



Dynamics in a transactions-based monetary growth model

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Abstract

This paper characterizes the dynamics of a monetary endogenous growth model in which money is introduced into the system via a transactions-cost technology. A monetary equilibrium that either satisfies the Friedman rule of the optimum quantity of money or accommodates the zero-inflation-rate policy is dynamically unstable. With Cagan-like hyperinflation, the monetary equilibrium may either be unstable or exhibit dynamic indeterminacy in which a variety of equilibrium outcomes emerge in transition. The rate of monetary expansion, the relative magnitudes of the intertemporal elasticity of substitution and the production technological parameter are crucial for determining the stability property of the model. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

The study of the effects of money growth on capital accumulation has attracted much attention in dynamic macroeconomics since the seminal work of

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Tobin (1965) and Sidrauski (1967).¹ Under the neoclassical growth framework, the steady-state rate of economic growth is determined by exogenous demographic and technological parameters and thus the effect of money growth is pertained to the level macroeconomic aggregates. Recent literature of new growth theory, pioneered by Lucas (1988), Romer (1986) and Rebelo (1991), has generated renewed interest in the study of money and growth for at least two reasons. On the one hand, its analytical framework is available for examining the long-run effects of monetary expansion on the endogenously determined rate of economic growth (e.g., Rebelo, 1988; Gylfason, 1991; Wang and Yip, 1992a; Gomme, 1993; van der Ploeg and Alogoskoufis, 1994; Palivos and Yip, 1995). On the other hand, the possible growth effect of money leads to a natural re-evaluation of the welfare costs of inflation (e.g., Lucas, 1993; Ireland, 1994; Chari et al., 1995; Jones and Manuelli, 1995; Wang and Yip, 1995; Wu and Zhang, 1998). However, without exceptions, all studies focus exclusively on the real effects of money and inflation in long-run balanced growth, leaving the *transitional dynamics* of the monetary equilibrium completely unexplored.

It is not surprising that the dynamics of monetary endogenous growth models remain a virgin land for exploration. There are at least two technical barriers for studying the dynamics analytically in any of the aforementioned models. First, with two state variables, capital and real monetary balances, the natural presence of a distortionary tax (due to non-zero inflation) can result in complex dynamics (see Bond et al., 1996).² Second, to allow for real growth effects of monetary expansion, the dynamics of the price (intertemporal relative price and factor prices) and the quantity (both controls and states) variables are mutually dependent. This generates a high dimension of the dynamical system, unable to reduce to a subsystem in a fashion analogous either to Benhabib and Perli (1994), with factor reallocation variables, or to Bond et al. (1996), with intertemporal relative price variables. To overcome these difficulties, it is necessary to depart from the structures of standard models in the existing literature while still maintaining the conventional role of money within the dynamic general equilibrium framework. The present paper is, to our knowledge, the first attempt at such an endeavor.

The central feature of this paper is to introduce money into an endogenously growing economy via its transactions role so that the underlying dynamics can be characterized analytically. To circumvent the analytical complexity

¹ For a survey of the literature on the neoclassical models of money and growth, see Dornbusch and Frenkel (1973) and Wang and Yip (1992b).

² In the class of two state variable, endogenous-growth models with constant returns production technologies, Bond, Wang and Yip show the presence of 'polarization' of the dynamics in the sense that a stable price adjustment is always associated with an unstable quantity adjustment, and vice versa. This polarization property may, however, break down in an economy with distortionary taxes.

encountered by the general two-sector model in Wang and Yip (1995), we consider an endogenous growth model of the AK-variation in which the aggregate production function is linear, depending on a unified general capital stock.³ Due to costly transactions, a fraction of real output is devoted to transactions. Money, by facilitating transactions, reduces the resources costs of transactions services, following in the spirit of the seminal work by Saving (1971). To allow for balanced growth, we assume that this fraction of transactions-used output is a decreasing function of the real balances–consumption ratio. This simple but plausible structure links the price dynamics to the dynamics of two transformed ratios (consumption–capital and real money balances–consumption ratios). It therefore enables us to completely characterize the balanced growth equilibrium and the underlying transitional dynamics in a straightforward 2×2 system in these transformed ratios.

By characterizing the balanced growth equilibrium, this paper has the following results. First, a higher rate of monetary expansion unambiguously reduces the common, balanced growth rate of (per capita) consumption, output, real money balances and capital accumulation. Second, the effects of a higher rate of money growth are to reduce both the real balances–consumption and consumption–capital ratios. These results support, in part, the transactions-cost effect (monetary expansion retards the rate of capital accumulation) and, in part, the Tobin asset substitution effect (higher anticipated inflation encourages a portfolio shift from real balances to capital). Third, since the real rate of return on capital is positively related to the endogenous growth rate of the economy, money growth creates an adverse effect on the real interest rate. This allows for a less-than-one-to-one adjustment of the nominal interest rate to anticipated inflation, consistent with Irving Fisher's conjecture and Summers' (1983) empirical evidence.

More importantly, the main contributions of the paper are the complete characterization of the dynamics of the monetary equilibrium in the presence of sustained economic growth. We find that a monetary equilibrium satisfying the Friedman rule (optimum quantity of money) is *dynamically unstable*. A zero-inflation monetary policy is locally unstable and thus *cannot* be attained by gradual adjustment. Moreover, we show that in a Cagan hyperinflationary environment, the optimizing monetary dynamics may be unstable *without relying on any ad hoc expectations mechanism*; it may also generate, *in the absence of positive externalities*, dynamic indeterminacy in that equilibrium outcomes may vary dramatically in transition toward the balanced growth path. Furthermore, we provide necessary and sufficient conditions for the monetary equilibrium to exhibit saddle-path stability. This is important, because a valid calibration exercise with shock perturbation and/or policy simulation must have

³ In particular, Wang and Yip (1995) rely on numerical analysis to pin down the property of the underlying dynamics of their model.

parameterization consistent with this stability condition. Finally, we generalize the basic model by considering a convex production technology with endogenous labor-leisure. We find that while variable capital-output ratios tend to serve as a stabilizing force, all but the result of instability under Friedman's rule remain qualitatively unchanged.

The remainder of the paper is organized as follows. Section 2 develops a transactions-based monetary endogenous growth model. In Section 3, we show the existence and uniqueness of the balanced growth equilibrium. Section 4 examines the stability properties whereas Section 5 characterizes the dynamics of the saddle-path equilibrium. In Section 6, we extend the basic model by considering a convex production technology with elastic labor. Finally, concluding remarks and possible extensions are presented in Section 7.

2. The model

We construct a benchmark endogenous growth model augmented with money. The economy consists of perfectly competitive goods and factor markets. The population growth rate is normalized to zero. The unified consumers–producers have perfect foresight and are infinitely lived and endowed with a general (physical, human and knowledge) capital of $k_0 > 0$ and a nominal money stock of $M_0 > 0$. In the benchmark model, the production technology is linear in capital whereas the labor supply is inelastic, normalized as unity (both assumptions to be relaxed in Section 6 below). Each agent makes intertemporal decisions, incorporating a budget constraint and determining consumption (c) allocation and capital (k) accumulation over his entire lifetime.⁴

The representative agent is assumed to have a lifetime preference exhibiting constant intertemporal elasticity of substitution

$$W = \int_0^{\infty} \frac{c^{1-\sigma}}{1-\sigma} e^{-\rho t} dt, \quad (1)$$

where ρ is the rate of time preference and $\sigma > 1$ is the inverse of the intertemporal elasticity of substitution.⁵ The case of log-linear utility is excluded to maintain homogeneity of the preferences.⁶ Denote the maximum sustainable

⁴ The dependence of the variables on time, t , will be suppressed for notational convenience.

⁵ Our assumption, $\sigma > 1$, is consistent with the empirical evidence presented by Hall (1988), whose results suggest that the inverse of the intertemporal elasticity of substitution is much less than one.

⁶ The homogeneity property allows us to have a monotone relationship between the endogenous rate of growth and individual welfare, as well as to fully characterize the underlying transitional dynamics. For a detailed discussion of this assumption, the reader is referred to Bond, et al. (1996).

rate of consumption growth as θ_C^{\max} . To ensure that the lifetime utility is bounded, we impose the Brock–Gale condition.

Condition U (Bounded lifetime utility). $\rho > (1 - \sigma)\theta_C^{\max}$.

The individual production of a single consumption/investment good takes a linear, constant-returns-to-scale technology,

$$Y = Ak, \tag{2}$$

where A is a constant scaling factor measuring the productivity of the general capital.

To introduce money into the above prototypical model of endogenous growth, we consider pecuniary costs associated with individual transactions. Denote M and m , respectively, as nominal and real money balances. The real resource costs required to facilitate transactions services in the economy is denoted by T and is given by $T = S(m, c)Y$, where S is the fraction of final output Y devoted to transactions. Following in the spirit of Saving (1971), we assume that this transactions cost function, $S(m, c)$, is strictly decreasing and convex in m , strictly increasing and convex in c , and twice continuously differentiable in m and c . To ensure existence of a BGP, we further assume that $S(m, c)$ is homogeneous of degree zero in m and c , i.e., $S(m, c) = s(m/c)$.⁷ In summary, $s(m/c)$ possesses the following properties:⁸

$$\begin{aligned} s' &< 0, \\ s'' &\geq 0, \\ \lim_{m/c \rightarrow 0} s(m/c) &= 1, \\ \lim_{m/c \rightarrow 1} s(m/c) &= \bar{s} < 1. \end{aligned}$$

Notice that the higher the ratio of real money balances to consumption, the lower will the fraction of output devoted to transactions be. However, the marginal benefit of holding money for transactions purposes exhibits diminishing returns. The limit conditions are used to ensure the existence of an interior demand for real money balances. During periods of accelerating money growth, individuals hold less real money balances since the opportunity cost of money holdings increases. Hence, as the money growth rate rises, a larger fraction (s) of real resources is allocated for costly transactions. An example of a transactions cost function satisfying the above properties is given by: $s(m/c) = 1 - a(m/c)^b$, where $0 < a < 1$ and $0 < b \leq 1$.

⁷ We are not aware of any other degree of homogeneity for s which will ensure existence of a BGP.

⁸ When s becomes a step function in which $s(m/c) = s_L$ for $m/c \geq 1$ and s_H for $m/c < 1$, with $s_L < s_H$, our structure mimics the cash-in-advance model.

Under the linear production technology (2), the budget constraint facing each agent is given by

$$\dot{k} + \dot{m} = Ak(1 - s) - c - \pi m + \tau, \tag{3}$$

where π is the inflation rate and τ measures lump-sum real money transfers from the government. Following the conventional wisdom, money supply is assumed to expand at a constant rate, μ . In money market equilibrium, money demand equals money supply and thus the evolution of real money balances can be expressed as

$$\dot{m} = m(\mu - \pi). \tag{4}$$

Equilibrium money market together with balanced government budget imply: $\tau = \mu m$.

The representative agent’s intertemporal optimization is to maximize the lifetime utility W specified in (1), subject to the budget constraint (3) and the initial conditions,

$$k(0) = k_0 > 0 \text{ and } M(0) = M_0 > 0.$$

Thus, this is a standard one-control, two-state and one-evolution equation optimal control problem, which can be more conveniently solved by transforming the problem using a slack variable $q = dm/dt$. Let the current value Hamiltonian be

$$H = \frac{c^{1-\sigma}}{1-\sigma} + \lambda_1 \{Ak(1 - s) - c - \pi m - q + \tau\} + \lambda_2 q,$$

where λ_1 and λ_2 are the co-state variables associated with Eq. (3) and the slack variable equation, respectively.

By straightforward application of the Pontryagin Maximum Principle, the first-order necessary conditions can be derived as

$$c^{-\sigma} + \lambda_1 \{Aks'mc^{-2} - 1\} = 0, \tag{5}$$

$$\lambda_1 = \lambda_2, \tag{6}$$

$$\frac{\dot{\lambda}_1}{\lambda_1} = \rho - A(1 - s), \tag{7}$$

$$\frac{\dot{\lambda}_2}{\lambda_2} = \rho + Aks'c^{-1} + \pi. \tag{8}$$

Eq. (5) is the analogue of the Keynes–Ramsey condition, except that the marginal utility of consumption now depends on an additional term which represents the marginal cost of consumption attributable to transactions costs. In equilibrium, the shadow price of the two stores of value, physical capital and money, must be equal, which is given by (6). Eqs. (7) and (8) are the Euler

equations for physical capital and money, respectively. In Eq. (8), the second term represents the marginal benefit of holding money. Under the specification of preferences, transactions and production technology, these conditions, (3) and (5)–(8), are sufficient for optimization provided that the following transversality conditions hold:

$$\lim_{t \rightarrow \infty} \lambda_1 k(t) e^{-\rho t} = 0,$$

$$\lim_{t \rightarrow \infty} \lambda_2 m(t) e^{-\rho t} = 0.$$

3. Balanced growth path

In this section, we establish the existence and uniqueness of a *balanced growth path* (BGP). This is an important preliminary step towards analyzing the dynamics of the system. A *monetary equilibrium* is a set of paths $\{c, k, m, s, \pi\}_{t=0}^{\infty}$ that solves the optimization problem by maximizing the lifetime utility W in (1) subject to the budget constraint (3) for given initial conditions, in which the money market equilibrium condition (4) holds. Thus a monetary equilibrium solves (3), (4) as well as the first-order necessary conditions (5)–(8), and satisfies the initial and the transversality conditions. A (non-degenerate) *BGP monetary equilibrium* is a set of monetary equilibrium paths $\{c, k, m, s, \pi\}$ such that each of the quantity variables, c, k and m , grows at a constant rate. It is easily shown that on the BGP, the following relationships hold:⁹

$$\frac{\dot{c}}{c} = \frac{\dot{k}}{k} = \frac{\dot{m}}{m} = \theta \text{ and } \dot{s} = 0. \tag{9}$$

That is, along a BGP, the economy exhibits *common growth* in which consumption, capital and real money balances all grow at a common rate θ . Moreover, the fraction of real output devoted to transactions services is constant on a BGP.

Taking log derivatives of (5) and combining with (7), we derive the modified golden rule:

$$\theta = \frac{A(1-s) - \rho}{\sigma}. \tag{10}$$

⁹ From (7) and the definition of a BGP, s is determined to be constant, which implies that m/c is a constant. Hence, c and m grow at the same rate on the BGP. Examining the goods market equilibrium condition, which is given by $\dot{k} = Ak(1-s) - c$, we observe that if physical capital grows at a constant rate on the BGP and s is constant, as determined above, then c/k is a constant. Therefore, c, k, m , and Y all grow at a common rate, θ , on the BGP.

Eq. (10) states that in equilibrium, the gross return to consumption should equal to the return to investment net of the transactions cost, which pins down the rate of consumption growth. To ensure a non-degenerate BGP, we assume,

Condition G (Positive growth) $A(1 - \bar{s}) > \rho$.

Recall that \bar{s} is the constant upper-bound of the transactions cost function, s . This condition is an analogy to the Jones–Manuelli condition which guarantees the endogenously determined rate of economic growth to be always positive. From (7) and (8), we obtain

$$A(k/c)(-s') = A(1 - s) + \pi = i. \tag{11}$$

Eq. (11) is a *no-arbitrage condition* stating that the marginal benefit of holding money from reducing transactions costs (i.e., the term on the LHS), equals the marginal cost of holding money (i.e., the forgone nominal interest $i = A(1 - s) + \pi$ on the RHS in accord with the Fisher equation).¹⁰

We can combine (9)–(11) and the goods market equilibrium condition, to reduce to a two-equation, two-unknown system in terms of $Z_1 = c/k$ and $Z_2 = m/c$:

$$(\sigma - 1)A(1 - s(Z_2)) - \sigma Z_1 = -\rho, \tag{12}$$

$$-As'(Z_2)/Z_1 - Z_1 = \mu. \tag{13}$$

Fig. 1 plots Eqs. (12) and (13). The two loci provide the combinations of Z_1 and Z_2 for which the growth rates of consumption, physical capital, real balances and level of transaction costs are stationary. For both linear and non-linear transactions cost functions, we can utilize Figs. 1a and b to show that there exists a unique balanced growth monetary equilibrium.

Theorem 1. Under Conditions U and G, there exist a unique balanced growth monetary equilibrium.

Proof. Under Condition U, (3)–(8) can be used to solve for the BGP monetary equilibrium and the transversality conditions are clearly satisfied. The locus given by (12) is monotonically increasing and concave in (Z_2, Z_1) -space, whereas the locus given by (13) is monotonically decreasing and may be concave or convex depending on the third derivative of the transaction cost function s . From (12), we have

$$Z_2 \rightarrow 0 \Rightarrow Z_1 \rightarrow \frac{\rho}{\sigma},$$

$$Z_2 \rightarrow 1 \Rightarrow Z_1 \rightarrow \frac{\rho + (\sigma - 1)A\bar{s}}{\sigma},$$

¹⁰ More precisely, the Fisher equation is given by $i = r + \pi$, where $r = A(1 - s)$ is the real return to capital.

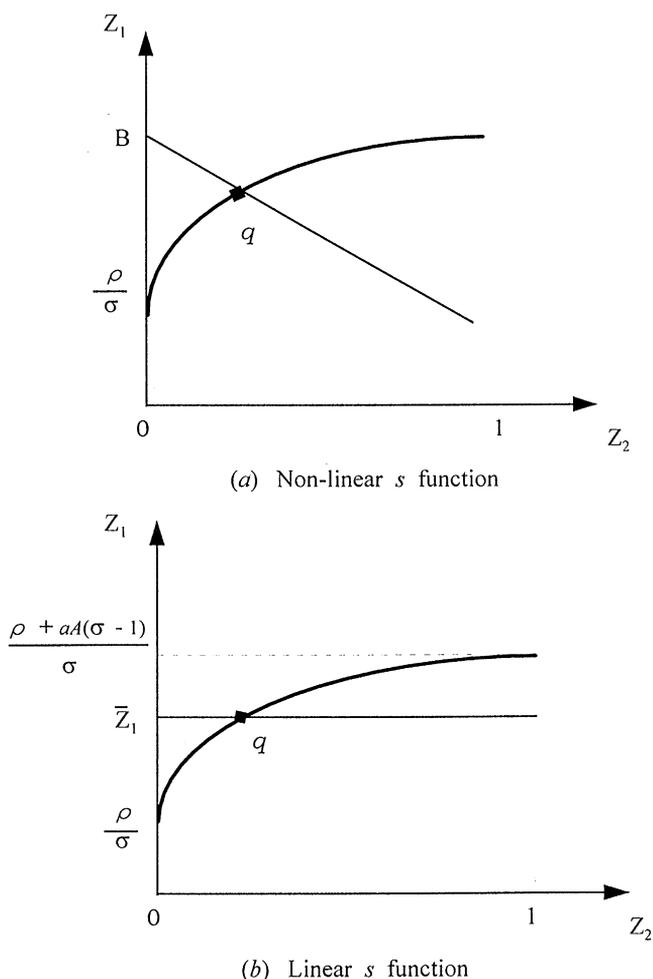


Fig. 1. Existence of BGP.

Let Z_1^{\max} and Z_1^{\min} denote the corresponding values determined by evaluating s' at $Z_2 = 0$ and 1 , respectively, and solving the quadratic equation (13). Thus, the downward sloping locus described by (13) has to have $Z_1^{\max} > \rho/\sigma$ in order for the two loci to intersect with each other, which is automatically satisfied given $\sigma > 1$. This is illustrated by Fig. 1a. The case of linear s is depicted in Fig. 1b, which can be regarded as a special case of the above analysis. Finally, the non-degenerate property is ensured by Condition G. \square

Totally differentiating (10), (12) and (13), we can obtain the comparative-static results with respect to the two transformed ratios, Z_1 and Z_2 , and the

endogenous growth rate, θ , in response to the money growth rate (μ):¹¹

$$\frac{dZ_1}{d\mu} = \frac{1}{(-1 + As'Z_1^{-2})} < 0, \tag{14a}$$

$$\frac{dZ_2}{d\mu} = \frac{\sigma}{(\sigma - 1)A(-s')} \left(\frac{dZ_1}{d\mu} \right) < 0, \tag{14b}$$

$$\frac{d\theta}{d\mu} = \frac{1}{\sigma - 1} \left(\frac{dZ_1}{d\mu} \right) < 0, \tag{14c}$$

$$\frac{ds}{d\mu} = \frac{-\sigma}{(\sigma - 1)A} \frac{dZ_1}{d\mu} > 0. \tag{14d}$$

Since the effect of a higher money growth rate is to discourage the real money holding, it results in a lower real balances–consumption ratio and lead to a larger fraction of real output devoted to transactions services (and away from accumulating the productive capital). Thus, higher monetary growth suppresses the endogenous rate of economic growth. Interestingly, the consumption to capital ratio decreases in response to a higher monetary growth rate, analogous to the ‘asset substitution effect’ or the Tobin effect referred to in the neoclassical money and growth literature.¹² However, monetary expansion in the long run is unambiguously growth-retarding.

In addition, we can derive the relationship between money growth and the nominal interest rate (i) as well as the inflation rate (π). By (4) and the common growth property, we learn that along the BGP, the inflation rate is pinned down by, $\pi = \mu - \theta$. This, together with the Fisher equation specified above, implies $i = A(1 - s) + \mu - \theta$. Totally differentiating these two expressions and utilizing (14c) and (14d) yields the following comparative statics regarding the inflation rate and the nominal interest rate:¹³

$$\frac{d\pi}{d\mu} = 1 - \frac{d\theta}{d\mu} = 1 + \frac{1}{\sigma - 1} \frac{c}{c + ik} > 1, \tag{15a}$$

$$\frac{di}{d\mu} = 1 + \frac{dZ_1}{d\mu} = \frac{ik}{ik - c} < 1. \tag{15b}$$

Due to the negative effect of monetary expansion on the endogenous growth rate, the inflation rate must respond more than proportionately to a rise in the

¹¹ The results are obtained based on the assumption that $\sigma > 1$ (i.e., the inverse of the intertemporal elasticity of substitution is less than unity).

¹² Of course, in the exogenous growth framework, the Tobin effect implies that money growth results in higher level of capital stock accumulated. In our balanced growth setup, the levels of quantities are unbounded asymptotically and such an effect is reflected in a rise in the capital stock.

¹³ In deriving the results, we have used (11) which implies $-As' = i/Z_1 = ik/c$.

money growth rate. Accordingly, we expect a partial (less-than-one-to-one) adjustment in the nominal interest rate to anticipated inflation.

Finally, it is worth discussing the effect of money growth on the income velocity of money since it may shed some light on explaining the empirically observed positive correlation between money growth and velocity found by Bordo and Jonung (1990). The implications of an increase in the money growth rate for income velocity can be determined by examining the ratio of income to real balances which is given by

$$v = \frac{A}{Z_1 Z_2}. \quad (16)$$

As the money growth rate rises, the opportunity cost of holding money increases. Thus, both the consumption to capital ratio (Z_1) and the real balances to consumption ratio (Z_2) decrease due to the intertemporal substitution effect and the asset substitution effect, respectively. These lead to an unambiguous increase in the money velocity, v . Our model illuminates the long-run channels through which the money growth rate influences income velocity positively. In summary, we have:

Proposition 1. In balanced growth monetary equilibrium, the effects of money growth are to

- (i) decrease the real balances to consumption ratio, the consumption to capital ratio, the real balances to capital ratio and the rate of common growth;
- (ii) increase the real resources allocated to transactions and the income velocity; raise the rate of inflation more than proportionately and the rate of nominal interest less than proportionately.

4. Stability analysis

In this section, we investigate the stability properties of the dynamical system in the neighborhood of the BGP. As can be seen from Section 3 above, the dynamical system reduces in a block-recursive manner to a 2×2 system in terms of the two transformed stationary ratios, Z_1 and Z_2 . Denote the BGP values of these two transformed variables as Z_1^* and Z_2^* , respectively. The local stability properties can therefore be obtained by examining a linearized system of the dynamics of Z_1 and Z_2 around their BGP values, Z_1^* and Z_2^* . To simplify the analysis of the underlying equilibrium dynamics, we assume that the transactions cost function, $s(m/c)$, is quasi-linear in the sense that its second and higher-order derivatives are sufficiently small.¹⁴

¹⁴ This assumption is consistent with the regularity conditions imposed on the transactions cost technology by Saving (1971) and Wang and Yip (1992b).

We first reduce the system to two differential equations for Z_1 and Z_2 , delineating the dynamics of the c/k and m/c ratios. Define $x = 1 - AZ_1^{-1}Z_2s' > 1$. Taking log derivatives of (5) and utilizing (7), we obtain

$$\frac{\dot{c}}{c} = [\rho - A(1 - s)] \left[\frac{x}{x(2 - \sigma) - 2} \right] + \left[\frac{x - 1}{x(2 - \sigma) - 2} \right] \left[\frac{\dot{k}}{k} + \frac{\dot{m}}{m} \right]. \tag{17}$$

Next, rewriting (3) and combining (4) and (11) yield

$$\frac{\dot{k}}{k} = A(1 - s) - \frac{c}{k}, \tag{18}$$

$$\frac{\dot{m}}{m} = \mu + A(1 - s) + \frac{As'}{c/k}. \tag{19}$$

Using the definitions of Z_1 and Z_2 and noting that s is a function of Z_2 alone, we can manipulate Eqs. (17)–(19) to derive the 2×2 differential equation system of Z_1 and Z_2 :

$$\begin{aligned} \frac{\dot{Z}_1}{Z_1} &= [x(2 - \sigma) - 2]^{-1} [A(1 - s)x(\sigma - 1) + \rho x - Z_1(1 + x(\sigma - 1)) \\ &\quad - A^2(s')^2 Z_2 Z_1^{-2} + (x - 1)\mu], \end{aligned} \tag{20}$$

$$\begin{aligned} \frac{\dot{Z}_2}{Z_2} &= - [x(2 - \sigma) - 2]^{-1} \{ A(1 - s)x(\sigma - 1) + \rho x + AZ_2s' \\ &\quad + [x(\sigma - 1) + 1](\mu + As'Z_1^{-1}) \}. \end{aligned} \tag{21}$$

Linearizing (20) and (21) around the BGP values of Z_1 and Z_2 yields

$$\begin{bmatrix} \dot{Z}_1 \\ \dot{Z}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} Z_1 - Z_1^* \\ Z_2 - Z_2^* \end{bmatrix}, \tag{22}$$

where

$$\begin{aligned} a_{11} &= [As'Z_1^{-2}(1 - x) - 1 - x(\sigma - 1)]jZ_1, \\ a_{12} &= -As'(\sigma - 1)xjZ_1, \\ a_{21} &= As'Z_1^{-2}Z_2^2 \{ -Z_1 + [x(\sigma - 1) + 1]Z_2^{-1} \}j, \\ a_{22} &= As'(\sigma - 1)xjZ_2, \\ j &= [x(2 - \sigma) - 2]^{-1}. \end{aligned}$$

Denote the determinant and the trace of the pre-multiplied matrix (D) on the RHS of (22) as $\text{Det}(D)$ and $\text{Tr}(D)$, respectively. Prior to our formal stability analysis, we would like to remind the reader that a unique BGP exists. It is also important to remind the reader that both Z_1 and Z_2 are jumping variables.

However, recall that $Z_1 Z_2 = M/(PK)$, where M and K cannot jump and P should be fixed on impact from any perturbation of the money expansion rate for any ‘honest government’ (Auernheimer, 1974).¹⁵ Therefore, the value of $Z_1 Z_2$ must be fixed on impact. This limits the short-run movements of the two transformed ratios in the sense that only one of them is a free jumping variable while another must change according to the above constraint. Thus, our 2×2 dynamical system displays the same feature as the conventional one: the *saddle-path* case has one root with negative and one with positive real part. That containing both roots with negative real parts becomes a *sink* which displays *dynamic indeterminacy*, whereas that having both roots with positive real parts is a *source* which implies *dynamic instability*. In the case of saddle-path stability, the intersection of the $Z_1 Z_2 = M/(PK)$ locus and the stable saddle represent the initial point, which has a unique transition path converging to the BGP.

To facilitate further analysis, we consider the following conditions.

Condition D (Positive determinant) $\sigma > 1 + \max\{A, 1/A\}$.

Condition H (Hyperinflationary economy). $\theta \rightarrow 0$ and $i \rightarrow \pi \rightarrow \mu$.

Condition D sets an upper bound for the intertemporal elasticity of substitution (σ^{-1}) that is more restrictive than the bounded lifetime utility condition (Condition U). Under Condition H, growth ceases and the rate of inflation is driven completely by the monetary expansion rate. This is an environment Cagan (1956) sets up for the study of hyperinflation.¹⁶ These conditions are shown, in the following lemmas, to be useful in determining the properties of the determinant and the trace of the matrix D.

Lemma 1. Under Condition D, $\text{Det}(D) > 0$.

Proof. From (22), we have

$$\begin{aligned} \text{Det}(D) = & -j^2 Z_2 s'(\sigma - 1)x[\sigma\{i(1 + A) - \mu A\} - iZ_2\{i[A - (\sigma - 1)] \\ & + (i - \mu)[1 - (\sigma - 1)A]\}], \end{aligned}$$

where $i = \theta(\sigma - 1) + \rho + \mu > 0$. Under Condition D, the second term in the square bracket on the RHS of the above expression is negative. Since $A > 0, i > \mu, \sigma > 1$ and $s' < 0$, it is verified that $\text{Det} > 0$. \square

¹⁵ P is the price of the consumption good. In particular, the normalized price level, P/M , is the relevant ‘stationary’ variable (in the growth rate sense), whose growth path must be fixed on impact so that the government does not receive extra inflationary tax revenues from the jump of the price growth path given sustained economic growth. Since the growth rate of nominal money balances is fixed on impact, the price path must be fixed as well.

¹⁶ We would like to remind the reader that under Condition H, growth ceases and the endogenous-growth related conditions are no longer applicable.

Lemma 2. If $\mu \leq 0$ and $\sigma \geq 2$, then $\text{Tr}(D) > 0$.

Proof. From (22), it is shown that

$$\text{Tr}(D) = j[i^2 Z_2 - (i - \mu)\{\sigma + i(\sigma - 1)Z_2\} - i(i - \mu)(\sigma - 1)(1 + iZ_2)Z_2].$$

When $\sigma \geq 2, j < 0$. If $\mu \leq 0$, then the square bracket term on the RHS of the above expression is negative, which implies that $\text{Tr} > 0$. \square

Lemma 3. Under Condition H, if $\sigma \geq 2$ then $\text{Tr}(D) < 0$.

Proof. Given $i = \mu$, the trace of D simplifies to $\text{Tr}(D) = j i^2 Z_2$, which is negative since $j < 0$ under $\sigma \geq 2$.

Remark. Notice that Condition D implies $\sigma \geq 2$, but not vice versa.

We are now ready to establish the main results regarding the stability properties of the dynamical system. We assume without further mentioning that Conditions U and G are satisfied throughout this section. To begin, we consider the Friedman rule of the optimum quantity of money in that the social cost of money holding measured by the nominal rate of interest is driven down to zero (i.e., $i = 0$).

Proposition 2. A monetary equilibrium satisfying the Friedman rule is locally unstable.

Proof. If the Friedman rule is implemented, then $i = 0$. In this case, the determinant and the trace of D reduce to

$$\text{Det} = j^2 Z_2 A'(\sigma - 1)\mu\sigma > 0,$$

$$\text{Tr} = j\mu\sigma > 0,$$

where $j = -1/\sigma < 0$ and $\mu < 0$. Thus, both characteristic roots are positive, implying dynamic instability. \square

This proposition implies that there exist no gradual adjustments for a monetary economy to achieve the Friedman rule.

The next proposition concerns the stability property of a monetary equilibrium with non-positive rates of money growth.

Proposition 3. Under Condition D, if $\mu \leq 0$, then the monetary equilibrium is locally unstable.

Proof. From Lemmas 1 and 2 and the fact that $\max\{A, 1/A\} \geq 1$, the presumptions imply $\text{Det}(D) > 0$ and $\text{Tr}(D) > 0$. Thus, the dynamical system (22) contains two characteristic roots with positive real parts. \square

Thus, we learn that a zero-inflation policy or a deflationary policy is not sustainable, since any small perturbation would shift the economy away from the original balanced growth monetary equilibrium. This may explain why, in practice, central banks exercise with a positive rate of monetary expansion, despite the associated welfare cost of inflation. We provide the following example for illustrative purposes.

Example 1. Consider the following parameterization: $A = 2$, $\mu = -0.05$, $\sigma = 5.5$, $s' = -0.439$ and $\rho = 0.03$. In this case, the eigenvalues are $1.08 + 1.03i$ and $1.08 - 1.03i$ [$\tau = (-1)^{1/2}$], which indicates that the steady state is a source (unstable spiral).

In the next proposition, we study a case that is particularly relevant for a Cagan-like hyperinflationary environment.

Proposition 4. Under Conditions D and H, dynamic indeterminacy emerges in which there exists a continuum of transition paths converging to the unique BGP.

Proof. From Lemmas 1 and 3, $\text{Det}(D) > 0$ and $\text{Tr}(D) < 0$, implying that both characteristic roots have negative real parts and hence the dynamic equilibrium is a sink. \square

However, Proposition 4 does not necessarily rule out the possibility of dynamic instability of monetary equilibria in hyperinflationary environment. To illustrate this, we consider,

Example 2. Let $i \cong \mu$, $A = 2.5$, $\mu = 0.8$, $\sigma = 4$, $s' = -0.25$ and $\rho = 0.03$. In this case, the eigenvalues are 0.49 , $0.91i$ and $0.49 - 91i$, which indicates that the steady state is a source (unstable spiral).

In a hyperinflationary episode delineated by Cagan (1956), economic growth ceases, the real returns to capital is nil and the rates of monetary expansion and inflation are about the same. Thus, the nominal interest rate approaches the money growth rate. Cagan models the hyperinflationary phenomenon as a monetary equilibrium of instability, in which the instability is based on an ad hoc adaptive expectations mechanism. In contrast, our result indicates that in such a situation, it is possible that the monetary equilibrium is dynamically unstable in an intertemporally optimizing, perfect foresight framework. Furthermore, we point out that in a hyperinflationary environment, dynamic indeterminacy may occur despite the absence of positive externality that is commonly believed to be necessary for this transitional non-uniqueness (see Benhabib and Perli, 1994). As a consequence, it is likely that the equilibrium outcomes in transition be very divergent as in many observed hyperinflationary episodes (such as the post-WWI

German, the post-WWII Chinese and the post-1980 Latin American and Israeli hyperinflation experiences).¹⁷

Finally, we seek for necessary and sufficient conditions to ensure saddle-path stability. Consider,

Condition S (Saddle-path stability)

$$\sigma \left[\frac{i}{A} + (i - \mu) \right] < AiZ_2 \left[\frac{i}{A^2}(A - (\sigma - 1)) + \frac{(i - \mu)}{A} \left\{ \frac{1}{A} - (\sigma - 1) \right\} \right]$$

Proposition 5. Under Condition S, the monetary equilibrium is saddle-path stable.

Proof. From the proof of Lemma 1, it is straightforward to see that Condition S ensures that $\text{Det} < 0$, which implies that there is one characteristic root with positive and one with negative real parts. □

Example 3. Let $i \cong \mu$, $A = 1.5$, $\mu = 0.8$, $\sigma = 1.4$, $s' = -0.25$ and $\rho = 0.03$. In this case, the eigenvalues are -80 and 43 , which indicates that the steady state is a saddle point.

In the money and endogenous growth literature, calibration exercises are frequently used to evaluate the growth effects of money and the welfare costs of inflation. Such quantitative analyses are usually based on the assumption that the system is saddle-path stable (and dynamically determinate) without a formal proof. Our proposition provides a necessary and sufficient condition for saddle-path stability. This is important, because a valid calibration exercise with shock perturbation and/or policy simulation must have parameterization consistent with this or an analogous stability condition. For example, we can study the stability properties in our economy under the parameterization in Chari et al. (1995) and Jones and Manuelli (1995): $\rho = \theta = 2\%$ and $\sigma = 2$ (which are commonly used in many endogenous growth models).

Example 4. Given $\rho = \theta = 0.02$ and $\sigma = 2$, one obtains $j = -0.5$ and $i - \mu = (\sigma - 1)\theta + \rho = 4\%$. As a consequence, we have $\text{Det}(D) = -0.25Z_2s'x[\sigma\{(\mu + 0.04) + 0.04A\} + Z_2\mu(\mu + 0.04)^2(1 - A)]$ and $\text{Tr}(D) = -0.5[(\mu + 0.04)^2Z_2 - 0.04\{1 + [1 + (\mu + 0.04)Z_2]^2\}]$. If $A \leq 1$, then $\text{Det}(D) > 0$ for all $\mu \geq 0$ and the BGP equilibrium is a sink or a source, depending on the sign of $\text{Tr}(D)$ (e.g., a source with $\mu = 0.05$ and $Z_2 = 0.2$; a sink with $\mu = 1.5$ and $Z_2 = 0.05$). If $(A - 1)[Z_2\mu(\mu + 0.04)^2 - 0.08] > 2(\mu + 0.08)$, the BGP equilibrium is a saddle (e.g., with $A = 3.2$ and $\mu = Z_2 = 1$).

¹⁷ For example, see Tallman and Wang (1995) for the German and Chinese and Rogers and Wang (1993) for the Israeli and Latin American cases. Similar evidence from various cases is also found in Bruno and Easterly (1996).

Finally, we can approximate a cash-in-advance type economy by assuming $Z_2 = 1$ (i.e., $c = m$). Numerical exercises show that under $\rho = \theta = 2\%$ and $A \leq 1$, a saddle emerges only when the intertemporal elasticity of substitution ($1/\sigma$) is sufficiently high.

Example 5. Given $\rho = \theta = 2\%$, $\sigma = 0.01$, $\mu = 0.02$ and $A = 0.02244$ (so that $1 - s = 0.9$), one obtains $i = 0.0202$ and $i - \mu = 0.0002$ and Condition S is satisfied.

The above arguments are summarized by

Theorem 2. Under Conditions U and G, the monetary equilibrium may be dynamically unstable (a source), indeterminate (a sink) or saddle-path stable, depending crucially on the rate of monetary expansion, the intertemporal elasticity of substitution and the production technological parameters.

5. Characterization of the saddle-path dynamics

In this section, the comparative (local) dynamics of the saddle-path stable equilibrium is analyzed. In particular, we examine the dynamic responses of the consumption to capital ratio (Z_1), money to consumption ratio (Z_2), the inflation rate (π), and growth rates of real balances (θ_m), capital stock (θ_k) and consumption (θ_c), to an unanticipated permanent increase in the money growth rate (μ).

Since the monetary equilibrium is a saddle point, the associated characteristic roots can be denoted as: $\psi_1 < 0$ and $\psi_2 > 0$. The stable dynamic paths of Z_1 and Z_2 are governed by

$$Z_1(t) = Z_1^* + (Z_1(0) - Z_1^*)e^{\psi_1 t}, \tag{27}$$

$$Z_2(t) = Z_2^* + \left(\frac{a_{21}}{\psi_1 - a_{22}} \right) (Z_1(t) - Z_1^*), \tag{28}$$

Eqs. (27) and (28) enable us to determine the slope of the stable arm, which is given by

$$\frac{Z_2(t) - Z_2^*}{Z_1(t) - Z_1^*} = \frac{a_{21}}{(\psi_1 - a_{22})}. \tag{29}$$

The stable arm is positively sloped if $\sigma \geq 2$.

Consider the case where the economy is initially in a balanced growth equilibrium at point E. The dynamics of Z_1 and Z_2 are described by a saddle point in (Z_1, Z_2) space. Now, suppose that the monetary authority impose an unanticipated permanent increase in μ at a particular date, say $t = 0$, and that the values of the state variables in the model previous to $t = 0$ are at their initial

balanced growth equilibrium levels. An increase in the money growth rate has been shown to lead to a long-run decrease in the consumption–capital ratio (Z_1) and real balances–consumption ratio (Z_2). The BGP equilibrium therefore shifts from E to E' in the phase diagram (Fig. 2).

As mentioned in Section 4, the growth path of real balances (M/P) must be fixed on impact at $t = 0$ according to the ‘honest government’ assumption. The short-run movements of Z_1 and Z_2 are pinned down by the intersection of the $Z_1 Z_2 = M/(PK)$ locus and the new saddle path, $X'X'$. The economy jumps to point P in the immediate (instantaneously short) run. The transitional adjustments of Z_1 and Z_2 are represented by a movement from P to E' along $X'X'$ in Fig. 2, which results in a drop of Z_1 and Z_2 .

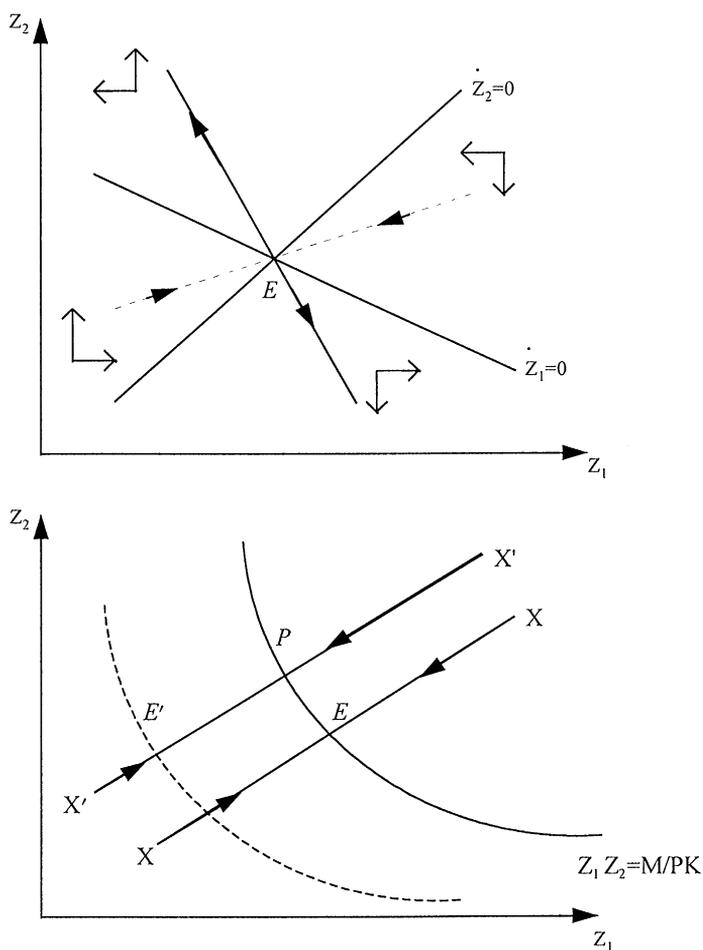


Fig. 2. Phase diagrams.

Instantaneously, the growth rate of real balances remains constant and $Z_1(0)Z_2(0) = Z_1^*Z_2^*$. Therefore, according to the ‘honest government’ argument adopted in the previous section, both real balances and capital grow at the same rate at point P. Since the growth rate of real balances remains constant between points E and P, the inflation rate must also be unchanged. Finally, as the consumption-capital ratio (Z_1) drops and the real balances-consumption ratio (Z_2) rises instantaneously, the growth rate of consumption must be below the growth rates of capital and real balances at $t = 0$.

To determine the transitional dynamics from point P to the new steady state at point E', we linearize (19), (18), and (4) around Z_1^*, Z_2^* and substitute in the linearized adjustment paths of Z_1 and Z_2 , given by (27) and (28), yielding the following expressions:

$$\begin{aligned} \theta_m(t) - \theta_m^{**} &= -s'A(Z_1(t) - Z_1^{**})\left(\frac{a_{21}}{\psi_1 - a_{22}}\right) \\ &\quad - s'AZ_1^{*-2}(Z_1(t) - Z_1^{**}) > 0 \quad \forall t \geq 0 \end{aligned} \quad (30)$$

$$\theta_k(t) - \theta_k^{**} = -(Z_1(t) - Z_1^{**})\left(\frac{s'Aa_{21}}{\psi_1 - a_{22}} + 1\right), \quad (31)$$

$$\begin{aligned} \pi(t) - \pi^{**} &= s'A(Z_1(t) - Z_1^{**})\left(\frac{a_{21}}{\psi_1 - a_{22}}\right) \\ &\quad + s'AZ_1^{*-2}(Z_1(t) - Z_1^{**}) < 0 \quad \forall t \geq 0. \end{aligned} \quad (32)$$

The complete characterization of the time paths of Z_1 , Z_2 , θ_m , θ_k , θ_c and π are illustrated in Fig. 3.

As shown in Fig. 2, as μ increases, both the consumption to capital ratio (Z_1) and the money to consumption ratio (Z_2) decline along the transition path from P to E'. This then implies the growth rate of capital stock must be greater than the growth rate of consumption, which in turn is greater than the growth rate of real money balances from P to E', i.e., $\theta_m < \theta_c < \theta_k$. From (30) and (32), we know that θ_m and π converge to their new BGP equilibrium values monotonically. Thus, the growth rates of consumption and capital stock must be above their new BGP equilibrium value (θ^{**}) in transition from P to E'.

Before exiting this section, we would like to make a remark concerning the jumps in the growth rate of consumption. As μ rises, our analysis above indicates that θ_c has to jump down instantaneously at P and then jumps back up again at a level between θ_m and θ_k . The volatility of θ_c in our model is similar to that in the level of consumption in Ramsey-type exogenous growth models. To sum up, we present our main findings regarding saddle-path dynamics in the following Proposition.

Proposition 6. Under Condition S, the transitional dynamics of a monetary equilibrium exhibit the following properties.

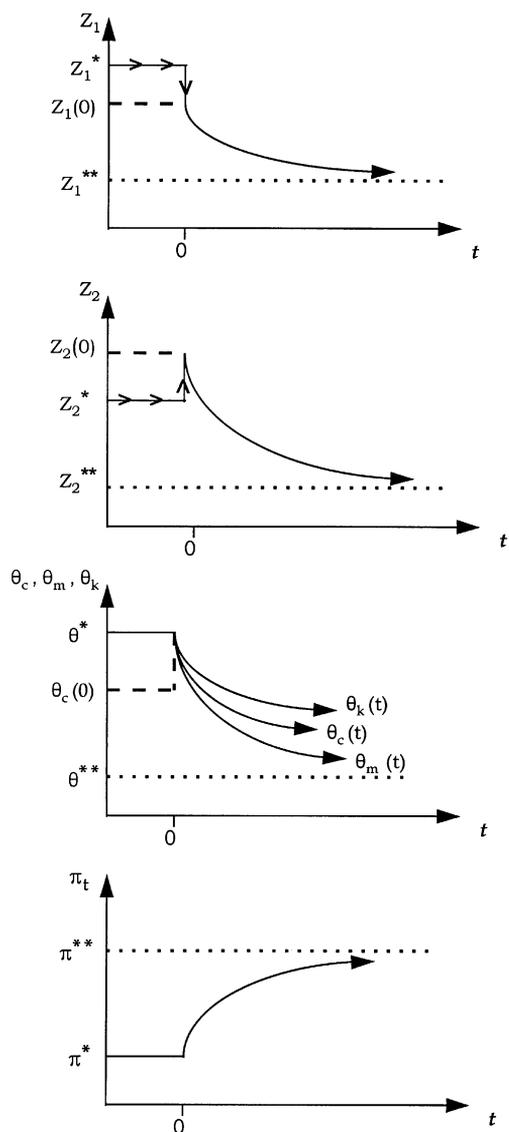


Fig. 3. Transition dynamics.

(i) Instantaneously, an increase in the nominal money growth rate leads to a drop in the consumption–capital ratio and a rise in the real balance–consumption ratio. The growth rates of capital and real balances, as well as the inflation rate, remain unchanged while the growth rate of consumption falls below the initial BGP equilibrium level.

(ii) *In transition, both the consumption–capital and real balance–consumption ratios decline. The growth rate of real balances (the inflation rate) falls (rises) monotonically toward its new BGP equilibrium level. While the growth rate of capital is higher than the consumption growth rate, both rates are higher than that of real balances.*

6. Convex Production technology and elastic labor supply

In this section, we examine the robustness of our results with regards to a generalized convex production technology and an endogenous labor-leisure choice. Following Jones and Manuelli (1990), we generalize the AK technology by appending a Solovian diminishing-return portion. This setup in a simple one-sector endogenous growth model without money is shown to lead to a saddle-path stable balanced growth equilibrium. Thus, one may wonder in our monetary economy whether a source or a sink may still emerge. Moreover, in a human capital-based endogenous growth model, Benhabib and Perli (1994) argue that endogenous leisure may increase the likelihood of dynamic indeterminacy without requiring strong intertemporal substitution. It would be interesting to see if this argument applies to our monetary framework. We can undertake both tasks under one unified structure. In particular, we modify the production technology to: $Y = Ak + Bk^\alpha L^{1-\alpha}$, where L denotes the labor supply, $B > 0$ and $0 < \alpha < 1$. The preference specification follows Rogerson (1988) with the felicity function given by: $c^{1-\sigma}/(1-\sigma) - vL$, where $v > 0$. Therefore, the optimization problem of the representative agent becomes: $\max \int_0^\infty [c^{1-\sigma}/(1-\sigma) - vL]e^{-\rho t} dt$, subject to

$$c + \dot{k} + \dot{m} = (1-s)(Ak + Bk^\alpha L^{1-\alpha}) - \pi m + \tau. \tag{33}$$

Defining $Z_3 = L/k$, the first-order necessary conditions are

$$c^{-\sigma} = \lambda_1 \left[1 - \frac{s'(Z_2)Z_2}{Z_1} (A + BZ_3) \right], \tag{34}$$

$$\lambda_1 = \lambda_2, \tag{35}$$

$$\dot{\lambda}_1/\lambda_1 = \rho - (1-s)(A + \alpha BZ_3^{1-\alpha}), \tag{36}$$

$$\dot{\lambda}_2/\lambda_2 = \rho + \pi + \frac{s'(Z_2)}{Z_1} (A + BZ_3^{1-\alpha}), \tag{37}$$

$$v = \lambda_1(1-s)(1-\alpha)BZ_3^{-\alpha} \tag{38}$$

which together with the constraint (33) and the transversality conditions are sufficient for the optimization.

Asymptotically, the above system will converge to the same BGP equilibrium as in the Ak model where c , k and m are growing at the same constant rate and L is constant (hence, as $t \rightarrow \infty$, we have $Z_3 \rightarrow 0$). Thus, the marginal product of capital approaches the constant A . Eqs. (34), (36) and (37) then converge to (5), (7)

and (8), so that the 2×2 BGP system of (12) and (13) can be obtained. Also, by totally differentiating the system and then substituting in the asymptotic value for Z_3 , we get a 2×2 differential equation system identical to (20) and (21), which can be used to analyze the stability properties. To be consistent with the BGP equilibrium, we impose the restriction of Xie (1991) so that $\sigma = \alpha$. Thus, we need to have $\sigma < 1$ and hence some previous stability properties should be re-done (as Condition D is no longer valid). First, we can combine (34) and (38) to yield the BGP equilibrium value of L

$$v = \frac{(1 - \alpha)[1 - s(Z_2)]BL^{-\alpha}Z_1^{-\sigma}}{1 - As'Z_2/Z_1} \tag{39}$$

Then, by totally differentiating (38) with respect to time and utilizing (20) and (21), we get the following equation governing the dynamics of labor supply:

$$\begin{aligned} \frac{\dot{L}}{L} = & \frac{\rho - (1 - \alpha)(1 - s)A}{\alpha} - Z_1 + \frac{s'(Z_2)Z_2}{\alpha(1 - s)[x(2 - \alpha) - 2]} \\ & \{ \rho x - (x - 1)Z_1 - A(1 - s)x(1 - \alpha) \\ & + [1 - x(1 - \alpha)](\mu - (x - 1)Z_2^{-1}) \} \end{aligned} \tag{40}$$

Clearly, labor supply dynamics have no influence on the property of the dynamical system and the dynamics of Z_1 and Z_2 can be characterized by the linearized system (22) with $\sigma = \alpha$.

We can establish:

Proposition 7. If $x > 1/(1 - \alpha)$, then the monetary equilibrium is always unstable.

Proof. The determinant and the trace of the pre-multiplied matrix D , $\text{Det}(D)$ and $\text{Tr}(D)$, are

$$\begin{aligned} \text{Det}(D) = & j^2 Z_2 s'(1 - \alpha)x\alpha\{i + (i - \mu)A\} - iZ_2\{i[A + (1 - \alpha)] \\ & + (i - \mu)[1 + (1 - \alpha)A]\} \end{aligned}$$

$$\text{Tr}(D) = j[i^2 Z_2 - (i - \mu)\{\alpha - i(1 - \alpha)Z_2\}] + i(i - \mu)(1 - \alpha)(1 + iZ_2)Z_2,$$

where $i = [\rho - (1 - \alpha)\theta] + \mu \geq 0$ and $i - \mu > 0$ under Condition U. Since $\text{sign } j = \text{sign } [x - 2/(2 - \alpha)]$, which is positive under $x > 1/(1 - \alpha)$. Moreover, we have: $\text{sign}(a_{11}) = \text{sign}(a_{21}) = \text{sign}(a_{22}) = \text{sign}(j) > 0$ and $\text{sign}(a_{12}) = -\text{sign}(j) < 0$, implying $\text{Det}(D) > 0$ and $\text{Tr}(D) > 0$ and so the equilibrium is a source. \square

Proposition 8. A monetary equilibrium satisfying the Friedman rule is saddle-path stable.

Proof. Under Friedman’s rule ($i = 0$), we have: $\text{Det}(D) = j^2 Z_2 s'(1 - \alpha)x\alpha A[\rho - (1 - \alpha)\theta] < 0$, implying a saddle. \square

Proposition 9. Under Condition H, if $x < 2/(2 - \alpha)$ and $iZ_2[A + (1 - \alpha)] > \alpha$, then dynamic indeterminacy emerges.

Proof. Under Condition H, we have $\text{Tr}(D) = j^2 Z_2$ and $\text{Det}(D) = -j^2 Z_2 s'(1 - \alpha)ix\{iZ_2[A + (1 - \alpha)] - \alpha\}$. When $x < 2/(2 - \alpha)$ and $iZ_2[A + (1 - \alpha)] > \alpha$, it is easily seen that $\text{Det}(D) > 0$ and $\text{Tr}(D) < 0$, implying that the dynamic equilibrium is a sink. \square

Compared to Propositions 2–4, the above Propositions convey some useful information. On the one hand, in hyperinflationary episodes, the possibility of dynamic indeterminacy is robust (and as before, dynamic instability may also occur). On the other hand, under a convex production technology with elastic labor, the monetary equilibrium satisfying Friedman's rule is a saddle (rather than a source). Intuitively, the generalization induces an additional dynamic adjustment due to a variable capital–output ratio along transition. Since this additional adjustment is a stabilizing force, it overcomes the destabilizing forces in the unstable case of Friedman's rule, leading to saddle-path stability. Obviously, adding an additional stabilizing force would only increase the possibility of a sink and thus the dynamic indeterminacy result is robust. More importantly, Theorem 2 still remains valid, as the monetary equilibrium may be a source, a saddle or a sink.

7. Conclusions

This study has shown that in a simple endogenous growth model, with money entering the model through a transactions service technology, a tractable dynamical system arises. In contrast to previous studies, the tractability of the dynamical system allows us to completely characterize the transitional dynamics. We find that a balanced growth monetary equilibrium either satisfying the Friedman rule of the optimum quantity of money or accommodating the zero-inflation-rate policy is dynamically unstable. In a Cagan-like hyperinflationary environment, the monetary equilibrium may be unstable without relying on ad hoc adaptive expectations mechanism; it is also possible that dynamic indeterminacy emerges without relying on positive externalities. Under proper parameterization, the monetary equilibrium is saddle-path stable. The rate of monetary expansion, the magnitude of the intertemporal elasticity of substitution and the production technological parameter are crucial for determining the stability property of the model.

There are several ways to extend our benchmark framework and for brevity we only discuss three possibilities. First, it is possible to formally contrast our results with those in a cash-in-advance model. To simplify the analysis, one may specify the felicity function as $(\min\{c, \eta m\})^{1-\sigma}/(1-\sigma)$ and set $s = s' = 0$, where $\eta \geq 1$ captures the feature of fractional cash-in-advance as in Palivos, Wang,

and Zhang (1993). It would be interesting to see if dynamic indeterminacy and instability may still emerge under this alternative setting. Second, one may extend the deterministic structure to allow for stochastic movements in the production technology and the money growth process in a spirit similar to Turnovsky (1993). Our study of transitional dynamics can be generalized to investigate the stochastic economy, enabling an integrated analysis of (stochastic) trend and cycles and the short- and long-run welfare consequences of inflation.

Finally, it may be informative to relax our perfect foresight assumption by allowing for an array of expectations schemes, from the irrational adaptive expectations (slow adjustment in expectations) to a limit of short-run perfect foresight (instantaneous adjustment in expectations). In a generalized Tobin model, Benhabib and Miyao (1981) study whether the dynamics depends crucially on the extent to which the short-run price expectations of agents adjust. They show that instability occurs no matter how quickly prices adjust. It may be interesting to examine if their conclusion holds in our transactions-based model rather than in the Tobin-like asset substitution framework. In particular, the long-run ‘complementarity’ between capital and real money balances may serve as a stabilizing force and price expectations may matter as they can affect the degree of short-run substitution toward the long-run equilibrium path.

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