Housing Dynamics: Theory Behind Empirics

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Abstract: We construct a two-sector optimal growth model of housing where housing is produced by both land and housing structure/household durables. We explicitly model locational choice. Housing services derive positive utility but are decayed away from the city center. Our model enables a full characterization of the dynamic paths of housing as well as housing and land prices. The model is particularly designed to be calibrated to fit some important stylized facts, including faster growth of housing structure/household durables than housing, faster growth of land prices than housing prices, and a locationally steeper land rent gradient than the housing price gradient. The calibrated model is then used to quantitatively assess the long-run and short-run consequences of changes in preferences and technologies.

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1 Introduction

The housing sector is very significant in size. While the value of the American housing stock accounts for more than 30% of national wealth, the housing-related expenditure is about one-fourth of the total household spending. Yet, not until the turn of the century, the housing sector has largely been ignored by macroeconomists.[1] Given its significant size, a thorough study of the housing sector in a dynamic general equilibrium framework is valuable because housing activity can generate large macroeconomic effects and important policy implications.[2]

Housing is not just a type of capital or a form of durable goods; rather, it has some distinct features. Because a house is tied to a plot of land at a specific location usually close to the occupant's workplace, it is locationally immobile and the consumption set of housing is nonconvex.[3] In a recent insightful study, Davis and Heathcote (2007) find that properly decomposing a house into housing structure and land enables better understanding of the time series movements and cross location variation in housing prices. In the present paper, we construct a dynamic general equilibrium model of housing, explicitly modeling the evolution of housing structures and the separate role played by land and explicitly taking the households' locational choice decision into account.

What are the stylized facts of our specific interest? Based on the U.S. observations, we selectively highlight just three:

- Measured by housing structures plus household durables, the housing durable out-grows housing.
- Housing prices grow at much lower rates than land rents.
- By putting aside urban ghettos, both housing price and land rent gradients are downward-sloping away from urban centers (or subcenters), though the land rent gradient is much steeper.

Our dynamic general equilibrium model of housing is particularly designed to be calibrated to fit all these facts.

Specifically, we construct a two-sector optimal growth model with a composite final good sector and a housing sector. The composite final good can be used for consumption or for capital in-

[1] See Leung (2004) for a critical survey, documenting clearly such an ignorance in the literature.

[2] As an example, Case, Quigley and Shiller (2005) find rather large effects of housing wealth on household consumption using a panel of 14 developed countries over the period of 1975-1999 and a panel of U.S. states over the period of 1982-1999.

[3] While a house in San Francisco and a house in New York are both in the consumption set, a convex combination of a fraction of a house in San Francisco and a fraction of a house in New York is not.

vestment. In addition to composite good consumption, housing services also derive positive utility. While housing is more luxurious than the composite good consumption, its services are decayed away from the city center. Housing is produced by land and housing structures/durables, in which there a minimum structure required for a house. Both housing structures/durables and the composite good are produced with the use of physical capital. In equilibrium, both goods and land market clear (no vacant land) and no household has incentive to relocate (locational no-arbitrage). We begin by solving the social planner's problem and then decentralize it using location-dependent redistributions (taxes and subsidies). Upon obtaining the steady-state competitive spatial equilibrium, we derive a basket of analytical comparative statics and then calibrate the theoretical model to fit the average U.S. data over 1960-2000 to further quantify our analysis.

The main analytic findings of our paper are summarized as follows. First, an increase in the housing production technology or in the supply of land raises housing quantity but reduces the relative price of housing. Second, under the assumption that housing is more luxury than the composite consumption good, an increase in the consumption good production technology lowers the cost of producing the consumption good and enables reallocation of resources to housing production, thus raising both the quantity and the relative price of housing. Third, when agents become more impatient, housing, which requires continual inflows to maintain its adequate service, falls in response, though its net effect on housing price is ambiguous due to two conflicting forces – while a reduction in housing production tends to raise housing price, the endogenous response of a higher real interest rate tends to lower housing price.

By calibration analysis, we reach several noteworthy conclusions. First, our model fits well the three stylized facts highlighted above, namely, faster growth of housing structure/household durables than housing, faster growth of land prices than housing prices, and a locationally steeper land rent gradient than the housing price gradient. Second, housing exhibits much higher cross-location variations than consumption and housing durable schedules. Third, along a dynamic path with accumulation of capital and housing durables, the prices of housing durables exhibit a slight downward trend over time, corroborating with findings in the home production literature without spatial considerations. Fourth, a reduction in the luxury nature of housing or in locational discounting, or an advancement in the housing durable technology, raises the out-skirt to inner city ratios of the quantity of housing and housing durables, the prices of land, housing and housing durables, as well as the housing expenditure and the housing capital shares. By contrast, changes in the minimum housing structure requirement has little influence on the cross-location variations in housing, housing durables and their prices. Finally, to support an optimal city, the redistribution scheme in our benchmark model features taxing on those residing in the inner city and subsidizing those residing in outskirts.

In Section 2, we describe the spatial setup and dynamic evolution of the two-sector optimal growth model. Section 3 solves the optimization problem and derives the steady-state equilibrium. We then calibrate the theoretical model to fit the post-WWII U.S. in Section 4, followed by a numerical characterization of the transitional dynamics of the model economy in Section 5. In Section 6, we provide a sensitivity analysis for alternative parametrization and alternative model specification. Section 7 concludes the paper and provides possible avenues for future research.

Related Literature

There are two streams of conventional research: one is the durable housing literature in regional science and urban economics and another is the microfinance literature. Because these studies do not focus on macroeconomic issues, we would not discuss the details but simply refer the reader to the survey by Leung (2004).

More recently, there is a small but growing literature of housing that is macro-based. Kan, Kwong and Leung (2004) study the upward trend of residential and commercial property prices and the relative volatility. Davis and Heathcote (2005) examine the movements in housing construction and other related macro aggregates over the business cycle. Ortalo-Magné and Rady (2006) model the trade-up of houses over a household's life cycle facing borrowing constraints. Bajari, Chan, Krueger and Miller (2008) and Flavin and Nakagawa (2008) study housing demand and asset portfolio in a world with income or asset return uncertainty. While Davis and Martin (2008) investigate whether the home production model of housing can explain equity or value premium puzzles, Davis and Ortalo-Magné (2008) examine cross-MSA variation in housing rentals and household wages.

In these papers, housing is introduced with its service entering the utility function either directly (cf. Leung 2001; Kan, Kwong and Leung 2004; Davis and Heathcote 2005; Ortalo-Magné and Rady 2006; Bajari, Chan, Krueger and Miller 2008; Davis and Ortalo-Magné 2008; Flavin and Nakagawa 2008) or indirectly via a consumption aggregator and home production (cf. Davis and Martin 2008). In most studies, housing is produced by capital or labor or a combination of the two. The only exceptions are Leung (2001) and Davis and Heathcote (2005) in which land is considered as an input of new house production. None of these models accounts for the location-specific feature of land, thereby ignoring endogenous locational choice.[4] Moreover, a complete characterization of

[4]Berliant, Peng and Wang (2002), Lucas and Rossi-Hansberg (2002), Xie (2008) and Lin, Mai and Wang (2004) allow for endogenous locational choice. However, the first three papers are static, whereas the last paper only considers a unified household capital without separating residential and nonresidential uses. Moreover, all of these studies focus

the dynamic paths of housing as well as housing and land prices remain unexplored. Our paper attempts to fill these important gaps.

2 The Model

Let the city (or MSA) be situated in a segment of real line, $[-1, 1]$, with location 0 representing the central business district (CBD). Let the land supply be distributed along the real line according to an exogenous density function $T(z)$, for $z \in [-1, 1]$, where z indexes a location. We assume $T'(z) > 0$ to capture the fact that land is more abundant away from the city center. Moreover, we assume that the land supply at $z = 0$ is positive ($T(0) > 0$).

For convenience, the population of agents is assumed constant over time with mass two. Further assume that each agent supplies labor inelastically at $\frac{1}{2}$. Thus, the aggregate labor supply in the economy is one. We will focus on a symmetric equilibrium in which locational choice yields a uniform distribution of households over $[-1, 1]$.[5]

Our spatial economy has two theaters of production activities: one produces a composite final good and another accumulates housing durables. Production of both of these mobile goods take place at the CBD to which workers commute.

2.1 The Housing Sector

Housing at location z is specified as:

$$H_z = T_z^{\gamma} \left(D_z - \theta\right)^{1-\gamma} \tag{1}$$

where T_z is the use of land, D_z is the housing structure and household durable component of the house, and θ is introduced to capture the idea that a minimum structure is needed to be called a house. Notice that the minimum requirement for land is far less important and hence set to zero – think of a high rise in the central city where land per house can be very small. Hence, for simplicity, we will not impose one. The Cobb-Douglas form ensures that land and housing structures/durables are Pareto complement in the sense that an increase in one input raises the marginal product of another.

on the issues of urban land use and internal structure of cities, which are very different from ours.

[5] As illustrated by Berliant, Peng and Wang (2002), this symmetry or uniformity assumption may be viewed as an equilibrium selection. In our paper, this equilibrium selection simplifies our analysis greatly.

The output of housing durable investment at location z is produced with the use of physical capital:

$$X_z = BK_z^{\beta}$$

where $\frac{\dot{B}}{B} = G(t)$ with $G(t) \geq g$, $G' \leq 0$ and $\lim_{t\to\infty} G(t) = g$ for any z. In the steady-state equilibrium, we will set $G(t) = g = 0$. Thus, the stock of housing durables evloves according to,

$$\dot{D}_z = X_z - \delta D_z = BK_z^{\beta} - \delta D_z \tag{2}$$

where δ denotes the demolishment rate of housing structure/household durables and $D_z(0) = d \geq \theta$ for any z.

2.2 The Composite Final Good Sector

The composite final goods sector features the following Cobb-Douglas production function:

$$Y = AK_c^{\alpha} L^{1-\alpha} \tag{3}$$

where labor, L, is inelastically supplied at one and $\frac{\dot{A}}{A} = g$. In the steady-state equilibrium, we will set $g = 0$.

This output can then be used for consumption or capital investment (assuming zero depreciation for the sake of simplicity), implying:

$$\dot{K} = AK_c^{\alpha} L^{1-\alpha} - \int_{-1}^{1} c_z dz \tag{4}$$

The total stock of capital, K, can be allocated as follows:

$$K = K_c + \int_{-1}^{1} K_z dz \tag{5}$$

where K is equally owned by all the agents.

2.3 Preferences

The lifetime utility function of an individual residing at location z is specified as:

$$\int_0^{\infty} u(c_z, \phi(z)H_z)e^{-\rho t} dt \tag{6}$$

where $\rho > 0$ is the subjective rate of time preference and $\phi(z)$ is a spatial discounting function capturing the idea that the further away the house is from the CBD, the lower the utility one derives from the house. Part of the reduction in utility may be thought of capturing the detrimental effect from commuting. With spatial discounting, it is not necessary to consider a separate resource cost of commuting, which we assume. Without loss of generality, we normalize $\phi(0) = 1$.

The point-in-time utility function takes the following form:

$$u(c_z, \phi(z)H_z) = c_z^{\sigma} \left(\phi(z)H_z + \eta\right)^{1-\sigma}, \ \sigma \in (0,1) \tag{7}$$

where $\eta > 0$ captures the luxury good nature of housing relative to the composite consumption good. Moreover, the Cobb-Douglas form ensures that composite good consumption and housing service are Pareto complement.[6]

2.4 Locational Choice

Given the ex ante symmetry between all agents, it has to be the case that in equilibrium, $u(c_z, \phi(z)H_z)$ is independent of z. In other words, the following locational no-arbitrage condition holds:

$$u(c_z, \phi(z)H_z) = u(c_0, H_0) \tag{8}$$

Thus, in equilibrium, individual agents feel indifferent in residing in any location.

3 Equilibrium Analysis

In this section, we solve the optimization problem and then derive the steady-state equilibrium. In the absence of distortions, we can focus on solving the central planner's problem instead of solving the competitive equilibrium directly. We will identify a necessary redistribution scheme later to support the decentralization of the optimal allocation obtained from the central planner's problem.

3.1 Optimization

For convenience, we define:

$$\Psi_z\left(D_0, D_z\right) \equiv \frac{T_0^{\gamma}\left(D_0-\theta\right)^{1-\gamma}+\eta}{\phi(z)T_z^{\gamma}\left(D_z-\theta\right)^{1-\gamma}+\eta}$$

which is increasing in D_0 but decreasing in D_z, satisfying $\Psi_0\left(D_0, D_0\right) = 1$. We can then simplify the central planner's problem by utilizing (7) and (8) to express the locational no-arbitrage condition in forms of final good consumption:

$$c_z = c_0 \Psi_z\left(D_0, D_z\right)^{\frac{1-\sigma}{\sigma}} \tag{9}$$

[6] An alternative to allow housing services to be unnecessary is to use the constant elasticity of substitution form with the elasticity less than one. However, this implies that composite good consumption and housing are Pareto substitutes, which is unrealistic.

That is, Ψ_z governs relative composite good consumption across locations.

Using (9), we can write the central planner's problem as:

$$\max \int_0^\infty c_0^\sigma \left(T_0^\gamma (D_0-\theta)^{1-\gamma}+\eta\right)^{1-\sigma} e^{-\rho t} dt$$

subject to

$$\dot{K} = A\left(K-\int_{-1}^{1} K_z dz\right)^\alpha L^{1-\alpha} - \int_{-1}^{1} c_0 \Psi_z (D_0, D_z)^{\frac{1-\sigma}{\sigma}} dz \tag{10}$$

$$\dot{D}_z = BK_z^\beta - \delta D_z \text{ for all } z \tag{11}$$

This optimization problem can be solved by setting the current-value Hamiltonian,

$$\begin{aligned}\mathcal{H} \;=\; & \max_{c_0, K_z} c_0^\sigma \left(T_0^\gamma (D_0-\theta)^{1-\gamma}\right)^{1-\sigma} \\ & +\lambda\left[A\left(K-\int_{-1}^{1} K_z dz\right)^\alpha L^{1-\alpha} - \int_{-1}^{1} c_0 \Psi_z (D_0, D_z)^{\frac{1-\sigma}{\sigma}} dz\right] \\ & +\int_{-1}^{1} \mu_z \left[BK_z^\beta - \delta D_z\right] dz\end{aligned}$$

where λ and μ_z are co-state variables.

We next define:

$$\Gamma = \int_{-1}^{1} \Psi_z (D_0, D_z)^{\frac{1-\sigma}{\sigma}} dz \tag{12}$$

which is indeed the endogenous social welfare weight on those residing at location 0.[7] The first-order conditions with respect to c_0 and K_z are:

$$\sigma c_0^{\sigma-1} \left(T_0^\gamma (D_0-\theta)^{1-\gamma}+\eta\right)^{1-\sigma} = \lambda\Gamma \tag{13}$$

$$\beta \mu_z B K_z^{\beta-1} = \alpha\lambda A \left(K-\int_{-1}^{1} K_z dz\right)^{\alpha-1} L^{1-\alpha} \tag{14}$$

While (13) equates the marginal benefit from raising location-0 resident's consumption and the marginal cost from reducing others' consumption, (14) equates the value of marginal product of capital between the two sectors. From (14), we have:

$$K_z = \left(\frac{\mu_z}{\mu_0}\right)^{1/(1-\beta)} K_0 \tag{15}$$

That is, the ratio of capital allocated to the housing sector between two locations depends positively on the ratio of the shadow value of housing durables.

[7] This can be easily verified by maximizing the social welfare function given by $\int_{-1}^{1} \Omega_z u(c_z, \phi(z) H_z) dz$, subject to (2) and (4). Applying Negishi (1960), we can compute the social welfare weights consistent with the decentralized equilibrium allocation, yielding: $\Omega_0 = \Gamma$.

The Euler equations with respect to K and D_z are given by,

$$\begin{aligned}\dot{\lambda} &= \rho\lambda - \alpha\lambda A\left(K - \int_{-1}^{1} K_z dz\right)^{\alpha-1} L^{1-\alpha} \\ \dot{\mu}_z &= (\rho+\delta)\mu_z - \lambda\left[(1-\gamma)\frac{1-\sigma}{\sigma}c_0\Pi_z\left(D_z\right)\Psi_z\left(D_0, D_z\right)^{\frac{1-\sigma}{\sigma}}\right]\end{aligned}$$

where $\Pi_z\left(D_z\right) \equiv \frac{1}{D_z-\theta}\frac{\phi(z)T_z^{\gamma}(D_z-\theta)^{1-\gamma}}{\phi(z)T_z^{\gamma}(D_z-\theta)^{1-\gamma}+\eta}$ is decreasing in D_z. By rewriting these above expressions using the first-order conditions, (13) and (14), we obtain:

$$\frac{\dot{\lambda}}{\lambda} = \rho - \alpha A\left(K - \int_{-1}^{1} K_z dz\right)^{\alpha-1} L^{1-\alpha} \tag{16}$$

$$\frac{\dot{\mu}_z}{\mu_z} = (\rho+\delta) - \frac{\beta B K_z^{\beta-1}}{\alpha A\left(K - \int_{-1}^{1} K_z dz\right)^{\alpha-1} L^{1-\alpha}}(1-\gamma)\frac{1-\sigma}{\sigma}c_0\Pi_z\left(D_z\right)\Psi_z\left(D_0, D_z\right)^{\frac{1-\sigma}{\sigma}} \tag{17}$$

The (shadow) relative price of housing can then be defined as $P_{D_z} = \frac{\mu_z}{\lambda}$. The above two expressions thereby lead to an intertemporal no-arbitrage condition:

$$\frac{\dot{P}_{D_z}}{P_{D_z}} = \alpha A\left(K - \int_{-1}^{1} K_z dz\right)^{\alpha-1} L^{1-\alpha} - \left[\frac{\beta B K_z^{\beta-1}(1-\gamma)\frac{1-\sigma}{\sigma}c_0\Pi_z\left(D_z\right)\Psi_z\left(D_0, D_z\right)^{\frac{1-\sigma}{\sigma}}}{\alpha A\left(K - \int_{-1}^{1} K_z dz\right)^{\alpha-1} L^{1-\alpha}} - \delta\right] \tag{18}$$

That is, if the net return on capital (first term on the right hand side) exceeds the net return on housing durables, the latter must have a capital gain ($\frac{\dot{P}_{D_z}}{P_{D_z}} > 0$) in order for both sectors to maintain operative (see Bond, Wang and Yip 1996).

3.2 Steady-State Equilibrium

We now examine the steady-state equilibrium in a baseline setting with $G(t) = g = 0$ (to be relaxed when we conduct calibration analysis). Again, we focus on a symmetric equilibrium uniformly distributed households for all z.

From (16) as well as (10) and (11), we obtain the following three steady-state relationships:

$$K_c = K - \int_{-1}^{1} K_z dz = \left(\frac{\alpha A}{\rho}\right)^{\frac{1}{1-\alpha}} \tag{19}$$

$$c_0 = \frac{1}{\Gamma}A\left(K - \int_{-1}^{1} K_z dz\right)^{\alpha} = \frac{A\left(\frac{\alpha A}{\rho}\right)^{\frac{\alpha}{1-\alpha}}}{\Gamma} \tag{20}$$

$$K_z = \left(\frac{\delta D_z}{B}\right)^{\frac{1}{\beta}} \tag{21}$$

Clearly, a higher composite good technology or a lower time preference rate raises consumption as well as capital allocated to the composite good sector. Moreover, a higher demolishment rate requires more capital to be allocated to the housing sector to maintain the need for housing services.

Substituting (20) into (9), we thus have:

$$c_z = \frac{A\left(\frac{\alpha A}{\rho}\right)^{\frac{\alpha}{1-\alpha}}}{\Gamma}\Psi_z\left(D_0, D_z\right)^{\frac{1-\sigma}{\sigma}} \tag{22}$$

The above equations can then be combined with (17) to yield,

$$\frac{\beta B}{\rho}\left(\frac{\delta D_z}{B}\right)^{\frac{\beta-1}{\beta}}\left[(1-\gamma)\frac{1-\sigma}{\sigma}\frac{A\left(\frac{\alpha A}{\rho}\right)^{\frac{\alpha}{1-\alpha}}}{\Gamma}\Pi_z\left(D_z\right)\Psi_z\left(D_0, D_z\right)^{\frac{1-\sigma}{\sigma}}\right] = \rho + \delta \tag{23}$$

Notice that, at $z = 0$, (23) reduces to an expression for solving uniquely $D_0(\Gamma)$ which turns out to be a decreasing function. This can then be substituted into (23) to derive all housing durables $D_z(\Gamma)$, which are all decreasing in Γ as well. Next, substituting $D_z(\Gamma)$ into the definition (12) yields a fixed point mapping in Γ. Once the fixed point of Γ is obtained, it can then be plugged into $D_z(\Gamma)$ to solve for D_z for all z, and then into (19), (20), (21) and (22) to solve for K, c_0, K_z and c_z. Finally, using (1) and (3), we obtain the steady-state value of housing and the composite good output, H and Y.

The involvement of Γ in all the location-specific variables makes the steady-state equilibrium too complicated to be characterized analytically. In particular, all the preference and technology parameters of interest, $(A, B, \eta, \rho, \theta, T)$, will affect the fixed point of Γ ambiguously due to their opposing effects on $\Psi_z\left(D_0, D_z\right)$ via $D_0(\Gamma)$ and $D_z(\Gamma)$. Thus, we will instead perform comparative-static exercises only under the baseline one-location setup, while conducting the equilibrium characterization of the general model only numerically.

3.3 Characterization of the Steady-State Equilibrium

In order to perform comparative statics with respect to $(A, B, \eta, \rho, \theta, T)$, we utilize the "hat calculus" that has been frequently adopted by general equilibrium trade theorists. Denoting $\hat{X} = \frac{\dot{X}}{X}$, we can totally differentiate the key relationships in the baseline one-location setup and manipulate the expressions to derive the fundamental equation governing the changes in the housing quantity in response to changes in $(A, B, \eta, \rho, \theta)$ (see Appendix A):

$$\begin{aligned} \hat{H} \quad = \quad & \frac{1}{\Upsilon}\left\{\frac{1}{1-\alpha}\hat{A} + \frac{1}{\beta}\hat{B} - \frac{\eta}{H+\eta}\hat{\eta} - \left(\frac{1}{1-\alpha} + \frac{\rho}{\rho+\delta}\right)\hat{\rho}\right. \\ & \left. - \frac{1-\beta}{\beta}\frac{\theta}{D}\hat{\theta} + \frac{\gamma}{1-\gamma}\left(\frac{1-\beta}{\beta} + \frac{D}{D-\theta}\right)\frac{D-\theta}{D}\hat{T}\right\} \end{aligned} \tag{24}$$

where $\Upsilon \equiv \frac{1}{1-\gamma}\left(1+\frac{1-\beta}{\beta}\frac{D-\theta}{D}\right)-\frac{\eta}{H+\eta}>0$.

Similarly, we can define the (shadow) price of housing as:

$$P_H = \frac{R_H}{\rho} = \frac{1}{\rho}\frac{1-\sigma}{\sigma}\frac{c}{H+\eta} \tag{25}$$

We can then obtain the fundamental equation governing the changes in the housing price in response to changes in $(A, B, \eta, \rho, \theta, T)$ (see also Appendix A):

$$\begin{aligned}\hat{P}_H \;\; &= \;\; \frac{\Lambda}{1-\alpha}\hat{A} - \frac{1}{\beta}\frac{H}{H+\eta}\frac{1}{\Upsilon}\hat{B} - \frac{\eta\Lambda}{H+\eta}\hat{\eta} + \left[\frac{\rho}{\rho+\delta}\frac{H}{H+\eta}\frac{1}{\Upsilon} - \frac{1}{1-\alpha}\left(1-\frac{H}{H+\eta}\frac{1}{\Upsilon}\right)\right]\hat{\rho} \\ &\quad + \frac{1-\beta}{\beta}\frac{H}{H+\eta}\frac{1}{\Upsilon}\frac{\theta}{D}\hat{\theta} - \frac{\gamma}{1-\gamma}\frac{H}{H+\eta}\frac{1}{\Upsilon}\left(\frac{1-\beta}{\beta}+\frac{D}{D-\theta}\right)\frac{D-\theta}{D}\hat{T}\end{aligned} \tag{26}$$

where $\Lambda \equiv 1 - \frac{H}{H+\eta}\frac{1}{\Upsilon}$, which is expected to be positive if η is not too large compared to H. Based on these two fundamental equations, we can summarize the comparative static results in the following table:

	A	B	η	ρ	θ	T
Housing Quantity (H)	+	+	−	−	−	+
Housing Price (P_H)	+	−	−	?	+	−

Intuitively, an increase in the housing production technology (B) lowers the cost of producing housing, thus raising housing quantity but reducing housing price. The responses of housing quantity and price to an increase in the supply of land are similar. We next examine what happens to an increase in the minimum structure requirement for housing (higher θ). Since such a requirement raises the cost of producing a house, housing price rise while housing supply decreases in response. In response to an increase in the luxury good nature of housing relative to the consumption good (higher η), individual preferences shift away from housing and as a result both housing quantity and housing price are lower. Notably, while an increase in B or T or a decrease in θ capture a prototypical outward shift in housing supply, a decrease in η indicate a prototypical outward shift in housing demand.

Turning now to time discounting (ρ), we can see that more impatience discourages allocation of resources for the future. Since housing requires continual inflows to maintain its adequate service, it falls in response to an increase in time discounting. While such a reduction in housing production tends to raise housing price, the resulting increase in the real interest rate tends to lower housing price. The net effect of impatience on housing price is therefore ambiguous. Notice that in partial equilibrium setups adopted by conventional housing models, rising time discounting would reduce housing price unambiguously.

Finally, an increase in the consumption good production technology (A), in addition to a positive wealth effect (demand effect), lowers the cost of producing the consumption good and increases the relative price of housing. As a consequence, it enables reallocation of resources to housing production and raises the quantity of housing (supply effect). Such an effect only arises in multi-sectoral setups within the general equilibrium framework.

It is noted that equations (24) and (26) are useful not only for deriving comparative statics but also for numerically decomposing changes in the quantity and the price of housing once we have calibrated the model economy, to which we now turn.

4 Quantitative Analysis

We now calibrate the model to fit with the average U.S. data over 1960-2000. We then use the calibrated model to perform various numerical analyses. Additionally, we check the robustness of our main quantitative findings using a gammaville.

4.1 Calibration

Under our theoretical framework, the total population is two. Denote c as the per capita flow of non-housing related consumption good, D as the per capita stock of housing structure plus household durables (called housing durable), X as the per capita output of the housing durables sector and H as housing per capita (all without the location subscript z). We specify the land supply as a simple quadratic function: $T(z) = (b + q\,|z|)^2$, where b measures the land supply at the CBD and $q > 0$ reflects increasing land supply away from the CBD. We further specify the spatial discounting function in a linear form given by: $\phi(z) = 1 - a\,|z|$, where a measures the locational discount rate. We normalize $b = 1$ so that the amount of land at the CBD is $T(0) = 1$. We then select $a = 0.3$ and $q = 0.3$, under which those at city border discount housing consumption by 30% compared to a resident at the CBD and land supply at city border is 69% more than at the CBD. In computing aggregate variables, the per capita land supply is set as: $T = \int_0^1 (1 + 0.3z)^2 dz = 1.33$.

In the macroeconomics literature, the time preference rate is taken to be between 2% and 5%; we thus set $\rho = 0.035$. Also in compliance with the literature, we choose the capital income share as one-third (implying $\alpha = 1/3$). The overall depreciation of housing structure and household durables considered herein includes both demolishment of housing structure and depreciation of household durables. While Greenwood and Hercowitz (1991) document the household durables depreciation rate as 7.8%, Davis and Heathcote (2005) computes the housing demolishment rate as 1.57%. It is reasonable to assume that the latter accounts for 75% of the overall depreciation, which yields $\delta = 0.0313$.

The calibration analysis is conducted using a simpler version of the model in which there is one location, namely all households are situated in location $z = 0$. By choosing units, we normalize one of the two technological scaling factors by setting $A = 1$. Let $\zeta = \rho D/c$ measure the housing durable flow to non-housing consumption ratio. The capital share of housing sector is denoted by s_K. Further denote the capital-output ratio in the housing durable sector as $\chi = K_d/X$. In the steady state, $X = \delta D$, which implies: $K_d = \delta\chi\zeta c/\rho$. In the home production literature (e.g. Benhabib, Rogerson and Wright 1991; Greenwood and Hercowitz 1991), the housing durable flow is regarded as large as non-housing consumption; we thus set $\zeta = 1$. The economy-wide capital-output ratio in the U.S. usually falls in the range from 2.5 to 3. We take an average, setting $\chi = 2.75$ as the benchmark. Based on our steady-steady relationships, we can then obtain:

$$\begin{aligned} K_c &= \left(\frac{\alpha A}{\rho}\right)^{\frac{1}{1-\alpha}} = 36.9806 \\ c &= \frac{1}{2} A K_c^{\alpha} = 1.5430 \end{aligned}$$

Subsequently, the capital stock devoted to the housing durable sector, the housing capital share and the steady-state value of housing durables can be computed as:

$$\begin{aligned} K_d &= \frac{\delta\chi\zeta c}{\rho} = 3.7948 \\ s_K &= \frac{2K_d}{2K_d + K_c} = 0.2052 \\ D &= \frac{\zeta c}{\rho} = 44.0867 \end{aligned}$$

That is, about 20% of the aggregate capital stock is allocated to producing housing durables.

According to Davis and Ortalo-Magné (2008), the expenditure share of housing is about 24% ($s_H = 0.24$). Over the four decades between 1960 and 2000, we can use the data provided by Davis and Heathcote (2007) to compute housing growth rate at 1.8% ($g_H = 0.018$), the housing structure growth rate at 2.4% ($g_D = 0.024$), the housing structure price growth rate at 0.68% ($g_{R_D} = 0.0068$) and the land price growth rate at 4.33% ($g_{R_T} = 0.0433$). Moreover, the average land value to housing value share is about 36% ($s_T = 0.36$). Using non-durable consumption as a proxy, we compute the non-housing consumption good growth rate as 3% ($g_c = 0.03$).

These ratios and growth rates can then be used to calibrate some key parameters in our model. From our model,

$$R_D = \frac{(1-\gamma) R_H H}{D - \theta}$$

$$R_H = \frac{1-\sigma}{\sigma} \frac{c}{H+\eta}$$

and the land rent can then by defined based on the bid rent concept,

$$R_T = \frac{R_H H - R_D D}{T}$$

where the first expression yields a useful relationship governing the prices of housing durables and housing

$$R_D D = (1-\gamma) \frac{D}{D-\theta} R_H H$$

The land value to housing value share is then defined as: $s_T = \frac{R_T T}{R_H H}$.

Assuming fixed land supply over time, we totally differentiate the above three price relationships around the steady state to obtain:

$$\hat{R}_D = \hat{R}_H + \hat{H} - \frac{D}{D-\theta}\hat{D} \tag{27}$$

$$\hat{R}_H = \hat{c} - \frac{H}{H+\eta}\hat{H} \tag{28}$$

$$\hat{R}_T = \frac{R_H H}{R_H H - R_D D}\left(\hat{R}_H + \hat{H}\right) - \frac{R_D D}{R_H H - R_D D}\left(\hat{R}_D + \hat{D}\right)$$

Straighforward manipulations lead to,

$$\hat{R}_T = \left(\hat{R}_H + \hat{H}\right) + \frac{(1-\gamma)\frac{D}{D-\theta}\frac{\theta}{D-\theta}}{1-(1-\gamma)\frac{D}{D-\theta}}\hat{D} \tag{29}$$

$$s_T = \frac{R_T T}{R_H H} = 1 - (1-\gamma)\frac{D}{D-\theta} \tag{30}$$

Let the rates of changes of all price and quantity variables capture their respective transitional growth rates, $(g_{R_D}, g_{R_H}, g_{R_T}, g_D, g_H, g_c)$.[8] From (27) and (28), we have:

$$\frac{\theta}{D} = 1 - \frac{g_D}{g_H + g_{R_H} - g_{R_D}} \tag{31}$$

$$\frac{\eta}{H} = \frac{g_H}{g_c - g_{R_H}} - 1 \tag{32}$$

We utilize (30) to write $(1-\gamma)\frac{D}{D-\theta} = 1 - s_T$, which, together with (29) and (31), gives:

$$g_{R_H} = s_T g_{R_T} + (1-s_T)(g_{R_D} + g_D) - g_H = 0.0173$$

[8] These transitional changes are consequences of transitional changes in $G(t)$. We do not model these changes as permanent because we must otherwise construct specific unbalanced endogenous growth models which often require adding a third sector with two of the three sectors growing at different rates but balancing each other in aggregation (see Kongsamut, Rebelo and Xie 2001, Bond, Trask and Wang 2003 and Acemoglu and Guerrieri 2008). Adding such a sector would make the analysis more difficult without generating further insight over our simple optimal growth structure.

We can now use (30) and (31) to compute:

$$
\begin{aligned}
\theta &= \left(1 - \frac{g_D}{g_H + g_{R_H} - g_{R_D}}\right) D = 6.9611 \\
\gamma &= 1 - \frac{1 - s_T}{\frac{D}{D-\theta}} = 0.4611
\end{aligned}
$$

Thus, the minimum structure requirement for housing is about one-sixth of the amount of housing durables, which seems quite reasonable. Applying the functional form of housing $H = T^{\gamma}\left(D - \theta\right)^{1-\gamma} = 7.9997$ and the land supply schedule, we can then utilize (32) to calibrate:

$$
\eta = \left(\frac{g_H}{g_c - g_{R_H}} - 1\right) H = 3.3385
$$

Finally, from the first-order condition governing consumption and housing demand, we have:

$$
s_H = \frac{R_H H}{c + R_H H} = \frac{1}{1 + \frac{\sigma}{1-\sigma}\frac{H+\eta}{H}}
$$

which yields,

$$
\sigma = \frac{\left(\frac{1}{s_H} - 1\right)\frac{H}{H+\eta}}{1 + \left(\frac{1}{s_H} - 1\right)\frac{H}{H+\eta}} = 0.6908
$$

Furthermore, from the steady-state relationship $BK_d^{\beta} = \delta D$, we can write:

$$
B = \frac{\delta D}{K_d^{\beta}}
$$

Substituting this expression into another steady-state relationship,

$$
\frac{\beta B}{\rho}\left(\frac{\delta D}{B}\right)^{\frac{\beta-1}{\beta}}(1-\gamma)\frac{1-\sigma}{\sigma}\frac{A\left(\frac{\alpha A}{\rho}\right)^{\frac{\alpha}{1-\alpha}}}{2}\frac{1}{D-\theta}\frac{H}{H+\eta} = \rho + \delta,
$$

leads to a single equation in β. This gives the calibrated value $\beta = 0.9021$, which can be plugged back into the previous expression to calibrate $B = 0.4143$.

4.2 Numerical Results

It may be noted that one may compute the point-in-time price of housing durables as:

$$
P_D = \frac{\mu}{\lambda} = \frac{\rho}{\beta B K_d^{\beta-1}}
$$

It can be verified that in the steady state the housing durable price satisfies $R_D = (\rho + \delta) P_D$ and the housing price satisfies $R_H = \rho P_H$. That is, the capitalization of housing durables and housing differs by the demolishment factor δ. Since both ρ and δ are constant over time and across locations, we can examine the dynamic and spatial patterns of housing and housing durable prices by using their corresponding rental price measures (R_H and R_D), which are in comparable units to the land rent.

Now going back to the original spatial structure, we have:

$$\begin{aligned} R_{H_z} &= \frac{1-\sigma}{\sigma}\frac{\phi(z)c_z}{\phi(z)H_z+\eta} \\ R_{D_z} &= \frac{(1-\gamma)\,R_{H_z}H_z}{D_z-\theta} \\ R_{T_z} &= \frac{R_{H_z}H_z-R_{D_z}D_z}{T_z} \end{aligned}$$

It is important to verify that these are location-specific supporting prices to the allocation derived from the central planner problem under an appropriate redistribution scheme. Specifically, consider a uniform tax τ_z (subsidy if negative) on capital rental and consumption together with a housing tax ν_z (subsidy if negative). Let w denote the wage rate and r denote capital rental rate (which equals ρ in the steady state). Given the total population mass of two, each agent's wealth is measured by,

$$\Omega_z = \frac{K}{2} + (1-\nu_z)\,P_{H_z}H_z$$

and it evolves according to,

$$\dot{\Omega}_z = \frac{1}{2}(w+rK) - (1+\tau_z)(c_z+rK_z)$$

To satisfy locational no-arbitrage, it must be that $\Omega_z = \Omega_0$ and $\dot{\Omega}_z = 0$ in the steady state for all z. These enable us to solve the redistribution scheme.

In our benchmark case, the redistribution scheme features imposing both taxes on those in inner city $[-0.5665, 0.5665]$ and providing both subsidies to those in outskirts $[-1, -0.6004] \cup [0.6004, 1]$. Those residing at midtown $[-0.6004, -0.5665] \cup [0.5665, 0.6004]$ pay housing taxes but receive consumption/capital rental subsidies. The redistributive tax/subsidy schedules over the right half of the city, $[0, 1]$, are plotted in Figure 1. Intuitively, the consideration of locational discounting $\phi(z)$ can be thought of regarding the CBD as a public good whose services decay with distance. Thus, one would expect that those enjoying more of such public good services (in the inner city) would be taxed. Notably, the redistribution scheme features a higher tax/subsidy rate on housing (solid line) than on capital rental/consumption (dashed line). Moreover, such tax rates are all less than 1.2% and subsidy rates all less than 2%. As a result, the after-tax housing price and land price schedules are very close to the pre-tax schedules. As a by-product of this exercise, we can compute the wealth share of housing as 41.6%. Based on the 2000 Census, such a share without including household durables is 32.3%. With the consideration of household durables, our result is viewed as reasonably consistent with the data.

Using calibrated parameter values, we can further compute 3 quantity and 3 price ratios across locations in the city, plus 3 aggregate shares/ratios, the housing expenditure share (s_H), the housing capital share (s_K) and the ratio of aggregate housing durables to housing (D/H). The results are reported below:

$\frac{c_1}{c_0}$	$\frac{H_1}{H_0}$	$\frac{D_1}{D_0}$	$\frac{R_{H_1}}{R_{H_0}}$	$\frac{R_{T_1}}{R_{T_0}}$	$\frac{R_{D_1}}{R_{D_0}}$	s_H	s_K	$\frac{D}{H}$
1.0352	1.2701	0.9957	0.7829	0.5874	0.9995	0.24	0.2052	5.4291

Thus, the quantity of housing at the city fringe is about 27% more than at the CBD (the amount of land is by construction almost 70% more). While the land rent is about 41% lower, the housing price is only about 22% less at the border compared to the center. In Figure 2, we plot the schedule of each endogenous quantity or price over the right half of the city, $[0, 1]$. As one can see clearly, while housing schedule shows significant cross-location variations, consumption and housing durable schedules are rather flat. Moreover, the land rent schedule is much steeper than the housing rental price schedule, whereas the housing durable rental price schedule is essentially flat. Intuitively, land is entirely immobile while housing durables are fully mobile. It is expected that the greater the degree of mobility is, the less the cross-location variation will be, thereby explaining our results.

We can also compute the housing quantity and price elasticities with respect to various parameter changes, reported in the table below:

	A	B	η	ρ	θ	T
Housing Quantity (H)						
Housing Price (P_H)						

We next turn to conducting comparative-static exercises quantitatively. We are particularly interested in the responses of the above cross-location ratios and the three aggregate shares/ratios to a 10% increase in each of four key preference and technology parameters, η, θ, a and B. Such responses in percentage are reported as follows.

%	$\frac{c_1}{c_0}$	$\frac{H_1}{H_0}$	$\frac{D_1}{D_0}$	$\frac{R_{H_1}}{R_{H_0}}$	$\frac{R_{T_1}}{R_{T_0}}$	$\frac{R_{D_1}}{R_{D_0}}$	s_H	s_K	$\frac{D}{H}$
η	−0.08	−0.23	−0.36	−0.24	−0.61	−0.04	−3.07	−2.70	−1.08
θ	−0.00	−0.00	0.00	0.00	−0.02	0.01	−0.00	1.32	1.60
a	1.30	−0.17	−0.27	−0.19	−0.46	−0.03	−0.61	−0.51	−0.21
B	0.04	0.13	0.18	0.11	0.32	0.02	1.62	0.20	3.57

Thus, when housing becomes more luxurious (higher η), the out-skirt to inner city ratios of consumption, the quantity of housing and housing durables, and the rental prices of land, housing and housing durables are all lower. Intuitively, when housing becomes less necessary, housing demand must fall. In terms of the production of housing, the derived demand for housing durables will also fall, though normally by not as much.[9] Our quantitative results suggest that while housing expenditure and housing capital shares fall sharply, the ratio of aggregate housing durables to housing falls. Among all the cross-location ratios, housing, housing durables, housing rental prices and land rents are more responsive.

An increase in the minimum housing structure requirement (higher θ) has little influence on any of the cross-location ratios (with many of such changes less than 0.005%). In response to this increased minimum requirement, it is necessary to allocate more capital to housing capital to produce the required housing durables (i.e., the housing capital share must increase). As a result, both housing durable prices and housing prices rise, while the land rent falls. The former changes discourage housing demand, thereby lowering the housing expenditure share and raising the housing durables to housing ratio. Our quantitative results suggest that while the housing expenditure share drops negligibly, both the housing capital share and the aggregate housing durables to housing ratio rise sharply.

Except for the effect on the cross-location consumption ratio, the change in spatial discounting generates qualitatively identical effects to the change in the luxury good nature of housing. Intuitively, in response to higher spatial discounting (higher a in the spatial discounting function, $\phi(z)$), agents are less willing to reside at outskirts, thereby reducing housing demand and housing durables demand as well as their prices and the land rent in the outer city. That is, both the ratios of housing and housing durables at the fringe to the center must fall. Our quantitative results suggest that the economy-wide housing durables to housing ratio decreases marginally. It is interesting to note that almost all the cross-location ratios (except housing durable prices) are most responsive to this spatial discounting perturbation.

Concerning an increase in the housing durable technology (higher B), all the responses are exactly reverse to an increase in the luxury good nature of housing. Such reversed effects are not surprising as one may view the luxury good nature of housing as a barrier to housing development, thereby having opposite impact to the productivity of housing durables. Because housing durable productivity has a direct positive impact on housing durables, it tends to increase the aggregate

[9] In trade theory, the finding that changes in output are larger than changes in inputs is usually referred to as the magnification effect in quantity.

housing durables to housing ratio. Our quantitative results show a sharp rise in both the housing expenditure share and the aggregate housing durables to housing ratio in response to an increase in the housing durable technology.

It is noted that in response to any of these parameter changes, land rents are always much more responsive than other rental prices, while housing is relatively less responsive than housing durables.

4.3 Gammaville

In so far, we have assumed for the sake of tractability that the population distribution within a city is uniform. We now relax this restriction by considering a gammaville as constructed in Riley (1973).

5 Transitional Dynamics

The dynamic spatial model constructed above with a continuum of locations is too complicated for a meaningful analysis of transitional dynamics. We thus shift our focus now to a simpler version of the model in which there is one location, namely all households are situated in location $z = 0$. The dynamics can be captured by the following equations (see derivation in Appendix B):

$$\dot{K} = A(K - 2F(K, \lambda, \mu))^{\alpha} - 2C(\lambda, D) \tag{33}$$

$$\dot{\lambda} = \rho\lambda - \alpha\lambda A\left(K - 2F(K, \lambda, \mu\right)^{\alpha-1} \tag{34}$$

$$\dot{D} = BF(K, \lambda, \mu)^{\beta} - \delta D \tag{35}$$

$$\dot{\mu} = (\rho + \delta)\mu - (1 - \gamma)\frac{1 - \sigma}{\sigma}\frac{2\lambda C(\lambda, D)}{D - \theta}\frac{T^{\gamma}\left(D - \theta\right)^{1-\gamma}}{T^{\gamma}\left(D - \theta\right)^{1-\gamma} + \eta} \tag{36}$$

where

$$C(\lambda, D) = \left(T^{\gamma}\left(D - \theta\right)^{1-\gamma} + \eta\right)\left(\frac{2\lambda}{\sigma}\right)^{1/(\sigma-1)}$$

and $K_d = F(K, \lambda, \mu)$ solves

$$K_d = \left(\frac{\beta\mu B}{2\alpha\lambda A}\right)^{\frac{1}{1-\beta}}\left(K - 2K_d\right)^{\frac{1-\alpha}{1-\beta}}$$

While $C(\lambda, D)$ is decreasing in λ and increasing in D, $F(K, \lambda, \mu)$ is decreasing in λ and increasing K and μ. The computation of the steady state values of K, λ, D, and μ can also be found in Appendix B.

Based on our calibrated economy, we can apply backward shooting method to this one-location setup to examine the transitional dynamics. Our numerical computations suggest that as the trajectory approaches the steady state, it oscillates in the space of (K, D). The intuition for oscillation can be illustrated using Figure 3 (a close-up near the steady state). Starting at point Q, $D = D^*$ but $K < K^*$, hence it is intuitive that a large fraction of capital would be allocated to the goods sector, implying $K_D < K_D^*$. As a result, $\dot{D} < 0$ at point Q. Since at point Q, the wealth of the representative agent is below that at the steady state, we must have $C_Q < C^*$ and the consumption is small enough to allow for capital accumulation, namely $\dot{K} > 0$ (see equation (33)). Hence, the trajectory from point Q is south-east. At point Q', $K = K^*$ but $D < D^*$, hence it is intuitive that a large fraction of capital would be allocated to durable structure production, namely, $K_D > K_D^*$, which implies that $\dot{D} > 0$ (see equation (35)). Although this means that $K_C < K_C^*$, but $C_{Q'}$ remains below C^*, making it possible for $\dot{K}$ to remain positive.

Of our particular interest, we can identify a transition path along which both K and D increase monotonically until they are close to the steady state (see Figure 4). Specifically, starting from $(K_0, D_0) = (24.2, 18.1)$, both K and D increase toward the steady state. As they approach the steady state (indicated by the big dot), an oscillation occurs as depicted in the three graphs in the lower panel of Figure 4: (i) K overshoots and then starts to fall while D continues to rise, (ii) both fall, and (iii) K then rises while D continues to fall. A repetition of such an oscillation continues until the steady state is reached (the close-up figure is not shown as it has already been illustrated in Figure 3). This path is mimicking the transition dynamics in an economy continuing to evolve by accumulating more capital and housing durables.

In addition to capital and housing durables, it is crucial to understand the transitional dynamics of the rental prices of housing, land and housing durables. One can clearly see from Figure 5 that along the transition, land rents (solid line) grow much more sharply (from 0.03 to 0.13) than housing rental prices (long-dashed line, from 0.058 to 0.062), whereas the rental price schedule of housing durables (short-dashed line) exhibits slight decline over time (from 0.01 to 0.0085). This latter finding is consistent with the home production literature, where cheaper household durables enable house wives to substitute out their time for participating in market activities.

Finally, we note that the presence of the luxury good nature of housing results in changes in the housing expenditure ratio over time. In our calibrated economy, this ratio increases moderately from 20.7% to 24% over the first 50 years and remain largely unchanged afterward. The moderate increase in the ratio is basically consistent with the evidence in the U.S. For example, Rogers (1988) documents that the ratio increased by 2.7% in urban areas and by 1.9% in rural areas from 1972/73 to 1985, whereas Davis and Martin (2008) finds that the ratio increased by 2.3% from 1975 to 1982 and then becomes relatively stable with a slight downward trend through 2007.

6 Alternative Parametrization and Model Specification

In this section, we will perform sensitivity analysis with regard to some parameter selections that are not entirely based on observations. We will also provide further discussion concerning particularly some key ingredients of our model specification.

6.1 Alternative Parametrization

In our calibration analysis, two parameter selections are not entirely based on observations: one is the ratio of housing durables to consumption (ζ, set as 1) and another is the housing-sector capital-output ratio (χ, set as 2.75). To check the robustness of our results, we change ζ up and down by 10% from its benchmark value (1) and χ from 2.5 to 3.0 (reasonable range used in the literature when calibrating the model to fit the U.S. data). We find that our main results are robust to all such changes. More specifically, both the dynamic patterns and the cross-locational patterns of our key variables are essentially unchanged. As reported in Appendix C, the only noticeable changes are the economy-wide capital share and housing durables to housing ratio in the steady state. Such changes are expected. When the model is calibrated with a higher housing durables to consumption ratio, both the housing capital share and the housing durable to housing ratio must rise. When the model is calibrated with a higher housing-sector capital-output ratio, the housing capital share must increase.

Our calibrated economy features increasing land supply away from the CBD where the relative supply at the fringe is about 70% more than at the center. In reality, such relative land supplies vary across different MSAs. We thus perform sensitivity analysis with respect to the land expansion rate away from the CBD (q in the land supply schedule, $T(z)$), changing it to 0.25 and 0.35 (deviating from its benchmark value of 0.30). We find that the dynamic patterns of our key variables are largely unchanged. In response to a steeper land expansion rate, all of the aggregate variables are essentially unchanged. Concerning the cross-locational patterns of our key variables, the most noticeable changes are steeper housing schedule and flatter housing price and land rent gradients away from the CBD (see Appendix C), which are not surprising given the increased supply of land toward fringes.

6.2 Alternative Model Specifications

There are three key factors driving some of the main results in the paper. The obvious one is the spatial structure captured by both spatial discounting and increasing land supply away from the CBD. These ensure reasonable housing ratios at the fringe relative to the center as well as a reasonable downward land rent gradient. In addition, there are two less obvious factors. One is the luxury good nature of housing relative to the composite good captured by $\eta > 0$; another is the minimum housing structure requirement captured by $\theta > 0$. The remainder of this section will be devoted to discussing the consequences of alternative model specification by setting each of these two parameters to zero.

6.2.1 Housing Is Not More Luxurious than Consumption

We abandon the luxury good nature of housing relative to the composite good (i.e., set $\eta = 0$), which does not affect any of the calibrated parameters except σ (whose recalibrated value becomes 0.76). The steady-state values of some key ratios are now recalculated below:

$\frac{c_1}{c_0}$	$\frac{H_1}{H_0}$	$\frac{D_1}{D_0}$	$\frac{R_{H_1}}{R_{H_0}}$	$\frac{R_{T_1}}{R_{T_0}}$	$\frac{R_{D_1}}{R_{D_0}}$	s_H	s_K	$\frac{D}{H}$
1.0318	1.2936	1.0245	0.7977	0.6155	1.0026	0.24	0.2052	5.5249

The most significant changes are that both the housing durables ratios and the housing durable price ratios at the fringe compared to at the center are now exceeding one. That is, agents residing in outskirts demand for more housing durables at higher prices. In terms of the dynamics, the non-housing consumption growth rate is now given by $g_c = 1.73\%$, much lower than the observed rate of 3%.

We also redo comparative statics, obtain the following results:

%	$\frac{c_1}{c_0}$	$\frac{H_1}{H_0}$	$\frac{D_1}{D_0}$	$\frac{R_{H_1}}{R_{H_0}}$	$\frac{R_{T_1}}{R_{T_0}}$	$\frac{R_{D_1}}{R_{D_0}}$	s_H	s_K	$\frac{D}{H}$
θ	0.00	0.00	**−0.04**	−0.00	**0.10**	−0.00	0.00	1.27	1.53
a	1.21	**0.59**	**0.93**	**0.60**	**1.51**	**0.10**	0.00	0.00	−0.01
B	0.00	−0.00	0.03	0.00	**−0.10**	0.01	0.00	**−1.17**	3.04

The most significant changes compared to the benchmark case are three folds. First, and perhaps the most undesirable outcome, the responses of housing-related quantity and price variables to a all have wrong signs. Specifically, greater spatial discounting away from the CBD should cause agents to be less willing to reside at outskirts, thereby reducing housing demand and housing durables demand as well as their prices and the land rent. With $\eta = 0$, agents turn out to be more willing to reside away from the CBD despite they have a stronger preference to be closer to the center.[10] This

[10] A by-product of this result is that the redistribution scheme for decentralization must now feature a housing tax on suburban residents and a housing subsidy to central-city residents. This redistribution scheme is also unlikely in the real world.

is because that, with $\eta = 0$, $\phi(z)$ becomes a common multiplier to both composite consumption and housing. In this case, adjustments in consumption may dominate the required adjustments in housing. leading to counter-intuitive results in the relative price of housing and the relative demands for housing. Second, the relative technological changes in the housing sector now have essentially no effect on any of the key ratios except the allocation of capital, which is unlikely in the real world. Indeed, the land rent gradient and the housing capital share respond negatively to a positive technology change in the housing sector, apparently counter-intuitive. Finally, although not reported in the table above, the housing expenditure share is entirely flat, not only over time but across locations within the city. The latter result is inconsistent with the U.S. data, where within the MSA variations are observed as documented by Davis and Ortalo-Magne (2008).

In summary, the consideration of the luxury good nature of housing is crucial for producing sensible comparative statics, particularly with respect to changes in locational preferences. It is also useful for obtaining a sharp upward trend in the land rent to housing durable price ratio and for the housing-related variables at different locations to respond differently to sector-specific technological changes.

6.2.2 Housing Requires No Minimum Structure

If we recalibrate the model by removing the minimum housing structure requirement (i.e., set $\theta = 0$), three calibrated parameters would change: $\gamma = 0.36$, $\eta = 35.3750$ and $\sigma = 0.4526$. The steady-state values of some key ratios become:

$\frac{c_1}{c_0}$	$\frac{H_1}{H_0}$	$\frac{D_1}{D_0}$	$\frac{R_{H_1}}{R_{H_0}}$	$\frac{R_{T_1}}{R_{T_0}}$	$\frac{R_{D_1}}{R_{D_0}}$	s_H	s_K	$\frac{D}{H}$
1.0622	1.1166	0.8844	0.7816	0.5164	0.9868	0.24	0.1693	3.2783

Most significant changes are the large drops in the housing and housing durables ratios as well as the land rent gradient and the housing capital share. Although there is no obvious problem associated with any of these changes, we shall point out the calibrated value of the preference bias parameter η appears unusually large relative to housing services $\phi(z)H_z$: the ratio $\frac{\eta}{\phi(z)H_z}$ ranges from 3.4 to 4.5 (much larger than the benchmark counterparts, 0.49 to 0.56). In terms of the dynamics, the housing durables growth rate is now given by $g_D = 3.65\%$, much higher than the observed rate of 2.4%.

We also redo comparative statics, obtain the following results:

%	$\frac{c_1}{c_0}$	$\frac{H_1}{H_0}$	$\frac{D_1}{D_0}$	$\frac{R_{H_1}}{R_{H_0}}$	$\frac{R_{T_1}}{R_{T_0}}$	$\frac{R_{D_1}}{R_{D_0}}$	s_H	s_K	$\frac{D}{H}$
η	−0.54	−1.33	−2.07	−0.97	−2.29	−0.23	−11.03	−11.40	−4.55
a	1.43	−2.40	−3.72	−1.77	−4.12	−0.44	−2.26	−2.36	−0.87
B	0.34	0.84	1.31	0.61	1.45	0.14	7.17	7.44	6.45

The outcomes are mixed. On the positive side, there are no wrong signs contradicting to the theory. On the negative side, several changes in response to a 10% increase in relative demand in the inner city (captured by higher a), a 10% decrease in city-wide demand for housing services (captured by an increase in the luxury good nature of housing η) and a 10% increase in city-wide supply (captured by higher B) seem too large quantitatively. For example, the more-than-proportional impacts of a 10% decrease in city-wide demand for housing services on the housing expenditure share and the housing capital share are unlikely to arise in the real world. Moreover, a 10% increase in housing durables production technology results in more than 7% increase in the housing expenditure share and the housing capital share, both very excessive to the reality. Moreover, since housing durables are mobile across locations, one would expect their cross-location ratios in quantities and prices not too responsive to locationally uniform changes (η and B). It is not the case under this model specification: a 10% decrease in city-wide demand for housing services leads to a 2.1% drop in the cross-location housing durables ratio, whereas a 10% increase in city-wide supply generates a 1.3% increase in the cross-location housing durables ratio.

In summary, the consideration of the minimum structure requirement for housing is most useful for creating a buffer that produces more plausible responses with respect to changes in city-wide parameters.

7 Concluding Remarks

We have developed a two-sector dynamic general equilibrium model explicitly accounting for locational choice and several special features of housing. The model has been calibrated to fit some important stylized facts, both across locations within an MSA and over time. Along these lines, perhaps the most important extension is to study the housing sector and its interplays with the non-housing sector over the business cycle. This may be done by introducing stochastic shocks to sector-specific technologies (A and B in our model). Another useful extension is to conduct normative analysis, studying the short-run and long-run effects of housing-related policy on the performance of the housing sector and the macroeconomy as a whole. Such policy may include property taxes and provision of public infrastructure that may affect housing development across different locations (such as highways, public transportation, and public utility).

Appendix A: Comparative-Static Analysis

The key relationships in the baseline one-location setup are summarized as follows:

$$\begin{aligned}
K_c &= \left(\frac{\alpha A}{\rho}\right)^{\frac{1}{1-\alpha}} \\
K_d &= \left(\frac{\delta D}{B}\right)^{\frac{1}{\beta}} \\
c &= \frac{1}{2}AK_c^{\alpha} = \frac{1}{2}A^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{\rho}\right)^{\frac{\alpha}{1-\alpha}} \\
H &= T^{\gamma}(D-\theta)^{1-\gamma} \\
P_H &= \frac{R_H}{\rho} = \frac{1-\sigma}{\sigma}\frac{1}{\rho}\frac{c}{H+\eta}
\end{aligned}$$

$$\frac{\beta B}{\rho}\left(\frac{\delta D}{B}\right)^{\frac{\beta-1}{\beta}}(1-\gamma)\frac{1-\sigma}{\sigma}\frac{A\left(\frac{\alpha A}{\rho}\right)^{\frac{\alpha}{1-\alpha}}}{2}\frac{1}{D-\theta}\frac{H}{H+\eta} = \rho+\delta$$

Utilizing the hat calculus, we first totally differentiate the above expressions to obtain:

$$\hat{K}_c = \frac{1}{1-\alpha}\left(\hat{A}-\hat{\rho}\right) \tag{37}$$

$$\hat{K}_d = \frac{1}{\beta}\left(\hat{D}-\hat{B}\right) \tag{38}$$

$$\hat{c} = \frac{1}{1-\alpha}\hat{A} - \frac{\alpha}{1-\alpha}\hat{\rho} \tag{39}$$

$$\hat{H} = \gamma\hat{T} + (1-\gamma)\left(\frac{D}{D-\theta}\hat{D} - \frac{\theta}{D-\theta}\hat{\theta}\right), \text{ or,} \tag{40}$$

$$\hat{D} = \frac{1}{1-\gamma}\frac{D-\theta}{D}\hat{H} - \frac{\gamma}{1-\gamma}\frac{D-\theta}{D}\hat{T} + \frac{\theta}{D}\hat{\theta} \tag{41}$$

$$\begin{aligned}
\hat{P}_H &= \hat{c} - \frac{H}{H+\eta}\hat{H} - \hat{\rho} - \frac{\eta}{H+\eta}\hat{\eta} \\
&= -\frac{H}{H+\eta}\hat{H} + \frac{1}{1-\alpha}\hat{A} - \frac{\eta}{H+\eta}\hat{\eta} - \frac{1}{1-\alpha}\hat{\rho}
\end{aligned} \tag{42}$$

$$\frac{1}{\beta}\hat{B} + \frac{1}{1-\alpha}\hat{A} - \left(\frac{1-\beta}{\beta} + \frac{D}{D-\theta}\right)\hat{D} + \frac{\theta}{D-\theta}\hat{\theta} + \frac{\eta}{H+\eta}\hat{H} - \frac{\eta}{H+\eta}\hat{\eta} = \left(\frac{1}{1-\alpha} + \frac{\rho}{\rho+\delta}\right)\hat{\rho} \tag{43}$$

Next, substituting (41) into (43) yields,

$$\begin{aligned}
&\frac{1}{\beta}\hat{B} + \frac{1}{1-\alpha}\hat{A} - \left(\frac{1-\beta}{\beta} + \frac{D}{D-\theta}\right)\left[\frac{1}{1-\gamma}\frac{D-\theta}{D}\hat{H} - \frac{\gamma}{1-\gamma}\frac{D-\theta}{D}\hat{T} + \frac{\theta}{D}\hat{\theta}\right] \\
&+\frac{\theta}{D-\theta}\hat{\theta} + \frac{\eta}{H+\eta}\hat{H} - \frac{\eta}{H+\eta}\hat{\eta} = \left(\frac{1}{1-\alpha} + \frac{\rho}{\rho+\delta}\right)\hat{\rho}
\end{aligned}$$

or, by rearranging terms, we obtain the fundamental equation governing the changes in the housing quantity (24). Finally, this latter fundamental equation can then be substituted into (42) to yield the fundamental equation governing the changes in the housing price (26).

Appendix B: The Dynamic System with One Location

To make the equilibrium properties consistent on average between this one location model and the multi-location model in the main text, we continue to assume that the population size equals 2 and the land per individual, T, stays the same, which requires:

$$T = \int_0^1 T(z)dz$$

While housing in this one location case is simply $H = T^\gamma (D-\theta)^{1-\gamma}$, the housing durable evolves according to $\dot{D} = BK_d^\beta - \delta D$ (with $D(0) \geq \theta$). The total labor supply L is assumed to be 1 (i.e., each individual supplies 1/2 unit of labor), so the aggregate capital stock evolves according to

$$\dot{K} = AK_c^\alpha L^{1-\alpha} - 2c$$

where $K = K_c + 2K_d$ is equally owned by all the agents.

Thus, the competitive equilibrium can be derived from solving the central planner's problem as follows:

$$\max \int_0^\infty c^\sigma \left(T^\gamma (D-\theta)^{1-\gamma} + \eta\right)^{1-\sigma} e^{-\rho t} dt$$

$$\text{subject to: } \dot{K} = A(K - 2K_d)^\alpha L^{1-\alpha} - 2c \tag{44}$$

$$\dot{D} = BK_d^\beta - \delta D \tag{45}$$

$$D(0) > \theta$$

The first-order conditions with respect to c and K_d are:

$$\sigma c^{\sigma-1} \left(T^\gamma (D-\theta)^{1-\gamma} + \eta\right)^{1-\sigma} = 2\lambda \tag{46}$$

$$\beta\mu B K_d^{\beta-1} = 2\alpha\lambda A (K - 2K_d)^{\alpha-1} L^{1-\alpha} \tag{47}$$

Euler equations with respect to K and D are given by,

$$\dot{\lambda} = \rho\lambda - \alpha\lambda A (K - 2K_d)^{\alpha-1} L^{1-\alpha}$$

$$\dot{\mu} = (\rho+\delta)\mu - (1-\gamma)\frac{1-\sigma}{\sigma}\frac{2\lambda c}{D-\theta}\frac{T^\gamma (D-\theta)^{1-\gamma}}{T^\gamma (D-\theta)^{1-\gamma} + \eta}$$

which can be rewritten using the first-order conditions as:

$$\frac{\dot{\lambda}}{\lambda} = \rho - \alpha A (K - 2K_d)^{\alpha-1} L^{1-\alpha} \tag{48}$$

$$\frac{\dot{\mu}}{\mu} = (\rho + \delta) - (1 - \gamma)\frac{1 - \sigma}{\sigma}\frac{c}{D - \theta}\frac{\beta B K_d^{\beta - 1}}{\alpha A (K - 2K_d)^{\alpha - 1} L^{1 - \alpha}}\frac{T^{\gamma}(D - \theta)^{1 - \gamma}}{T^{\gamma}(D - \theta)^{1 - \gamma} + \eta} \tag{49}$$

From (48) as well as (44) and (45), we obtain:

$$K_c = K - 2K_d = \left(\frac{\alpha A}{\rho}\right)^{\frac{1}{1-\alpha}} \tag{50}$$

$$c = \frac{1}{2}AK_c^{\alpha} = \frac{1}{2}A\left(\frac{\alpha A}{\rho}\right)^{\frac{\alpha}{1-\alpha}} \tag{51}$$

$$K_d = \left(\frac{\delta D}{B}\right)^{\frac{1}{\beta}} \tag{52}$$

These can then be used together with (49) to yield,

$$\frac{\beta B}{\rho}\left(\frac{\delta D}{B}\right)^{\frac{\beta-1}{\beta}}(1 - \gamma)\frac{1 - \sigma}{\sigma}\frac{A\left(\frac{\alpha A}{\rho}\right)^{\frac{\alpha}{1-\alpha}}}{2}\frac{1}{D - \theta}\frac{T^{\gamma}(D - \theta)^{1 - \gamma}}{T^{\gamma}(D - \theta)^{1 - \gamma} + \eta} = \rho + \delta \tag{53}$$

which solves uniquely D, which can then be plugged into (52) and (50) to solve for K_d and K.

Using (46) and (47), we can write in a recursive manner c as a function of (λ, D) and K_d as a function of (K, λ, μ):

$$c = \left(T^{\gamma}(D - \theta)^{1 - \gamma} + \eta\right)\left(\frac{2\lambda}{\sigma}\right)^{1/(\sigma - 1)} \equiv C(\lambda, D)$$

$$K_d = \left(\frac{\beta \mu B}{2\alpha \lambda A}\right)^{\frac{1}{1-\beta}}(K - 2K_d)^{\frac{1-\alpha}{1-\beta}}$$

where the latter yields a unique fixed point $K_d = F(K, \lambda, \mu)$. Once we obtain the steady state, we can then solve by backward shooting of the following system of four differential equations given by (33)-(36).

Appendix C: Sensitivity Analysis

We consider four sensitivity cases with respect to ζ (housing durable flow to consumption ratio) and χ (housing-sector capital-output ratio), adjusting one parameter each time while keeping another at its benchmark value. We then consider two more cases, adjusting q (land expansion rate away from the CBD) above and below its benchmark value.

	Benchmark	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
ζ	1	0.8	1.2	1	1	1	1
χ	2.75	2.75	2.75	2.5	3	2.75	2.75
q	0.3	0.3	0.3	0.3	0.3	0.25	0.35
η	3.3385	3.1542	3.5144	3.3385	3.3385	3.3261	3.4080
θ	6.9611	6.2649	7.6572	6.9611	6.9611	6.9611	6.9611
B	0.4143	0.4581	0.3676	0.4998	0.3409	0.4143	0.4143
β	0.9021	0.8119	0.9923	0.8201	0.9841	0.9021	0.9021
$\frac{c_1}{c_0}$	1.0352	1.0351	1.0353	1.0351	1.0352	1.0466	1.0244
$\frac{H_1}{H_0}$	1.2701	1.2705	1.2697	1.2705	1.2698	1.2240	1.3162
$\frac{D_1}{D_0}$	0.9957	0.9961	0.9952	0.9961	0.9952	0.9944	0.9969
$\frac{R_{H_1}}{R_{H_0}}$	0.7829	0.7827	0.7830	0.7827	0.7830	0.8110	0.7567
$\frac{R_{T_1}}{R_{T_0}}$	0.5874	0.5876	0.5872	0.5876	0.5873	0.6340	0.5459
$\frac{R_{D_1}}{R_{D_0}}$	0.9995	0.9991	0.9999	0.9991	0.9999	0.9994	0.9997
s_H	0.24	0.24	0.24	0.24	0.24	0.24	0.24
s_K	0.2052	0.1803	0.2118	0.1817	0.2104	0.1964	0.1963
$\frac{D}{H}$	5.4291	5.1810	5.6628	5.4380	5.4202	5.4406	5.3220

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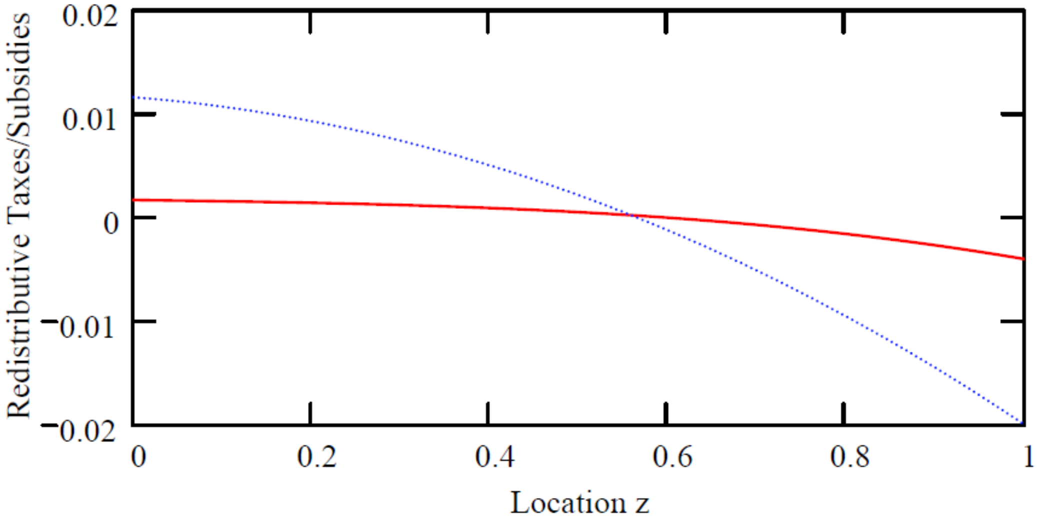

Figure1. Small Redistributive Taxes/Subsidies Needed for Decentralization

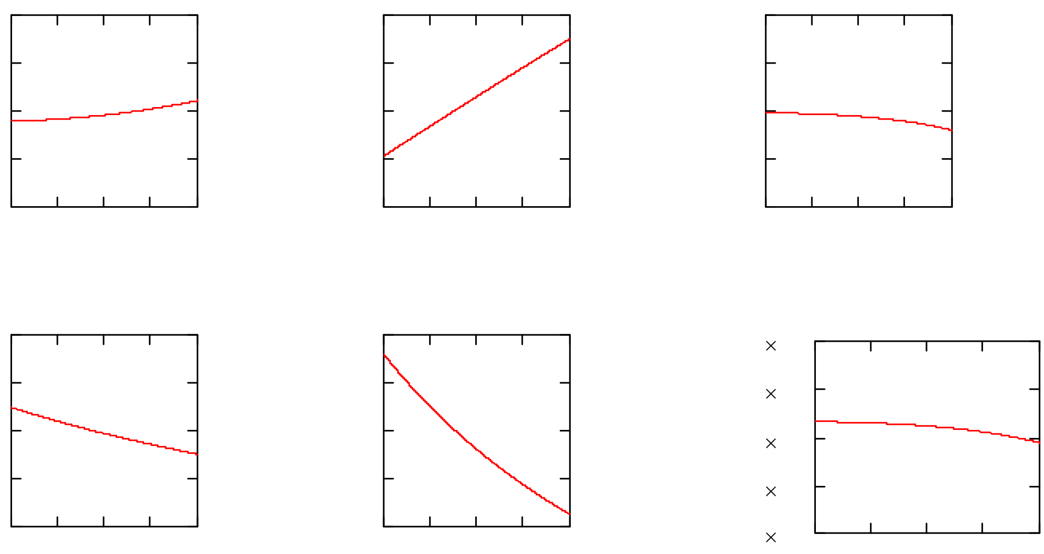

Figure 2. Housing and Land Rent Most Sensitive to Location

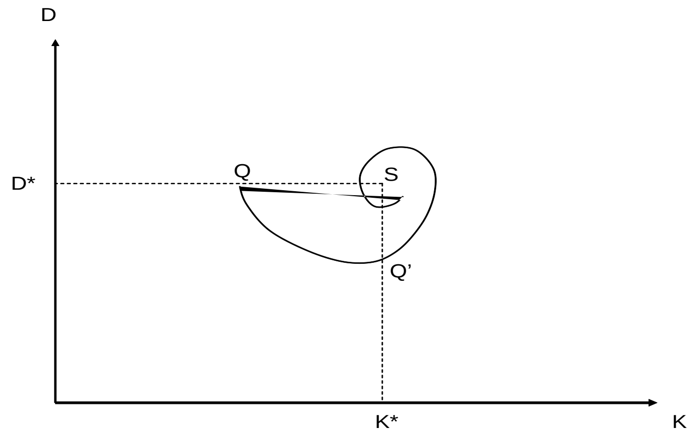

Figure 3. Oscillation Near the Steady State

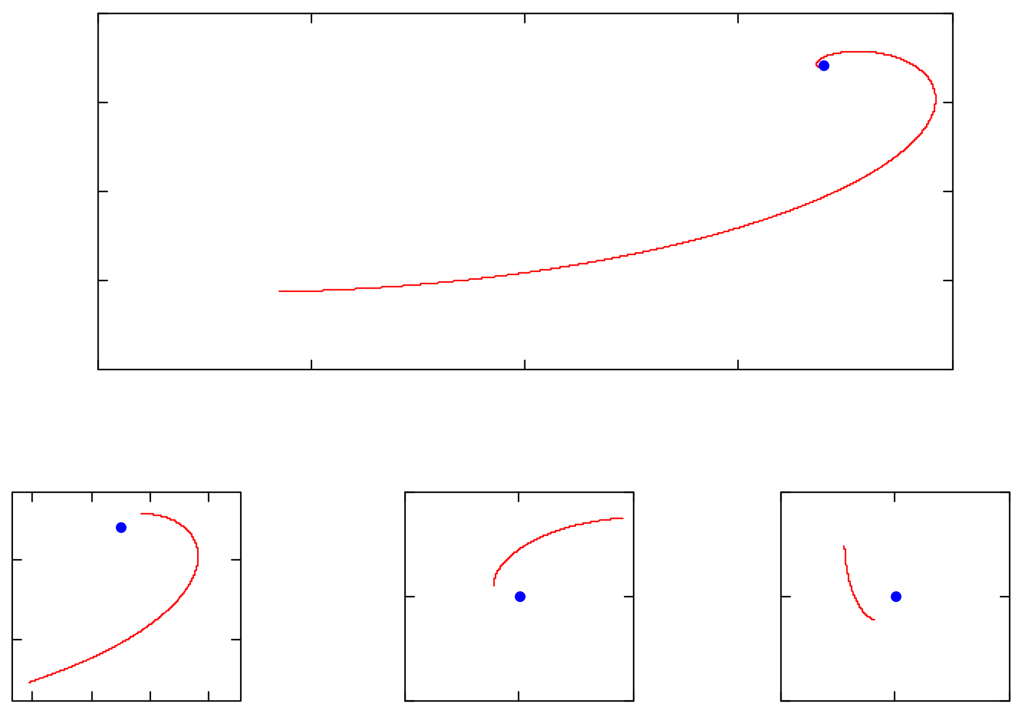

Figure 4. Equilibrium Trajectory

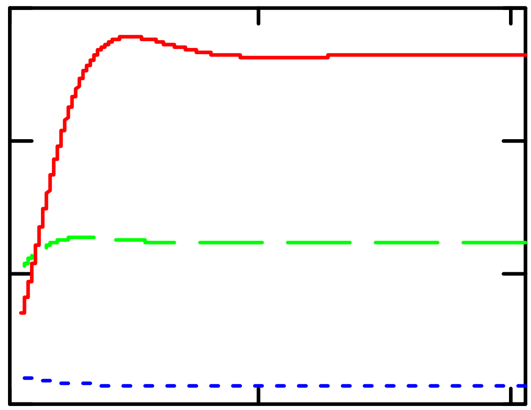

Figure 5. Rental Prices of House, Land, and Durables

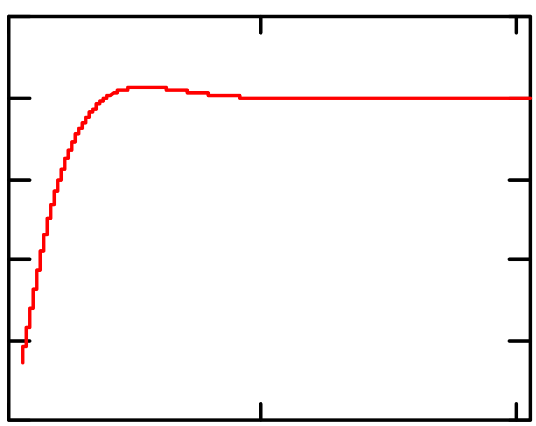

Figure 6. Housing Expenditure Share