta_{11}^{\widehat{\beta}_{11}}\beta_{21}^{\widehat{\beta}_{21}}} < \alpha_1 < \frac{s_1\left(\rho^{-1/(1-\sigma)}-1\right)}{\rho\beta_{11}^{\widehat{\beta}_{11}}\beta_{21}^{\widehat{\beta}_{21}}}$ and $\frac{s_2(0)}{\rho\beta_{12}^{\widehat{\beta}_{12}}\beta_{22}^{\widehat{\beta}_{22}}} < \alpha_2 < \frac{s_2\left(\rho^{-1/(1-\sigma)}-1\right)}{\rho\beta_{12}^{\widehat{\beta}_{12}}\beta_{22}^{\widehat{\beta}_{22}}}$.

Denote $\underline{1} \equiv (1,1)'$. Recall from (9) that we have: $\widehat{A}^{-1} = I_w\widehat{B}^{-1}I_p^{-1}$, which can be substituted into (26) to yield,

$$I_s p = \rho I_w \widehat{B}^{-1} I_p^{-1} p = \rho I_w \widehat{B}^{-1}\underline{1} = \rho I_w, \tag{28}$$

where in deriving the last equality we have used the property, $\widehat{B}\underline{1} = \underline{1}$ and hence, $\widehat{B}^{-1}\underline{1} = \underline{1}$. This implies, for $i = 1, 2$,

$$s_i(g)p_i = \rho w_i. \tag{29}$$

From (11), we can express real factor prices w_i/p_i as functions of relative prices $q \equiv p_1/p_2$ and plug the results into (29) to obtain:

$$\rho\overline{w}_1 q^{\widehat{\beta}_{21}/\widehat{\Delta}} - s_1(g) = 0, \quad \rho\overline{w}_2 q^{-\widehat{\beta}_{12}/\widehat{\Delta}} - s_2(g) = 0 \tag{30}$$

We begin by showing that under Assumption 1, Assumption 2 is sufficient for the balanced growth rate g^* to satisfy (27):

Lemma 1 *Under Assumptions* 1 *and* 2*, if the balanced growth rate g^* exists, it satisfies* (27) *and is unique and strictly positive.*

Notice that Lemma 1 implies that the BGP is nondegenerate, along which $s_i > 0$ for $i = 1, 2$.

Next, we show the existence and uniqueness of the relative price of outputs q^* on a BGP:

Lemma 2 *Under Assumption* 1*, the relative price of outputs on a balanced growth path q^* is uniquely determined and strictly positive.*

Finally, we prove the existence of the uniquely determined balanced growth rate:

Lemma 3 *Under Assumptions* 1 *and* 2*, the balanced growth rate g^* exists.*

3.2.2 Existence and uniqueness of normalized quantities

To obtain nondegenerate solution of the quantity variables, we establish:

Lemma 4 *Under Assumptions* 1 *and* 2*, normalized capitals on a balanced growth path are uniquely determined and strictly positive.*

Once the BGP values of $\{x_i, p_j, g\}$ are determined by Lemmas 1–4, $\{y_j, c, w_i/p_i, v_{ij}\}$ along the BGP can be pinned down in a recursive manner by

$$\begin{aligned} &y = A^{-1}x; \quad c = y_1 - (g^* + \delta)x_1; \quad w_i/p_i = s_i(g^*)/\rho; \\ &v_{ii} = \beta_{ii}\frac{y_i/x_i}{w_i/p_i}; \; v_{ij} = 1 - v_{ii}. \end{aligned} \tag{31}$$

This and Lemmas 1–4 imply,

Proposition 1 (Existence and uniqueness of the BGP) *Under Assumptions* 1 *and* 2, *a balanced growth path exists and is unique along which capital inputs, outputs and consumption all grow at a common balanced growth rate* $g^* \in (0, \rho^{-1/(1-\sigma)} - 1)$ *satisfying* $s_1(g^*)^{\widehat{\beta}_{12}} (s_2(g^*))^{\widehat{\beta}_{21}} - (\rho\overline{w}_1)^{\widehat{\beta}_{12}} (\rho\overline{w}_2)^{\widehat{\beta}_{21}} = 0$, *the balanced growth value of the relative price of outputs* $q^* > 0$ *is uniquely determined by* $\overline{w}_1 q^{*\widehat{\beta}_{21}/\widehat{\Delta}} - \overline{w}_2 q^{*-\widehat{\beta}_{12}/\widehat{\Delta}} + \delta_1 - \delta_2 = 0$, *and the balanced growth values of normalized quantities, factor allocation shares and real factor prices,* $\{x_i, y_j, c, v_{ij}, w_i/p_i\}$ *are all strictly positive, given by* (25) *and* (31).

4 Characterization of local dynamics

We now turn to characterizing the local dynamics of the recursive 4×4 dynamical system of $\{x_{1t}, x_{2t}, p_{1t}, p_{2t}\}$, governed by the two fundamental Eq. (20) and (21) around the BGP. It should be recognized that in this system that 4×4 dynamical system describes three dimensional dynamics because of the linear homogeneity of the system in $\{x_{1t}, x_{2t}, p_{1t}, p_{2t}\}$, and one of the characteristic roots at the steady state is always one as shown below. Thus, the dynamical system is *locally determinate* if there exist at least two unstable roots and it is *dynamically indeterminate* if there exist at least two stable roots.

4.1 Dynamical system

It is clearly seen from (20) and (21) that the price dynamics is determined by prices alone. Moreover, as we will establish in the case with partial capital depreciation, the price dynamics is related to the factor intensity ranking in the value or social sense. Thus, the associated characteristic roots are denoted by $\{\widehat{\lambda}_1, \widehat{\lambda}_2\}$ (corresponding to the determinant of the $\widehat{B}$ matrix). Likewise, since the quantity dynamics is related to the factor intensity ranking in the physical or private sense, the associated characteristic roots are denoted by $\{\lambda_1, \lambda_2\}$ (corresponding to the determinant of the B matrix).

Using (21), we can characterize the price dynamics by,

$$\frac{\partial p_{t+1}}{\partial p_t} = \frac{(1+g^*)^\sigma}{\rho} \left(I - I_\delta + \hat{A}'\right)^{-1}. \tag{32}$$

The eigen roots evaluated at the balanced growth path $\{\widehat{\lambda}_1, \widehat{\lambda}_2\}$ solve ($n = 1, 2$):

$$\det\left\{\frac{(1+g^*)^\sigma}{\rho}\left[I - I_\delta + \left(\widehat{A}'\right)^{-1}\right]^{-1} - \widehat{\lambda}_n I\right\} = 0,$$

or,

$$\det\left[\left(1+g^*\right)^\sigma I - \rho\widehat{\lambda}_n\left(I - I_\delta + \widehat{A}^{-1}\right)'\right] = 0. \tag{33}$$

Comparing (33) and (24) with $g = g^*$, it can be easily shown:

Lemma 5 *Under Assumptions* 1 *and* 2, *one of the eigen roots governing the local dynamics of the relative price of outputs is one* (*i.e.,* $\widehat{\lambda}_1 = 1$).

That $\widehat{\lambda}_1 = 1$ is a solution of (33) verifies our assertion above: the system exhibits one-dimensional linear dependency because prices are in nominal terms.

With respect to the quantity dynamics, we can obtain from (20) the following:

$$\frac{\partial x_{t+1}}{\partial x_t} = \left(1 + g^*\right)^{-1} \left[A^{-1} + (I - I_\delta)\right]. \tag{34}$$

The eigen roots evaluated at the balanced growth path $\{\lambda_1, \lambda_2\}$ solve $(n = 1, 2)$

$$\det\left[\left(1 + g^*\right)^{-1} \left(A^{-1} + I - I_\delta\right) - \lambda_n I\right] = 0. \tag{35}$$

As we will establish in the next subsection, the values of $\{\widehat{\lambda}_2, \lambda_1, \lambda_2\}$ and hence the property of local dynamics depend crucially on the rates of capital depreciation. This contrasts sharply with the continuous-time endogenous growth models of Bond et al. (1996), Benhabib et al. (2000) and Mino (2001).

4.2 Local stability

In this section we demonstrate that, as long as capital stocks depreciate partially, i.e., $0 < \delta_1, \delta_2 < 1$, the necessary and sufficient conditions for local determinacy and for dynamic indeterminacy obtained in continuous time are neither necessary nor sufficient in discrete time.

To begin, we examine how the price dynamics is related to the sign of the social factor share matrix, $\widehat{\Delta}$. For illustrative purposes, we rule out the cases of degenerate price paths (the first condition) and limit cycles (the second condition), which are possible for $\widehat{\Delta} < 0$:[10]

Assumption 3 $(1+g^*)^\sigma \neq \frac{\rho\left[(1-\delta_1)\widehat{\beta}_{21}+(1-\delta_2)\widehat{\beta}_{12}\right]}{\widehat{\beta}_{11}+\widehat{\beta}_{22}}$ and $(1+g^*)^\sigma \neq \rho\left[(1-\delta_1)\,\widehat{\beta}_{21} + (1-\delta_2)\,\widehat{\beta}_{12}\right]$.

Consider then following inequalities which are valid only under $\widehat{\Delta} < 0$ (and hence $\widehat{\beta}_{11} + \widehat{\beta}_{22} < 1$):

Condition P. $\rho\left[(1-\delta_1)\,\widehat{\beta}_{21} + (1-\delta_2)\,\widehat{\beta}_{12}\right] < (1+g^*)^\sigma < \frac{\rho\left[(1-\delta_1)\widehat{\beta}_{21}+(1-\delta_2)\widehat{\beta}_{12}\right]}{\widehat{\beta}_{11}+\widehat{\beta}_{22}}$.

We can then establish the following property concerning the price dynamics.

Lemma 6 *Under Assumptions* 1–3 *with partial capital depreciation* (*i.e.,* $0 < \delta_1, \delta_2 < 1$)*, the price dynamics possess the following properties:*

(i) *for* $\widehat{\Delta} > 0$*, the price dynamics is stable with* $0 < \widehat{\lambda}_2 < 1 = \widehat{\lambda}_1$*;*
(ii) *for* $\widehat{\Delta} < 0$*, the price dynamics is unstable with* $\widehat{\lambda}_1 = 1$ *and* $\widehat{\lambda}_2 < -1$ *if and only if Condition P holds and stable with* $\left|\widehat{\lambda}_2\right| < 1 = \widehat{\lambda}_1$ *if and only if Condition P fails.*

We next turn to examining how the quantity dynamics are related to the sign of the private factor share matrix, Δ.

[10] Formally speaking, when the first expression holds for equality, it is a flip bifurcation point; when the second expression holds for equality, it is a hopf bifurcation point.

Lemma 7 *Under Assumptions* 1–3 *with partial capital depreciation* (*i.e.,* $0 < \delta_1, \delta_2 < 1$)*, the quantity dynamics possess the following properties:*

(i) *for* $\Delta > 0$*, the quantity dynamics is unstable with* $|\lambda_1| > 1$ *and* $|\lambda_2| > 1$*;*
(ii) *for* $\Delta < 0$*, there exists* $0 < \rho_{\min} < 1$, $0 < \delta_{\max} < 1$ *and* $0 < \overline{\delta} < 1$ *such that for any* $\rho \in (\rho_{\min}, 1)$*,* $\delta_i \in (0, \delta_{\max})$*, and* $|\delta_1 - \delta_2| < \overline{\delta}$*, the quantity dynamics features a one-dimensional stable manifold with* $-1 < \lambda_2 < 1 < \lambda_1$.

An important message delivered by Lemmas 6 and 7 is that in contrast with continuous-time endogenous growth models, $\widehat{\Delta} > 0$ is *not necessary* for stable price adjustments (as shown by Part (ii) of Lemma 6) whereas $\Delta < 0$ is *not sufficient* for stable quantity adjustments (as shown by Part (ii) of Lemma 7). In the next two subsections, we establish conditions under which local determinacy and dynamic indeterminacy, respectively, emerge.

4.2.1 Local determinacy

While consistent factor intensity rankings in the private and social sense need not lead to local determinacy within our discrete-time endogenous growth framework, we consider a more special circumstance in which the production of the consumable good uses consumable (physical) capital more intensively than the production of the pure (human) capital good in both value (social) and physical (private) sense, that is,
Condition D $\Delta > 0$ and $\widehat{\Delta} > 0$.

Proposition 2 (Local determinacy under partial depreciation) *Under Assumptions* 1–3 *with partial capital depreciation* (*i.e.,* $0 < \delta_1, \delta_2 < 1$)*, the dynamical system is locally determinate if Condition D holds.*

It should be noted that since the proof applies to any dimensionality of $N \geq 2$, this local determinacy results is also general to any multi-sector endogenous growth models with socially constant-return sector-specific externalities.

In the continuous-time endogenous growth model, the necessary and sufficient condition for local determinacy is that the factor intensity rankings in the private and social sense are consistent, i.e., $sign(\Delta) = sign(\widehat{\Delta})$. Our Condition D is *stronger* than the conventional requirement, though it is *not necessary*.

We next illustrate why Condition D is not necessary for local determinacy. Should both Δ and $\widehat{\Delta}$ be negative, one may still establish:

Proposition 3 (Local determinacy under partial depreciation) *Under Assumptions* 1–3 *with partial capital depreciation* (*i.e.,* $0 < \delta_1, \delta_2 < 1$)*, there exists* $0 < \rho_{\min} < 1$, $0 < \delta_{\max} \leq 1$ *and* $0 < \overline{\delta} < 1$ *such that for any* $\rho \in (\rho_{\min}, 1)$*,* $\delta_i \in (0, \delta_{\max})$*, and* $|\delta_1 - \delta_2| < \overline{\delta}$*, the dynamical system is locally determinate if Condition P holds and Condition D fails.*

Both Propositions 2 and 3 establish conditions for local determinacy. Under the conditions stated in Proposition 2, price adjustments are stable but quantity adjustments are unstable; under the conditions stated in Proposition 3, price adjustments are unstable but quantity adjustments are stable. While Proposition 2 can be generalized to any dimensionality of $N \geq 2$, Proposition 3 cannot be.

Since Proposition 3 depends on not only the factor intensity ranking but the magnitudes of time preference and capital depreciation rates. For example, Let $\{\beta_{11}, \beta_{21}, \beta_{12}, \beta_{22}\} = \{0.5, 0.25, 0.6, 0.1\}$ under which $\Delta < 0$, and $\{\widehat{\beta}_{11}, \widehat{\beta}_{12}\} = \{0.7, 0.75\}$ so that $\widehat{\Delta} < 0$. Further, take $\delta_1 = \delta_2 = 0.1$, $\sigma = 1.05$, $\rho = 0.99$ and $\alpha_1 = \alpha_2 = 1$. In this case, the BGP is locally determinate.

Remark 2 It should be noted that if both capital stocks are fully depreciated in each period ($\delta_1 = \delta_2 = 1$), then the BGP is locally determinate, *regardless of factor intensity rankings*. Specifically, in the case of full depreciation, from (33) and (35) the price and quantity dynamics, respectively, determined by the roots of

$$\det\left[\rho^{-1}\left(1+g^*\right)^{\sigma}\widehat{A}'-\widehat{\lambda}I\right]=0 \quad \text{and} \quad \det\left[\left(1+g^*\right)^{-1}A^{-1}-\lambda I\right]=0.$$

We can confirm that both roots of the first equation in the above are in the unit circle, while those of the second equation are out of the unit circle. Thus the BGP is always locally determinate with full depreciation and partial depreciation is *necessary* for indeterminacy to arise. To gain intuition, consider an alternative way to transform the model into a stationary system using relative (rather than nominal prices), following the continuous-time analysis by Bond et al. (1996) and Mino (2001). More specifically, using the second good as the numeràire, the relative price dynamics under full depreciation are then governed by the following intertemporal no-arbitrage condition,

$$\frac{q_t}{q_{t+1}} = \frac{(1-\delta_1)+w_{1,t+1}/q_{t+1}}{(1-\delta_2)+w_{2,t+1}} = \frac{w_{1,t+1}/q_{t+1}}{w_{2,t+1}}, \tag{36}$$

By the Stolper–Samuelson theorem, the sign of the responses of the right-hand side of the relative price dynamics depends crucially on the factor intensity ranking in the social sense (positive for $\widehat{\Delta} > 0$ and negative for $\widehat{\Delta} < 0$). Yet, changes in the relative price always induce magnified changes in factor prices and, as a consequence, $\left|\frac{dq_t}{dq_{t+1}}\right|_{q_{t+1}=q_t=q} > 1$, which implies $\left|\frac{dq_{t+1}}{dq_t}\right|_{q_{t+1}=q_t=q} < 1$ and hence the price dynamics is always stable. By similar arguments, under full depreciation, the two capitals become essentially flow variables and the magnification effects from the Rybczynski theorem imply that the quantity dynamics is always unstable, reconfirming the local determinacy result. Note that the above argument is applicable to the case of partial depreciation as well, if both δ_1 and δ_2 are close to one. In this case $1-\delta_1$ and $1-\delta_2$ are sufficiently small, and thus the behavior of the relative price given by (36) is still stable while the quantity system remains unstable. Therefore, the equilibrium path would be locally determinate, if the capital depreication rates are relatively high.

4.2.2 Dynamic indeterminacy

We next turn to studying the possibility of dynamic (local) indeterminacy. Consider the necessary and sufficient condition for dynamic indeterminacy obtained in the continuous-time endogenous growth literature:
Condition I $\Delta < 0$ and $\widehat{\Delta} > 0$.

This condition states that the consumable good production uses pure (human) capital more intensively in the private (physical) sense but uses consumable (physical) capital intensive in the social (value) sense. We can now establish:

Proposition 4 (Dynamic indeterminacy under partial depreciation) *Under Assumptions* 1–3 *and Condition I with partial capital depreciation* (*i.e.,* $0 < \delta_1, \delta_2 < 1$)*, there exists* $0 < \rho_{\min} < 1$, $0 < \delta_{\max} < 1$ *and* $0 < \overline{\delta} < 1$ *such that for any* $\rho \in (\rho_{\min}, 1)$*,* $\delta_i \in (0, \delta_{\max})$*, and* $|\delta_1 - \delta_2| < \overline{\delta}$*, the dynamical system is locally indeterminate.*

Notably, Condition I is *neither sufficient nor necessary* for dynamic indeterminacy. On the one hand, Condition I is not sufficient for dynamic indeterminacy, because the magnitudes of time preference and capital depreciation rates both play essential roles in addition to the factor intensity rankings. In particular, to generate dynamic indeterminacy, we need both the time preference and the capital depreciation rates ($\rho^{-1} - 1$ and δ_i) to be sufficiently small and the depreciation rates of the two capitals to be sufficiently alike. To be more concrete, consider $\{\beta_{11}, \beta_{21}, \beta_{12}, \beta_{22}\} = \{0.5, 0.25, 0.6, 0.1\}$ and $\{\widehat{\beta}_{11}, \widehat{\beta}_{12}\} = \{0.7, 0.65\}$ so that $\Delta < 0$ and $\widehat{\Delta} > 0$ and set $\delta_1 = \delta_2 = 0.1$, $\sigma = 1.05$, $\rho = 0.99$, $\alpha_1 = 0.5$ and $\alpha_2 = 1$. In this case, the dynamical system is locally indeterminate (one-dimensional indeterminacy). Yet, by simply raising δ_2 to 0.2, the dynamical system becomes locally determinate. This finding confirms our conjecture pointed out in Remark 2: higher depreciation rates of capital stocks lowers the possibility of dynamic indeterminacy.

On the other hand, Condition I is not necessary for dynamic indeterminacy.

Proposition 5 (Dynamic indeterminacy under partial depreciation) *Under Assumptions* 1–3 *with partial capital depreciation* (*i.e.,* $0 < \delta_1, \delta_2 < 1$) *and with Condition S failing to hold, there exists* $0 < \rho_{\min} < 1$, $0 < \delta_{\max} < 1$ *and* $0 < \overline{\delta} < 1$ *such that for any* $\rho \in (\rho_{\min}, 1)$*,* $\delta_i \in (0, \delta_{\max})$*, and* $|\delta_1 - \delta_2| < \overline{\delta}$*, the dynamical system is locally indeterminate if Condition P fails.*

Again, to illustrate this possibility, consider the numerical example provided in the previous subsection: by simply reducing α_1 from 1 to 0.5, dynamic indeterminacy arises. Generally speaking, under the discrete-time setup, dynamic indeterminacy is more likely to occur as a consequence of *stable price dynamics* (Lemma 6) but less so as a result of *unstable quantity dynamics* (Lemma 7). Thus, Condition I turns out to be neither necessary (as the conditions for stable price adjustments become weaker) nor sufficient (as the conditions for stable quantity adjustments become stronger) for dynamic indeterminacy.

One may compare this latter result with that in the discrete-time exogenous growth framework developed by Benhabib et al. (2002). Let us relabel the consumption good sector in their paper as the consumable good sector and the investment good sector in their paper as the pure capital good sector. Then, they establish the conditions for dynamic indeterminacy (in our notation with $g^* = \delta_i = 0$): $\Delta < \beta_{11}/\rho$ and $\widehat{\Delta} > -\widehat{\beta}_{12}/2$. Although our conditions are very different, both suggest that even when Condition I is violated, dynamic indeterminacy may still emerge. Yet, while Condition I is sufficient for dynamic indeterminacy in the exogenous growth setting, it is not so in our endogenous growth framework. This is due to the *increased possibility of unstable quantity adjustments* under the sustained endogenous growth setup. In addition, the discussion in Sect. 4.2.1 has indicated that in our discrete-time setting the possibility of unstable behavior of the quantity system increases with the magnitudes of capital depreciation rates.

5 Concluding remarks

This paper has reexamined the property of local dynamics in a class of two-sector endogenous growth models with socially constant-returns production technologies with positive sector-specific externalities. Within this class of models in continuous time, two fundamental propositions have been established: (i) the balanced growth path is locally determinate if and only if the factor intensity rankings in the private and social sense are consistent; and, (ii) dynamic indeterminacy emerges if and only if the consumable good production uses pure (human) capital more intensively in the private (physical) sense but uses consumable (physical) capital intensive in the social (value) sense. This paper shows that in discrete time, both propositions fail to hold.

Our main finding is that the dynamic property is no longer determined by the factor intensity ranking exclusively. Under partial depreciation, local determinacy is obtained if the consumable good production uses consumable capital more intensively in both private and social sense. When the consumable good production uses pure capital more intensively in both private and social sense, the balanced growth path may be locally determinate or indeterminate, depending on the underlying preference and production parameters. When the consumable good production uses pure capital more intensively in the private sense but uses consumable capital intensive in the social sense, dynamic indeterminacy arises only if the time preference and the capital depreciation rates are sufficiently small and the depreciation rates of the two capitals are not too different. Similar conclusions hold if we replace sector-specific externalities by distortionary factor taxes.

Along these lines, one may revisit the issue raised by Benhabib and Perli (1994) that whether dynamic indeterminacy can arise in set of plausible parameters with moderate external effects. For example, if we select $\{\beta_{11}, \beta_{21}, \beta_{12}, \beta_{22}\} = \{0.3, 0.55, 0.25, 0.4\}$, $\{\widehat{\beta}_{11}, \widehat{\beta}_{12}\} = \{0.4, 0.35\}$, $\delta_1 = \delta_2 = 0.05$, $\sigma = 1.5$, $\rho = 0.97$, $\alpha_1 = 0.02$ and $\alpha_2 = 1$, the resulting real interest rate, balanced growth rate, and share of pure (human) capital devoted to final goods production are 5, 1.6 and 74%, respectively, fitting with the US data (though it should be noted that the selection of parameters is not unique). In this case, the dynamical system is locally determinate. Moreover, one may also find sets of parameters under which cycles may occur. Consider: $\{\beta_{11}, \beta_{21}, \beta_{12}, \beta_{22}\} = \{0.2091, 0.25, 0.6, 0.1\}$, $\{\widehat{\beta}_{11}, \widehat{\beta}_{12}\} = \{0.7, 0.75\}$, $\delta_1 = \delta_2 = 0.1$, $\sigma = 1.05$, $\rho = 0.99$, $\alpha_1 = 0.5$ and $\alpha_2 = 1$. In this case, one of the roots governing the quantity dynamics is zero and limit cycles arise. Of course, sector-specific externalities in the model economy must be the driving forces of cycles. Yet, under which circumstances cycles are likely to be present remains unexplored.[11] Finally, it may be interesting to characterize the property of local dynamics using a more general class of socially constant-return production technologies:

$$Y_{jt} = \alpha_j \Psi_j \left(\overline{v_{1jt} X_{1t}}, \overline{v_{2jt} X_{2t}} \right) G_j \left(v_{1jt} X_{1t}, v_{2jt} X_{2t} \right),$$

[11] As shown in Sect. 4.3, limit cycles are possible only for $\widehat{\Delta} < 0$. In the numerical example above, we have $\Delta < 0$ and $\widehat{\Delta} < 0$. It is possible to find cases exhibiting limit cycles with $\Delta > 0$. Thus, the consistency/inconsistency of the factor intensity rankings is not necessary for limit cycles to emerge.

where both Ψ_j and G_j are strictly increasing, strictly concave and twice continuously differentiable and $\Psi_j G_j$ exhibits constant-returns-to-scale. These are but three of many possible avenues for future research.

Appendix

Proof of Lemma 1 Define,

$$\Gamma(g) \equiv (s_1(g))^{\widehat{\beta}_{12}} (s_2(g))^{\widehat{\beta}_{21}} - (\rho \overline{w}_1)^{\widehat{\beta}_{12}} (\rho \overline{w}_2)^{\widehat{\beta}_{21}}$$

From (30), if a balanced growth rate g^* exists, it must satisfy $\Gamma(g^*) = 0$. Utilizing the definitions of s_i and $\overline{w}_i$ $(i = 1, 2)$ and $\sum_{i=1}^{2} \widehat{\beta}_{ij} = 1$, we can see that,

$$\Gamma(g) = \left[(1+g)^\sigma - \rho(1-\delta_1)\right]^{\widehat{\beta}_{12}} \left[(1+g)^\sigma - \rho(1-\delta_2)\right]^{\widehat{\beta}_{21}} - \left(\rho\alpha_1 \beta_{11}^{\widehat{\beta}_{11}} \beta_{21}^{\widehat{\beta}_{21}}\right)^{\widehat{\beta}_{12}} \left(\rho\alpha_2 \beta_{12}^{\widehat{\beta}_{12}} \beta_{22}^{\widehat{\beta}_{22}}\right)^{\widehat{\beta}_{21}},$$

where $\Gamma(g)$ is increasing in g and, by Assumption 2, $\Gamma(0) < 0$ and $\Gamma(\rho^{-1/(1-\sigma)} - 1) > 0$. Thus a solution g^* satisfying $\Gamma(g^*) = 0$ exists and is unique with $0 < g^* < \rho^{-1/(1-\sigma)} - 1$. □

Proof of Lemma 2 Defining the difference of the left hand sides of (30) as $f(q)$, we can use the definition of s_i to get:

$$f(q) \equiv \rho \left[\overline{w}_1 q^{\widehat{\beta}_{21}/\widehat{\Delta}} - \overline{w}_2 q^{-\widehat{\beta}_{12}/\widehat{\Delta}} + (\delta_1 - \delta_2)\right].$$

If $\widehat{\Delta} > 0$, f is an increasing function with $\lim_{q \longrightarrow 0} f(q) = -\infty$ and $\lim_{q \longrightarrow \infty} f(q) = \infty$; if $\widehat{\Delta} < 0$, f is a decreasing function with $\lim_{q \longrightarrow 0} f(q) = \infty$ and $\lim_{q \longrightarrow \infty} f(q) = -\infty$. Hence, $f(q) = 0$ has a unique solution $q^* > 0$. □

Proof of Lemma 3 From $f(q^*) = 0$ in the proof of Lemma 2,

$$\overline{w}_1 q^{\widehat{\beta}_{21}/\widehat{\Delta}} + \delta_1 = \overline{w}_2 q^{-\widehat{\beta}_{12}/\widehat{\Delta}} + \delta_2.$$

By combining this with (30), g^* is determined. □

Proof of Lemma 4 The key is to prove that the matrix $I - A(g^* I + I_\delta)$ is invertible and $\left[I - A(g^* I + I_\delta)\right]^{-1}$ in (25) is a positive matrix. From (24), we have,

$$\left[(1+g^*)^\sigma - \rho\left(1+g^*\right)\right] \widehat{A} p = \rho \left[I - \widehat{A}\left(g^* I + I_\delta\right)\right] p > 0. \quad (37)$$

Let $\widehat{\mu}$ be the Frobenius root of $\widehat{A}(g^* I + I_\delta)$. Then (37) implies $0 < \widehat{\mu} < 1$. Since $0 < A < \widehat{A}$, the Frobenius root μ of $A(g^* I + I_\delta)$ satisfies $0 < \mu < \widehat{\mu} < 1$. Thus, $\left[I - A(g^* I + I_\delta)\right]^{-1}$ exists and is a positive matrix. Hence, x is uniquely determined by (25) and $x > 0$. □

Proof of Lemma 6 First, we consider the eigen roots of (33) governing the price dynamics. Recalling from Lemma 5 that $\widehat{\lambda}_1 = 1$, the other root $\widehat{\lambda}_2$ is given by,

$$\begin{aligned}\widehat{\lambda}_2 &= \det\left[\rho^{-1}\left(1+g^*\right)^{\sigma}\left(I - I_\delta + \left(\widehat{A}'\right)^{-1}\right)^{-1}\right]\\ &= \rho^{-2}\left(1+g^*\right)^{2\sigma}\left|I - I_\delta + \left(\widehat{A}'\right)^{-1}\right|^{-1}\end{aligned}$$

or, applying (9),

$$\begin{aligned}\widehat{\lambda}_2 &= \rho^{-2}\left(1+g^*\right)^{2\sigma}\left|I - I_\delta + I_w\left(\widehat{B}'\right)^{-1} I_p^{-1}\right|^{-1}\\ &= \rho^{-2}\left(1+g^*\right)^{2\sigma}\det\begin{bmatrix}1-\delta_1+\widehat{\beta}_{22}\widehat{\Delta}^{-1}w_1p_1^{-1} & -\widehat{\beta}_{21}\widehat{\Delta}^{-1}w_1p_2^{-1}\\ -\widehat{\beta}_{12}\widehat{\Delta}^{-1}w_2p_1^{-1} & 1-\delta_2+\widehat{\beta}_{11}\widehat{\Delta}^{-1}w_2p_2^{-1}\end{bmatrix}^{-1}.\end{aligned}$$

Utilizing (29), $\sum_{i=1}^{2}\widehat{\beta}_{ij} = 1$, and the definition of s_i, one can then obtain:

$$\begin{aligned}\widehat{\lambda}_2 &= \frac{\left(1+g^*\right)^{2\sigma}}{\rho^2}\left[\left(1-\delta_1+\frac{\widehat{\beta}_{22}s_1}{\rho\widehat{\Delta}}\right)\left(1-\delta_2+\frac{\widehat{\beta}_{11}s_2}{\rho\widehat{\Delta}}\right)-\frac{\widehat{\beta}_{21}s_1}{\rho\widehat{\Delta}}\frac{\widehat{\beta}_{12}s_2}{\rho\widehat{\Delta}}\right]^{-1}\\ &= \left(1+g^*\right)^{2\sigma}\widehat{\Delta}\left\{\left(1-\delta_1\right)\left(1-\delta_2\right)\rho^2\widehat{\Delta}+s_1s_2+\rho\left[\widehat{\beta}_{22}\left(1-\delta_2\right)s_1\right.\right.\\ &\quad\left.\left.+\ \widehat{\beta}_{11}\left(1-\delta_1\right)s_2\right]\right\}^{-1}\\ &= \left(1+g^*\right)^{\sigma}\widehat{\Delta}\left\{\left(1+g^*\right)^{\sigma}-\rho\left[\left(1-\delta_1\right)\widehat{\beta}_{21}+\left(1-\delta_2\right)\widehat{\beta}_{12}\right]\right\}^{-1}.\end{aligned}$$

Moreover, we have:

$$\begin{aligned}\frac{1}{\widehat{\lambda}_2}-1 &= \frac{1}{\left(1+g^*\right)^{\sigma}\widehat{\Delta}}\left\{\left(1+g^*\right)^{\sigma}\left[1-\left(\widehat{\beta}_{11}\widehat{\beta}_{22}-\widehat{\beta}_{12}\widehat{\beta}_{21}\right)\right]-\rho\left[\left(1-\delta_1\right)\widehat{\beta}_{21}\right.\right.\\ &\quad\left.\left.+\left(1-\delta_2\right)\widehat{\beta}_{12}\right]\right\}\\ &= \frac{1}{\left(1+g^*\right)^{\sigma}\widehat{\Delta}}\left\{\left(1+g^*\right)^{\sigma}\left(\widehat{\beta}_{12}+\widehat{\beta}_{21}\right)-\rho\left[\left(1-\delta_1\right)\widehat{\beta}_{21}+\left(1-\delta_2\right)\widehat{\beta}_{12}\right]\right\}\\ &= \frac{1}{\left(1+g^*\right)^{\sigma}\widehat{\Delta}}\left(\widehat{\beta}_{21}s_1+\widehat{\beta}_{12}s_2\right),\end{aligned}$$

and,

$$\begin{aligned}\frac{1}{\widehat{\lambda}_2}+1 &= \frac{1}{\left(1+g^*\right)^{\sigma}\widehat{\Delta}}\left\{\left(1+g^*\right)^{\sigma}\left[1+\left(\widehat{\beta}_{11}\widehat{\beta}_{22}-\widehat{\beta}_{12}\widehat{\beta}_{21}\right)\right]\right.\\ &\quad\left.-\rho\left[\left(1-\delta_1\right)\widehat{\beta}_{21}+\left(1-\delta_2\right)\widehat{\beta}_{12}\right]\right\}\\ &= \frac{1}{\left(1+g^*\right)^{\sigma}\widehat{\Delta}}\left\{\left(1+g^*\right)^{\sigma}\left(\widehat{\beta}_{11}+\widehat{\beta}_{22}\right)-\rho\left[\left(1-\delta_1\right)\widehat{\beta}_{21}\right.\right.\\ &\quad\left.\left.+\ \left(1-\delta_2\right)\widehat{\beta}_{12}\right]\right\}.\end{aligned}$$

Hence, $sign(\frac{1}{\widehat{\lambda}_2}-1) = sign(\widehat{\Delta})$, but $sign(\frac{1}{\widehat{\lambda}_2}+1)$ depends both on $sign(\widehat{\Delta})$ and the magnitude of g^*, ρ and δ_i.

For $\widehat{\Delta} > 0$, $\widehat{\beta}_{12} + \widehat{\beta}_{12} < 1$. Thus, by Lemma 1,

$$\begin{aligned}\widehat{\lambda}_2 &\propto \left(1+g^*\right)^{\sigma} - \rho\left((1-\delta_1)\widehat{\beta}_{21} + (1-\delta_2)\widehat{\beta}_{12}\right)\\ &> \left(1+g^*\right)^{\sigma} - \rho\max\left\{(1-\delta_1),(1-\delta_2)\right\}\left(\widehat{\beta}_{21}+\widehat{\beta}_{12}\right)\\ &> \left(1+g^*\right)^{\sigma} - \rho\max\left\{(1-\delta_1),(1-\delta_2)\right\}\\ &= \min\{s_1, s_2\} > 0.\end{aligned}$$

Also, in this case, $\widehat{\lambda}_2 < 1$ and we establish: $0 < \widehat{\lambda}_2 < 1 = \widehat{\lambda}_1$ under $\widehat{\Delta} > 0$.

For $\widehat{\Delta} < 0$ and hence $\widehat{\beta}_{11} + \widehat{\beta}_{22} < 1$, we have to examine three possible cases. First, consider the case where Condition P holds. In this case, one can easily see that $\widehat{\lambda}_2 < -1$ and hence the price dynamics is unstable. Next, consider $(1+g^*)^{\sigma} < \rho\left[(1-\delta_1)\widehat{\beta}_{21} + (1-\delta_2)\widehat{\beta}_{12}\right]$ under which Condition P fails and $0 < \widehat{\lambda}_2 < 1$. This implies that price adjustments are stable and monotonically converge to the BGP. Finally, $(1+g^*)^{\sigma} > \frac{\rho\left[(1-\delta_1)\widehat{\beta}_{21}+(1-\delta_2)\widehat{\beta}_{12}\right]}{\widehat{\beta}_{11}+\widehat{\beta}_{22}}$ under which Condition P also fails, but now $-1 < \widehat{\lambda}_2 < 0$ and hence price adjustments are stable and oscillate in converging to the BGP. □

Proof of Lemma 7 Recall that the quantity dynamics are governed by (35). Define,

$$F(\lambda) = \det\left[\left(1+g^*\right)^{-1}\left(A^{-1} + I - I_{\delta}\right) - \lambda I\right].$$

First, consider the case of $\Delta > 0$. We evaluate:

$$F(1) = \left(1+g^*\right)^{-2}\det\left[A^{-1} - \left(g^* I + I_{\delta}\right)\right].$$

Using the transformed version of (5) along the BGP, $a_{ij} = \beta_{ij} p_j w_{it}^{-1}$, and (29), we can derive:

$$\begin{aligned}F'(1) &= -\left(1+g^*\right)^{-1}\left[(a_{11}+a_{22})|A|^{-1} - \left(2g^* + \delta_1 + \delta_2\right)\right]\\ &= -\left(1+g^*\right)^{-1}\left[\left(\beta_{11}p_2^{-1}w_2 + \beta_{22}p_1^{-1}w_1\right)\Delta^{-1} - \left(2g^* + \delta_1 + \delta_2\right)\right]\\ &= -\left(1+g^*\right)^{-1}\rho^{-1}\left[(s_2\beta_{11} + s_1\beta_{22})\Delta^{-1} - \rho\left(2g^* + \delta_1 + \delta_2\right)\right].\end{aligned}$$

Yet, from (25), $\left[A^{-1} - (g^* I + I_{\delta})\right]x = z$. Since $\Delta > 0$, $\left[A^{-1} - (g^* I + I_{\delta})\right]$ has negative off-diagonal elements and positive diagonal elements, implying that the matrix, $\left[A^{-1} - (g^* I + I_{\delta})\right]$, has a dominant diagonal with every principle minor being positive. This results in $F(1) > 0$ and $F'(1) = -(1+g^*)^{-1}[(a_{11} + a_{22})|A|^{-1} - (2g^* + \delta_1 + \delta_2)] < 0$. Let $d_1 + d_2 i$ be the roots of $F(\lambda) = 0$, i.e., $F(\lambda) = (\lambda - d_1 - id_2)(\lambda - d_1 + id_2) = 0$ (where $d_2 = 0$ means the roots are real). Then,

$$F'(1) = 2(1 - d_1) < 0.$$

Thus, the real part of two roots, d_1, are greater than 1 and $F(\lambda) = 0$ has two roots $\{\lambda_1, \lambda_2\}$, both outside the unit circle, $|\lambda_1| > 1$ and $|\lambda_2| > 1$.

Next, consider the case of $\Delta < 0$. Since $\det\left[I - A\left(g^* I + I_{\delta}\right)\right] > 0$ and

$$\det\left[1 - A\left(g^* I + I_{\delta}\right)\right] = |A|\det\left[A^{-1} - \left(g^* I + I_{\delta}\right)\right],$$

$\Delta < 0$ implies $|A| < 0$ and hence $\det\left[A^{-1} - (g^* I + I_\delta)\right] < 0$ must hold true. This gives,

$$F(1) = \left(1 + g^*\right)^{-2} \det\left[A^{-1} - \left(g^* I + I_\delta\right)\right] < 0.$$

Also $(1 + g^*)^{-1}\left[(a_{11} + a_{22})\,|A|^{-1} - (2g + \delta_1 + \delta_2)\right] < 0$ implies $F'(1) > 0$. Thus, $\lambda_1 > 1$ and $\lambda_2 < 1$. To see if λ_2 is larger or less than -1, we evaluate $F(\lambda)$ at $\lambda = -1$:

$$F(-1) = \left(1 + g^*\right)^{-2} \det\left[A^{-1} + I - I_\delta + \left(1 + g^*\right) I\right].$$

We rewrite $\det\left[A^{-1} + I - I_\delta + (1 + g^*) I\right]$ as follows:

$$\left(2 + g^* - \delta_1\right)\left(2 + g^* - \delta_2\right) + \frac{1}{\rho\Delta}\left\{\left[\beta_{11}\left(2 + g^* - \delta_1\right) s_2 + \beta_{22}\left(2 + g^* - \delta_2\right) s_1\right] + \frac{s_1 s_2}{\rho}\right\}.$$

Let assume that $\delta_1 = \delta_2 = \delta$. Then $s_1 = s_2 = s$ and the above expression becomes

$$\left(2 + g^* - \delta\right)^2 + \frac{1}{\rho\Delta}\left[(\beta_{11} + \beta_{22})\left(2 + g^* - \delta\right) s + \frac{s^2}{\rho}\right].$$

$F(-1) > 0$ for some $s > 0$ only if the following holds for some $s > 0$:

$$G(s) = s^2 + \rho\,(\beta_{11} + \beta_{22})\left(2 + g^* - \delta\right) s + \rho^2\left(2 + g^* - \delta\right)^2 \Delta < 0.$$

Note that

$$G(0) = \rho^2\left(2 + g^* - \delta\right)^2 \Delta < 0.$$

There are two cases to consider. First, if $\sigma > 1$, then the following inequality holds as long as $g^* > 0$.

$$\left(1 + g^*\right)^\sigma > \rho\left(1 + g^*\right).$$

In order for $g^* > 0$, s must satisfy

$$s > 1 - \rho\,(1 - \delta).$$

Hence, if $\rho\,(1 - \delta)$ is close to 1, then there exists some $s > 1 - \rho\,(1 - \delta)$ with $G(s) < 0$. Second, if $0 < \sigma < 1$, the transversality condition $(1 + g^*)^\sigma > \rho\,(1 + g^*)$ is equivalent to $s > \rho\,(g^* + \delta)$. If g^* and δ are both close to 0, then there exists some $s > \rho\,(g^* + \delta)$ such that $G(s) < 0$, where g^* can be arbitrarily small as ρ is close to 1, from Assumption 2 and the definition $\Gamma(g)$. By continuity, the same arguments apply for the case with non-zero but small $|\delta_1 - \delta_2|$, which completes the proof. □

Proof of Proposition 2 Since $\widehat{\Delta} > 0$, Lemma 6 implies: $0 < \widehat{\lambda}_2 < 1 = \widehat{\lambda}_1$. Moreover, under $\Delta > 0$, Lemma 7 implies $|\lambda_1| > 1$ and $|\lambda_2| > 1$. The dynamical system therefore has one root equal to 1, one stable root and two unstable root, implying local determinacy. □

Proof of Proposition 3 Since $\widehat{\Delta} < 0$, Lemma 6 implies $\widehat{\lambda}_1 = 1$ and $\widehat{\lambda}_2 < -1$ when Condition P holds. Under $\Delta < 0$, Lemma 7 implies that there exists $0 < \rho_{\min} < 1$, $0 < \delta_{\max} \leq 1$ and $0 < \overline{\delta} < 1$ such that for any $\rho \in (\rho_{\min}, 1)$, $\delta_i \in (0, \delta_{\max})$, and $|\delta_1 - \delta_2| < \overline{\delta}$, we have: $-1 < \lambda_2 < 1 < \lambda_1$. Thus, the dynamical system has one root equal to 1, one stable root and two unstable root, implying local determinacy. □

Proof of Proposition 4 From Lemma 6, the roots of the characteristic equation governing the price dynamics are $\widehat{\lambda}_1 = 1$ and $0 < \widehat{\lambda}_2 < 1$. By Lemma 7, there exists $0 < \rho_{\min} < 1$, $0 < \delta_{\max} \leq 1$ and $0 < \overline{\delta} < 1$ such that for any $\rho \in (\rho_{\min}, 1)$, $\delta_i \in (0, \delta_{\max})$, and $|\delta_1 - \delta_2| < \overline{\delta}$, we have: $-1 < \lambda_2 < 1 < \lambda_1$. The dynamical system therefore has one root equal to 1, two stable root and one unstable root, implying one-dimensional dynamic indeterminacy. □

Proof of Proposition 5 From Lemma 6, the roots of the characteristic equation governing the price dynamics are $\widehat{\lambda}_1 = 1$ and $\left|\widehat{\lambda}_2\right| < 1$. Using Lemma 7, there exists $0 < \rho_{\min} < 1$, $0 < \delta_{\max} \leq 1$ and $0 < \overline{\delta} < 1$ such that for any $\rho \in (\rho_{\min}, 1)$, $\delta_i \in (0, \delta_{\max})$, and $|\delta_1 - \delta_2| < \overline{\delta}$, one obtain: $-1 < \lambda_2 < 1 < \lambda_1$. The dynamical system therefore has one root equal to 1, two stable root and one unstable root, implying one-dimensional dynamic indeterminacy. □

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