Production externalities and local dynamics in discrete-time multi-sector growth models with general production technologies

Kazuo Nishimura*, Koji Shimomura† and Ping Wang‡

The present paper examines the dynamic properties of discrete-time, multi-sector growth models in the presence of sector-specific externalities. It extends the literature by allowing for multiple capital good sectors with general social constant returns production technologies. We establish conditions for the steady-state equilibrium to be locally determinate or locally indeterminate, depending crucially on the ratios of the social to private marginal products and the number of capital good sectors. We show that when the ratios of the social to private marginal products are uniform across all sectors, the steady state is always locally determinate in a two-sector model, although local indeterminacy might still arise when the economy features more than two sectors.

Key words sector-specific externalities, social constant returns, local dynamics

JEL classification D90, O41

1 Introduction

Over the past decade, there has be a growing literature examining under what circumstances dynamic indeterminacy might occur. When dynamic indeterminacy arises, a steady-state or balanced growth equilibrium is locally indeterminate in the sense of multiple, often a continuum of, converging transition paths. The vast interest in this dynamic property is a result of the observations that many macro variables are volatile over time and that economic growth rates are dispersed across countries. These observations might be partly explained by the presence of dynamic indeterminacy.

Previous studies conclude that dynamic indeterminacy mightoccur in continuous-time endogenous growth models with increasing returns, distortionary sector-specific factor

* Institute of Economic Research, Kyoto University, Kyoto, Japan. Email: nishimura@kier.kyoto-u.ac.jp

† Research Institute for Economics and Business Administration, Kobe University, Kobe, Japan. Email: simomura@rieb.kobe-u.ac.jp

‡ Department of Economics, Washington University, USA. pingwang@artsci.wustl.edu

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taxes, or sector-specific externalities. For example, Benhabib and Perli (1994) Boldrin and Rustichini (1994) and Xie (1994) identify such a possibility in endogenous growth models with increasing returns as a result of uncompensated positive spillovers. In an endogenously growing economy with private constant return production technologies and with distortionary sector-specific factor taxes, Bond *et al.* (1996) establish the "polarization theorem" when factor tax distortion is moderate such that the factor intensity rankings in the physical and value sense are consistent, the price and quantity dynamics are polarized, in that when one is stable, another becomes unstable, therefore implying saddle-path stability. When factor tax distortion is sufficiently strong, dynamic indeterminacy occurs if the final good sector is human capital intensive in the physical sense but physical capital intensive in the value sense.[1] Benhabib *et al.* (2000) and Mino (2001) consider social constant returns production technologies with sector-specific externalities within the endogenous growth setup and obtain similar conclusions to Bond *et al.* (1996), by recognizing that the factor intensity rankings in the physical and value sense are parallel to those in the private and social sense. Dynamic properties have also been examined in exogenous growth models, either in continuous time (Benhabib and Nishimura 1998, 1999) or in a *discrete-time two-sector* setting (Benhabib *et al.* 2002 and Nishimura and Venditti 2004). In all published papers but Bond *et al.* (1996), analysis has been conducted under Cobb-Douglas production specifications.

In the present paper, we extend this line of research by examining the dynamic properties of the class of discrete-time exogenous growth models with multiple capital good sectors and general social constant returns production technologies. Moreover, our approach also differs sharply from that in Benhabib *et al.* (2002) and Nishimura and Venditti (2003). In their papers, the local dynamics are characterized by a second-order difference equation in capital, which summarizes both price and quantity dynamics. In our paper, we separate the price and quantity dynamics, through which the analysis is greatly simplified and more intuition can be gained, even in a multi-sector setup with general production technologies. Specifically, we establish conditions for the steady-state equilibrium to be locally determinate or locally indeterminate, depending crucially on the ratios of the social to private marginal products and the dimensionality of the system of reproducible factors. We find that when the ratios of the social to private marginal products are uniform across all sectors, the steady state is always locally determinate in a two-sector model; however, local indeterminacy might still arise when the economy features more than two sectors.

2 The basic model

We consider a discrete-time model (indexed by $t = 0, 1, 2, \ldots$) with labor x_0 and n accumulated reproducible capitals (denoted x_i, $i = 1, \ldots n$) and $n + 1$ distinctive sectors (labeled by $j = 0, \ldots, n$). Therefore, sector 0 produces the final good for consumption

[1] The former factor intensity ranking ensures stable quantity dynamics and the latter guarantees stable price adjustments. Therefore, the balanced growth equilibrium becomes locally indeterminate. Similar results also obtain in a dynamic trade model with immobile capital, as established by Nishimura and Shimomura (2002).

and sectors $1, \ldots, n$ produce capital goods. Throughout the paper, we assume that total labor endowment x_{0t} is constant and equal to one.

2.1 Production

Denoting y_{jt} as the output of sector j in time t and x_{ijt} as the factor i (measured at the beginning of period t) devoted to sector j production, the production technologies with time-to-build is given by F_j, where sector-specific externalities are captured by the society's average values of sector-specific factors (labeled with bars). The production function of an individual firm producing good j is denoted as a function of the inputs of these primary factors (the time index t is suppressed whenever it is unnecessary),

$$y_j = F_j(x_{0j}, x_{1j}, \cdots, x_{nj}, \bar{x}_{0j}, \bar{x}_j, \cdots, \bar{x}_{nj}), \tag{1}$$

where $\bar{x}_{ij}$ is the average input of factor i in industry j, $j = 0, 1, \ldots, n$, which is an external term for an individual firm. We assume that (1) is linearly homogeneous in $(x_{0j}, x_{1j}, \cdots, x_{nj}, \bar{x}_{0j}, \bar{x}_{0j}, \cdots, \bar{x}_{nj})$, increasing in each x_{ij} and quasi-concave in $(x_{0j}, \cdots, x_{nj})$. Note that y_0 is the consumption good and x_{0j} is labor.

Consider the ratios of social to private marginal products,

$$s_{ij} \equiv \frac{\frac{\partial F^j}{\partial \bar{x}_{ij}}}{\frac{\partial F^j}{\partial x_{ij}}}. \tag{2}$$

Our basic assumption is:

Assumption 1 For each j, $j = 1 \cdots, n$, $(s_{0j}, \cdots, s_{nj})$ are independent of $(x_{0j}, \cdots, x_{nj})$ as long as $\bar{x}_{0j} = x_{0j}, \cdots, \bar{x}_{nj} = x_{nj}$ hold.

A similar assumption is used in Nishimura *et al.* (2004). Note that we have $x_{ijt} = \bar{x}_{ijt}$ in equilibrium, which will be formally defined later, and that (2) implies $\frac{\partial F}{\partial x_{ij}} + \frac{\partial F}{\partial \bar{x}_{ij}} = (1 + s_{ij})\frac{\partial F}{\partial x_{ij}}$.

Remark 1 For the Cobb-Douglas production function $y_j = \overset{n}{\underset{i=0}{\Pi}}{}^{x}_{ij}\, x_{ij}^{\beta_{ij}} \bar{x}_{ij}^{b_{ij}}$, $\frac{\partial F}{\partial \bar{x}_{ij}} = b_{ij} y_j / \bar{x}_{ij}$ and $\frac{\partial F}{\partial x_{ij}} = \beta_{ij} y_j / x_{ij}$. Therefore, $1 + s_{ij} = (\beta_{ij} + b_{ij})/\beta_{ij}$ for $x_{ij} = \bar{x}_{ij}$. Assumption 1 is satisfied.

Let c_t denote period t consumption of the final good and $x_{1t}, \ldots, x_{nt}$ denote the capital goods. For simplicity, we assume that all capitals are fully depreciated in one period. Therefore, the evolution of the capital stocks can be expressed as $(t = 0, 1, 2, \ldots)$:

$$\begin{aligned} 0 &= y_{0t} - c_t, \\ x_{1,t+1} &= y_{1,t}, \\ &\vdots \\ x_{n,t+1} &= y_{n,t}, \end{aligned} \tag{3}$$

where the initial stocks of both capitals, $x_{10}, \ldots, x_{n0}$, are historically given and positive.

2.2 Household optimization

The representative household has a time-additive preference with a subjective discount factor $\rho \in (0, 1)$ and a felicity function that is linear in consumption c_t. Her lifetime utility is, therefore, given by:

$$U = \sum_{t=0}^{\infty} \rho^t c_t. \tag{4}$$

Let Π_t be the distribution of profits from both sectors to the representative agent. Denote $w_{0,t}$ and $p_{0,t}$ as the nominal wage and the nominal price of the consumption good at time t, and $w_{i,t}$ and $p_{j,t}(i,\ j = 1, 2, \ldots, n)$ as the nominal factor prices and nominal good prices of the ith capital and jth output at time t, where all prices are measured in utils without accounting for intertemporal variation from time-discounting. Assuming that both capitals are owned by households (of mass one), the flow budget constraint facing the presentative household at time t can be written as:

$$w_{0,t}x_{0,t} + \sum_{i=1}^{n} w_{i,t}x_{i,t} + \Pi_t = p_{0,t}c_t + \sum_{j=1}^{n} p_{j,t}x_{j,t+1}, \tag{5}$$

where the left-hand side is the sum of factor incomes and profit redistributions and the right-hand side is the total expenditure (consumption plus gross investments in capitals).

Therefore, the representative household chooses $\{c_t, x_{1t}, \ldots, x_{nt}\}$ to maximize the lifetime utility, U, subject to the budget constraint (21). Because, input and output prices are defined in utils without time-discounting, the Lagrangean function can be specified as:

$$\mathcal{L} \equiv \sum_{t=0}^{\infty} \rho^t \left\{ c_t + w_{0,t}x_{0,t} + \sum_{i=1}^{n} w_{i,t}x_{i,t} + \Pi_t - p_{0,t}c_t - \sum_{j=1}^{n} p_{j,t}x_{j,t+1} \right\}.$$

The first-order necessary conditions are:

$$\frac{\partial \mathcal{L}}{\partial c_t} = \rho^t(1 - p_{0,t}) = 0, \tag{6}$$

$$\frac{\partial \mathcal{L}}{\partial x_{it}} = \rho^t w_{i,t} - \rho^{t-1} p_{i,t-1} = 0, \quad i = 1, \ldots, n, \tag{7}$$

which are sufficient for optimization under the transversality conditions given by $(i = 1, \ldots, n)$:

$$\lim_{t\to\infty} \rho^t p_{i,t}x_{i,t+1} = 0.$$

2.3 Full employment and competitive profit conditions

Let $x_t \equiv (x_{1,t}, \ldots, x_{n,t})$ and $y_t \equiv (y_{1,t}, \ldots, y_{n,t})$. Therefore, (3) implies:

$$x_{t+1} = y_t. \tag{8}$$

Let $p_t \equiv (p_{1,t}, \ldots, p_{n,t})$ and $w_t \equiv (w_{1,t}, \ldots, w_{n,t})$. From (6), $p_{0,t} = 1$ for all t. Updating (7) by one period yields:

$$p_t = \rho w_{t+1}. \tag{9}$$

Equations (8) and (9) constitute a $2n \times 2n$ dynamical system, governing the quantity and price dyanamics.

Under full employment, the quantity of factors used must add up to their total, therefore implying:

$$A_0 \begin{bmatrix} c \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ x \end{bmatrix}, \tag{10}$$

where

$$A_0 \equiv \begin{bmatrix} a_{00} & a_{0\cdot} \\ a_{\cdot 0} & a \end{bmatrix}, \tag{11}$$

and where $a_{0j} \equiv x_{0j}/c$, $a_{ij} \equiv x_{ij}/y_j$, a_{00} is a scalar and a is a $n \times n$ matrix. For simplicity, we restrict our attention to the case that A_0 is a non-singular positive matrix, which is guaranteed by the following two assumptions.

Assumption 2 $a_{00} > 0$ and $a_{ij} > 0$ for $i, j = 1, \ldots, n$.

Assumption 3 $A \equiv a - a_{\cdot 0} a_{00}^{-1} a_{0\cdot}$ is non-singular.

Therefore,

$$(A_0)^{-1} = \begin{bmatrix} (a_{00})^{-1}\left(1 + a_{0\cdot} A^{-1} a_{\cdot 0}\right) & -(a_{00})^{-1} a_{0\cdot} A^{-1} \\ -A^{-1} a_{\cdot 0} (a_{00})^{-1} & A^{-1} \end{bmatrix}. \tag{12}$$

By defining $z \equiv A^{-1} a_{\cdot 0} \, (a_{00})^{-1}$, we can write the output vector as:

$$y = A^{-1} x - z. \tag{13}$$

Next we look at the output price–factor price relationship. We maximize

$$\pi^j = p_j F_j(x_{0j}, \ldots, x_{nj}, \bar{x}_{oj}, \ldots, \bar{x}_{nj}) - w_0 x_0 - \sum_{i=1}^{n} w_i x_{ij}.$$

to get the factor demand equations: $p_j \frac{\partial F_j}{\partial x_{ij}} = w_i$. Using the linear homogeneity property of F^j and $\bar{x}_{ij} = x_{ij}$, we have:

$$(1 + s_{0j})\frac{\partial F_j}{\partial x_{0j}} x_{0j} + \sum_{i=1}^{n}(1 + s_{ij})\frac{\partial F_j}{\partial x_{ij}} x_{ij} = y_j.$$

Substituting the factor demand equations into the above expression and applying the definition of a_{ij}, we obtain the following competitive profit relationships:

$$(1 + s_{0j})w_0 a_{0j} + \sum_{i=0}^{n}(1 + s_{ij})w_i a_{ij} = p_j. \tag{14}$$

Let $\hat{a}_{ij} \equiv (1 + s_{ij})a_{ij}$ and

$$\widehat{A}_0 \equiv \begin{bmatrix} \widehat{a}_{00} & \widehat{a}_{0\cdot} \\ \widehat{a}_{\cdot 0} & \widehat{a} \end{bmatrix}.$$

Then the competitive profit conditions, (14), can be written in the matrix form:

$$(\widehat{A}_0)' \begin{bmatrix} w_0 \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ p \end{bmatrix}. \tag{15}$$

We impose

Assumption 4 $\widehat{A} \equiv \widehat{a} - \widehat{a}_{\cdot 0}\,(\widehat{a}_{00})^{-1}\,\widehat{a}_{0\cdot}$ is non-singular.

Under Assumptions 2 and 4, $\widehat{A}_0$ is non-singular and, hence, we can write the inverse of $(\widehat{A}_0)'$ as:

$$[(\widehat{A}_0)']^{-1} = \begin{bmatrix} (\widehat{a}_{00})^{-1}(1 + \widehat{a}_{0\cdot}\widehat{A}^{-1}\widehat{a}_{\cdot 0}) & -(\widehat{a}_{00})^{-1}\widehat{a}_{0\cdot}\widehat{A}^{-1} \\ -\widehat{A}^{-1}\widehat{a}_{\cdot 0}(\widehat{a}_{00})^{-1} & \widehat{A}^{-1} \end{bmatrix}'. \tag{16}$$

Defining $q \equiv (\widehat{a}_{00}^{-1}\widehat{a}_{0\cdot}\widehat{A}^{-1})'$, we can manipulate (15) and (16) to derive:

$$w = [(\widehat{A})']^{-1} p - q. \tag{17}$$

3 Dynamic competitive equilibrium

A *dynamic competitive equilibrium* is a tuple of positive quantities $\{c_t, y_t, x_{0t}, x_t\}_{t=0}^{\infty}$ and positive prices $\{p_t, w_{0t}, w_t\}_{t=0}^{\infty}$ satisfying (8)–(10) and (15) for given positive values of $x_{0,0}, x_{1,0}, \cdots$, and $x_{n,0}$.

Subtituting (13) and (17) into (8) and (9), we obtain the two fundamental equations governing the dynamical system in $\{x_t, p_t\}$:

$$x_{t+1} = A^{-1}x_t - z, \tag{18}$$

$$p_{t+1} = \rho^{-1}(\widehat{A})' p_t + (\widehat{A})' q. \tag{19}$$

3.1 Steady state

A *nondegenerate steady-state equilibrium* is a dynamic competitive equilibrium $\{c_t, y_t, x_{0t}, x_t, p_t, w_{0t}, w_t\}_{t=0}^{\infty}$ such that all quantities and prices are positive constants. Because our main focus is on the dynamic properties, we assume that a nondegenerate steady-state equilibrium exists and is unique.

From (18) and (19), the steady-state equilibrium values of $\{x, p\}$ are determined by:

$$[A^{-1} - I]x = z, \tag{20}$$

$$[\rho^{-1} I - (\widehat{A}^{-1})']p = -q. \tag{21}$$

We further impose

Assumption 5 $I - A$ is non-singular.

Then, x can be determined by inverting (20):

$$x = [I - A]^{-1} Az \quad \text{or} \quad x = [I - A]^{-1} a_{\cdot 0}(a_{00})^{-1}. \tag{22}$$

From $q = (\widehat{a}_{00}^{-1} \widehat{a}_{0\cdot} \widehat{A}^{-1})'$, (21) implies.

$$p = [I - \rho^{-1}(\widehat{A})']^{-1} \hat{a}'_{0\cdot}(\hat{a}_{00})^{-1}. \tag{23}$$

We must check whether the steady-state equlibrium is well-defined with positive quantities and prices. For convenience, we impose the following sufficient conditions:

Assumption 6 A has negative diagonal elements and positive off-diagonal elements; $\widehat{A}$ has diagonal elements less than ρ and positive off-diagonal elements.

The first part of this assumption implies that $I - A$ has positive diagonal elements and negative off-diagonal elements. Therefore, Assumption 5 holds true once Assumption 6 is imposed, under which $[I - A]^{-1}$ is a positive matrix and from (22), it satisfies

$$x = [I - A]^{-1} a_{\cdot 0}\, (a_{00})^{-1} > 0.$$

The second part of this assumption implies that $[I - \rho^{-1}(\widehat{A})']^{-1}$ is a positive matrix and, hence, from (23), we have:

$$p = [I - \rho^{-1}(\widehat{A})']^{-1} \hat{a}'_{0\cdot}\, (\hat{a}_{00})^{-1} > 0.$$

Then, applying (8) and (9) in the steady state, we must have: $y > 0$ and $w > 0$. It should be noted, however, that the conditions in Assumption 6 are strong and more than necessary. In the low dimension case (see Remark 2 of Section 4), we will discuss the implications of relaxing this assumption by allowing the lower (or upper) triangular off-diagonal elements of A to be negative.

We further impose:

Assumption 7

$$1 + a_{0\cdot} \left\{ \left(1 - a_{00}^{-1}\right) A^{-1} - I \right\} (I - A)^{-1} a_{\cdot 0} > 0.$$

Assumption 8

$$1 + \rho^{-1} \hat{a}'_{\cdot 0} \left\{ (1 - \hat{a}_{00}^{-1}) [\rho^{-1} (\widehat{A})']^{-1} - I \right\} [I - \rho^{-1} (\widehat{A})']^{-1} \hat{a}'_{0\cdot} > 0.$$

These assumptions are imposed to guarantee that c and w_0 are positive in the steady state.[2]

From (10), (12) and (22), we have:

$$\begin{aligned} c &= a_{00}^{-1} \left[1 + a_{0\cdot} A^{-1} a_{\cdot 0} - a_{0\cdot} A^{-1} x \right] \\ &= a_{00}^{-1} \left[1 + a_{0\cdot} A^{-1} \left\{ I - a_{00}^{-1} (I - A)^{-1} \right\} a_{\cdot 0} \right] \\ &= a_{00}^{-1} \left[1 + a_{0\cdot} a_{00}^{-1} A^{-1} \left\{ a_{00} (I - A) - I \right\} (I - A)^{-1} a_{\cdot 0} \right] \\ &= a_{00}^{-1} \left[1 + a_{0\cdot} a_{00}^{-1} A^{-1} \left\{ (a_{00} - 1) I - a_{00} A \right\} (I - A)^{-1} a_{\cdot 0} \right] \\ &= a_{00}^{-1} \left[1 + a_{0\cdot} \left\{ \left(1 - a_{00}^{-1}\right) A^{-1} - I \right\} (I - A)^{-1} a_{\cdot 0} \right], \end{aligned}$$

which is positive under Assumption 7.

Similarly, from From(15), (16) and (23), we derive:

$$\begin{aligned} w_0 &= \hat{a}_{00}^{-1} \left\{ 1 + \hat{a}'_{\cdot 0} [(\widehat{A})']^{-1} \hat{a}'_{0\cdot} - \hat{a}'_{\cdot 0} [(\widehat{A})']^{-1} p \right\} \\ &= \hat{a}_{00}^{-1} \left(1 + \hat{a}'_{\cdot 0} [(\widehat{A})']^{-1} \left\{ I - \hat{a}_{00}^{-1} [I - \rho^{-1} (\widehat{A})']^{-1} \right\} \hat{a}'_{0\cdot} \right) \\ &= \hat{a}_{00}^{-1} \left\{ 1 + \hat{a}'_{\cdot 0} [(\widehat{A})']^{-1} \left[I - \rho^{-1} (\widehat{A})' - \hat{a}_{00}^{-1} I \right] [I - \rho^{-1} (\widehat{A})']^{-1} \hat{a}'_{0\cdot} \right\} \\ &= \hat{a}_{00}^{-1} \left(1 + \rho^{-1} \hat{a}'_{\cdot 0} \left\{ (1 - \hat{a}_{00}^{-1}) [\rho^{-1} (\widehat{A})']^{-1} - I \right\} [I - \rho^{-1} (\widehat{A})']^{-1} \hat{a}'_{0\cdot} \right), \end{aligned}$$

which is positive under Assumption 8.

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[2] In the two-sector economy Assumption 7 is met under $1 - a_{00} > 0$ and $1 - a_{11} > 0$. Under Assumption 6, $A^{-1} < 0$ and $(1 - A)^{-1} > 0$ must hold. Therefore,

$$\begin{aligned} c &= a_{00}^{-2} A^{-1} (1 - A)^{-1} \left[a_{00} A (1 - A) - a_{10} a_{01} A + a_{01} a_{10} \left(1 - a_{00}^{-1}\right) \right] \\ &> a_{00}^{-2} (1 - A)^{-1} (a_{00} (1 - A) - a_{10} a_{01}) \\ &= a_{00}^{-2} (1 - A)^{-1} (1 - a_{11}) > 0. \end{aligned}$$

Moreover, Assumption 8 is met under $(\rho^{-1} \widehat{A})^{-1} (1 - \widehat{a}_{00^{-1}}) > 1$, which implies

$$\begin{aligned} w_0 &= (\hat{a}_{00})^{-1} \left\{ 1 + \hat{a}_{01} \widehat{A}^{-1} (\hat{a}_{10} - p) \right\} \\ &= (\hat{a}_{00})^{-1} \left\{ 1 + \hat{a}_{01} \hat{a}_{10} (1 - \rho^{-1} \widehat{A})^{-1} \rho^{-1} \left[\left(1 - \hat{a}_{00}^{-1}\right) (\rho^{-1} \widehat{A})^{-1} - 1 \right] \right\} \\ &> 0. \end{aligned}$$

3.2 Characterization of local dynamics

We are now prepared to characterize the local dynamic properties around the steady-state equilibrium.

It is clearly seen from (18) and (19) that the price dynamics is determined by prices alone. Moreover, the price dynamics is related to the factor intensity ranking in the value or social sense. Therefore, the associated characteristic roots are denoted by $\{\widehat{\lambda_1}, \ldots, \widehat{\lambda_n}\}$. Likewise, because the quantity dynamics is related to the factor intensity ranking in the physical or private sense, the associated characteristic roots are denoted by $\{\lambda_1, \ldots, \lambda_n\}$.

Using (19), we can characterize the price dynamics by,

$$\frac{\partial p_{t+1}}{\partial p_t} = \rho^{-1}(\widehat{A})'. \tag{24}$$

The eigen roots evaluated at the steady state $\{\widehat{\lambda_1}, \ldots, \widehat{\lambda_n}\}$ solve:

$$\det[\rho^{-1}(\widehat{A})' - \widehat{\lambda} I] = 0. \tag{25}$$

With respect to the quantity dynamics, we can obtain from (18) the following:

$$\frac{\partial x_{t+1}}{\partial x_t} = A^{-1}. \tag{26}$$

The eigen roots evaluated at the the steady state $\{\lambda_1, \ldots, \lambda_n\}$ solve:

$$\det[A^{-1} - \lambda I] = 0. \tag{27}$$

Note that the (i, j) element of $\hat{A}$ is:

$$(1 + s_{ij})a_{ij} - \frac{(1 + s_{i0})(1 + s_{0j})}{(1 + s_{00})} \cdot \frac{a_{i0}a_{0j}}{a_{00}}. \tag{28}$$

If (25) and (27) have more than n stable roots, we say that a steady state is *locally indeterminate* and if they have exactly n stable roots, we say that a steady state is *locally determinate*.

Let $\lambda_1, \ldots, \lambda_n$ be roots of A^{-1}. We consider the case roots of A^{-1} to be all stable; that is, $|\lambda_1|, \ldots, |\lambda_n| < 1$. Then we establish conditions under which local indeterminacy arises.

Theorem 1 *Let Assumptions 1–4 and 6–8 hold and suppose that all roots of A are outside the unit circle (i.e.* $|\lambda_1^{-1}|, \ldots, |\lambda_n^{-1}| > 1$). *Then by properly choosing* $s_{11}, \ldots, s_{nn} > 0$, *one can obtain a steady state that is locally indeterminate.*

PROOF: Let $S_{ij} \equiv \frac{(1+s_{i0})(1+s_{0j})}{(1+s_{00})(1+s_{ij})}$ and $M_{ij} \equiv \frac{a_{i0}a_{0j}}{a_{00}a_{ij}}$. Then $\widehat{A}$ can be written as:

$$\widehat{A} = \begin{pmatrix} (1+s_{11})\, a_{11}\, (1-S_{11}M_{11}) & \cdots & (1+s_{1n})\, a_{1n}\, (1-S_{1n}M_{1n}) \\ \vdots & \ddots & \vdots \\ (1+s_{n1})\, a_{n1}\, (1-S_{n1}M_{n1}) & \cdots & (1+s_{nn})\, a_{nn}\, (1-S_{nn}M_{nn}) \end{pmatrix}.$$

We consider two ways to make a root of $\widehat{A}$ stable. Start with the case that $\widehat{A}$ as well as A have negative diagonal elements and positive off-diagonal elements. We choose s_{ii}, $i = 0, \ldots, n$, so that $[I + \rho^{-1}\widehat{A}]$ has all positive diagonal elements. Then the Frobenius root of $[I + \rho^{-1}\widehat{A}]$ is positive, and the corrsponding root $\hat{\lambda}$ of $\rho^{-1}\widehat{A}$ is larger than -1. There are at least $n + 1$ stable roots of the system.

We can also choose s_{ij}, $i, j = 0, \ldots, n$ to make $\widehat{A}$ a positive matrix that satisfies:

$$[I - \rho^{-1}(\widehat{A})']\, p = \hat{a}_{01}'\, (\hat{a}_{00})^{-1} > 0.$$

This implies that the Frobenius root of $\rho^{-1}\widehat{A}$, which is positive, is less than 1 and all other roots $\hat{\lambda}$ of $\rho^{-1}\widehat{A}$ satisfy $1 > |\hat{\lambda}|$ (see Takayama 1974). For this case price dynamics has all stable roots and there are $2n$ stable roots in the system. Q.E.D.

Remark 1 Suppose that $\widehat{A} = A$, which is the case of no externality. Then if λ is a root of (27), $(\rho\lambda)^{-1}$ is a root of (25). This implies that a steady state is always locally determinate for $\rho = 1$, regardless of the factor intensity ranking.

We can establish a sufficient condition to rule out local indeterminacy, namely the ratios of social to private marginal products are uniform for all factors and across all sectors:

Theorem 2 *Let Assumptions 1–4 and 6–8 hold and suppose $s_{ij} = s_{00}$ for all $i, j = 0, \ldots, n$. Then the steady state equilibrium can not be locally indeterminate, regardless of the factor intensity rankings or the rate of time preferences.*

Proof: With $s_{ij} = s_{00}$ for all $i, j = 0, \ldots, n$, (28) becomes:

$$(1 + s_{00}) \left(a_{ij} - \frac{a_{i0}\, a_{0j}}{a_{00}} \right). \tag{29}$$

Moreover, we have:

$$\widehat{A} = (1 + s_{00})\, A, \tag{30}$$

and, therefore, if λ is a solution of (27), then $\rho^{-1}(1 + s_{00})\,\lambda^{-1}$ is a solution of $\rho^{-1}\widehat{A}$. Let $\hat{\lambda} = \rho^{-1}(1 + s_{00})\,\lambda^{-1}$. Then,

$$|\hat{\lambda}| \cdot |\lambda| = \rho^{-1}(1 + s_{00}) > 1. \tag{31}$$

This implies that there exists at most n stable roots among $\lambda_1, \ldots, \lambda_n, \hat{\lambda}_1, \ldots, \hat{\lambda}_n$. Hence, the steady state cannot be locally indeterminate. Q.E.D.

4 Two-sector economy ($n = 1$)

Let us now reexamine the two-sector case (sectors 0 and 1). Local indeterminacy in the two-sector model was studied by Benhabib *et al.* (2002) and Nishimura and Venditti (2003).

This exercise is useful because we allow for general socially constant returns production technologies (whereas Benhabib *et al.* (2002) and Nishimura and Venditti (2003) rely exclusively on the Cobb-Douglas specification).

We first check the existence of the steady-state equilibrium with positive quantities and prices. Note that in the two-sector case, $S_{11} = \frac{(1+s_{10})(1+s_{01})}{(1+s_{00})(1+s_{11})}$ and

$$\widehat{A} = (1 + s_{11})\left(a_{11} - S_{11}\frac{a_{10}a_{01}}{a_{00}}\right). \tag{32}$$

From (22) and (23) we have:

$$[1 - A]\,x = \frac{a_{10}}{a_{00}}, \tag{33}$$

$$[I - \rho^{-1}\widehat{A}]\,p = \frac{\widehat{a}_{01}}{\widehat{a}_{00}}. \tag{34}$$

We can establish necessary and sufficient conditions for the steady state to be locally indeterminate:

Proposition 1 *Let Assumptions 1–4 and 6–8 hold for $n = 1$ and suppose $(1 - S)\frac{a_{10}a_{01}}{a_{00}} > 1 - \frac{\rho}{1+s_{11}}$. Then local indeterminacy in this two-sector economy emerges iff*

$$-\frac{\rho}{1 + s_{11}} - (1 - S)\frac{a_{10}a_{01}}{a_{00}} < A < \min\left\{-1, \frac{\rho}{1 + s_{11}} - (1 - S)\frac{a_{10}a_{01}}{a_{00}}\right\}. \tag{35}$$

PROOF: Note that $p_0 = 1$ and x_0 is irreproducible labor. Using the root corresponding to sector 1 quantity $\lambda = A^{-1}$, we select $|A| > 1$ such that $|\lambda| < 1$. From (22), $A < 1$ must hold. If $A > 0$, then $\lambda = A^{-1} > 1$. This leads to a contradiction. Therefore, $A < 0$ (or $a_{00}a_{11}(a_{10}a_{01})^{-1} < 1$) and $-1 < \lambda = A^{-1} < 0$ must be the case.

From (25), the root corresponding to sector 1 price is $\hat{\lambda} = \rho^{-1}\widehat{A}$. If $|\rho^{-1}\widehat{A}| < 1$, then (23) implies $p > 0$. Let $S \equiv \frac{(1+s_{10})(1+s_{01})}{(1+s_{00})(1+s_{11})}$. Using $\hat{\lambda} = \rho^{-1}\widehat{A}$, in order for $|\hat{\lambda}| < 1$, we need

$$\left|a_{11} - S\frac{a_{10}a_{01}}{a_{00}}\right| < \frac{\rho}{1 + s_{11}}, \tag{36}$$

or

$$-\frac{\rho}{1 + s_{11}} - (1 - S)\frac{a_{10}a_{01}}{a_{00}} < A < \frac{\rho}{1 + s_{11}} - (1 - S)\frac{a_{10}a_{01}}{a_{00}}.$$

This, combined with the restriction $A < -1$, yields the required condition. Q.E.D.

Proposition 1 tells us that for local indeterminacy to arise, the ratios of social to private marginal products in the two sectors cannot be too much alike (otherwise, (35) can never be met). Consider the case with a sufficiently low subjective time-discount factor where indeterminacy cannot arise under Cobb-Douglas technologies (see Benhabib *et al.* 2002). We can show that even when the subjective time-discount factor converges to zero, (35) can still be valid if we choose $\{s_{ij}\}$ so that $S = 1 - \frac{a_{00}}{a_{10}a_{01}} > 0$ and, in this case, indeterminacy can still emerge.

The next proposition establishes a sufficient condition for the steady state to be locally determinate:

Proposition 2 *Let Assumptions 1–4 and 6–8 hold for $n = 1$ and suppose $s_{1j} = s_{0j}$ ($j = 1, 2$). Then the steady-state equilibrium in a two-sector economy is always locally determinate, regardless of the factor intensity rankings or the rate of time preferences.*

PROOF: With $s_{1j} = s_{0j} (j = 1, 2)$, we have $\widehat{A} = (1 + s_{01})A$ and, therefore,

$$\hat{\lambda}_1 = \frac{1 + s_{01}}{\rho} (\lambda_1)^{-1}, \tag{37}$$

or

$$|\hat{\lambda}_1| = \frac{1 + s_{01}}{\rho} |\lambda_1|^{-1} \quad \text{and} \quad |\lambda_1| = \frac{1 + s_{01}}{\rho} |\hat{\lambda}_1|^{-1}.$$

This implies that whenever λ_1 (or $\hat{\lambda}_1$) is within the unit circle, $\hat{\lambda}_1$ (or λ_1) is outside the unit circle. That is, the local dynamics is determinate. Q.E.D.

Therefore, dynamic indetermiacy cannot arise in the case where the ratios of social to private marginal products are uniform across both sectors, regardless of the factor intensity ranking or the time preference rate.[3]

Remark 2 It is important to note that the strong conditions imposed in Assumption 5 are more than necessary and that the condition for local determinacy cannot be generalized to the case of more than two sectors. Let $s_{ij} = s_{jj}$ for all $i, j = 0, \ldots, n$. Then (28) becomes:

$$(1 + s_{jj}) \left(a_{ij} - \frac{a_{i0}a_{0j}}{a_{00}} \right),$$

where it should be noted that $s_{00}, s_{11}, \ldots, s_{nn}$ can take different values. Therefore, we have:

$$\widehat{A} = AI_s, \tag{38}$$

where I_s is a diagonal matrix with diagonal elements $1 + s_{00}, \ldots, 1 + s_{nn}$. For illustrative purposes, we restrict our attention to a three-sector model (or $n = 2$). Let $\rho = 0.99$, $s_{00} = 0.5$, $s_{11} = 0.001$ and $s_{22} = 3$. Moreover, consider

$$A_0 = \begin{pmatrix} 0.1 & 0.5 & 1 \\ 0.5 & 1.7 & 5.72 \\ 1 & 4 & 5.9 \end{pmatrix}.$$

[3] We can show that this conclusion holds even with partial depreciation.

Therefore, we have relaxed the strong sufficient conditions imposed in Assumption 5 by allowing the lower triangular element of A to be negative:

$$A = \begin{pmatrix} -0.8 & 0.72 \\ -1 & -4.1 \end{pmatrix}.$$

We can use (38) to derive:

$$\rho^{-1}\left(\widehat{A}\right)' = \begin{pmatrix} -\frac{1+s_{11}}{\rho}\cdot 0.8 & -\frac{1+s_{22}}{\rho}\cdot 1 \\ \frac{1+s_{11}}{\rho}\cdot 0.72 & -\frac{1+s_{22}}{\rho}\cdot 4.1 \end{pmatrix} = \begin{pmatrix} -0.8089 & -1.0111 \\ 2.9091 & -16.5656 \end{pmatrix}.$$

In this case, all quantities and prices are positive, with,

$$x = \begin{pmatrix} 3.303 \\ 1.313 \end{pmatrix}, \ p = \begin{pmatrix} 0.912 \\ 1.669 \end{pmatrix}, \ c = 18.266, \ w_0 = 9.050,$$

and, hence, $y = x > 0$ and $w = \rho^{-1} p > 0$. The characteristic roots associated with the quantity dynamics governed by A^{-1} are: $\lambda_1 = -0.966$ and $\lambda_1 = -0.259$, whereas those with the price dynamics governed by $\rho^{-1}(\widehat{A})'$ are: $\hat{\lambda}_1 = -16.377$ and $\hat{\lambda}_2 = -0.998$. There are three negative real roots lying in $(-1, 0)$. Therefore, the steady-state equilibrium is locally indeterminate despite that the ratios of social to private marginal products are uniform across all three sectors.

Remark 3 For the Cobb-Douglas economy $y_j = x_{0j}^{\beta_{0j}} x_{1j}^{\beta_{1j}} \bar{x}_{0j}^{b_{0j}} x_{1j}^{b_{1j}}$, $j = 0, 1$, a_{ij} can be derived from $p_j \frac{\partial F}{\partial x_{ij}} = w_i$ and written $a_{ij} = p_j \beta_{ij}/w_i$. A charactristic root for this economy is obtained from (9) and (27). That is,

$$\begin{aligned} \lambda &= \left(a_{11} - a_{10}a_{01}a_{00}^{-1}\right)^{-1} \\ &= \rho^{-1}\beta_{00}\left(\beta_{00}\beta_{11} - \beta_{10}\beta_{01}\right)^{-1}. \end{aligned}$$

The other root is obtained from (9), (25) and (28). That is,

$$\hat{\lambda} = (1 + s_{11})\left(\beta_{00}\beta_{11} - S_{11}\beta_{10}\beta_{01}\right)/\beta_{00}.$$

5 Concluding remarks

We have examined the dynamic properties in a class of a discrete-time, multi-sector neoclassical growth models with sector-specific production externalities exhibiting social constant returns, relying on specific production functional forms. We have established conditions for the steady-state equilibrium to be locally determinate or locally indeterminate, which depends crucially on the ratios of the social to private marginal products. A natural extension is to allow for partial depreciation, which might be thought of as adding more stabilizing forces to capital adjustments and might, therefore, increase the likelihood of local indeterminacy.

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