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# Welfare analysis of the number and locations of local public facilities

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**Abstract**

We develop a model with a finite number of households and congestible local public goods where the level of provision, the number of facilities and their locations are all endogenously determined. We prove that an equal-treatment identical-provision second-best optimum exists, where all households are required to reach the same utility level, the provision of local public good is required to be the same at all facilities, and all facilities must serve the same number of consumers. Such an optimal public facility configuration may be concentrated (single site) or dispersed (multiple sites), depending on congestability, commuting cost and household preference parameters.


*JEL classification:* H41; R13; D61
*Keywords:* Congestible local public goods; Optimal public facility configurations

## 1. Introduction

Previous studies examining the effects of local public goods (LPGs) on the underlying spatial structures focus primarily on how the level of LPG provision and the locations of public facilities influence the locational choices of households and firms.[2] The presence of

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[1] Washington University and NBER, St. Louis, MO, USA.
[2] See, for example, Arnott and Stiglitz (1979), Ellickson (1979), Scotchmer (1986), Wildasin (1987), Thisse and Wildasin (1992), and those papers cited in a survey article by ReVelle (1987).

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LPGs serves as an agglomerative force, leading to the clustering of households and firms around a public facility. While this positive analysis is interesting, it is equally important to examine public facility configuration from a normative point of view. The number and locations of public facilities are crucial in determining the level of community welfare. However, due to technical barriers, this important task has not been explored completely. In this paper, we attempt such an endeavor by performing a general-equilibrium welfare analysis using a model with a finite number of households and congestible LPGs in which the number of facilities and their locations, as well as the level of LPG provision, are all endogenously determined.[3]

The joint determination of the optimal number and locations of public facilities and the optimal level of LPG provision is therefore the central feature of the present paper. As in the literature, we restrict our focus to "identical provision" of the LPG with the same level of the LPG provided in each site where each site is used by the same number of people. For example, Fujita (1986) and Sakashita (1987) feature identical provision while the locations of public facilities are endogenous, but the number of public facilities is exogenously given as one (Fujita) or as one or two (Sakashita) without any endogenous mechanism to pin down the number of public facilities within the system. Furthermore, the existing literature on public facility locations assumes that the level of provision is exogenously fixed. In contrast, we endogenize the optimal allocation of resources between the public and the private goods but establish a sufficient condition such that the level of provision of the LPG is independent of the configuration of public facilities and the allocation of the private good.

The second special feature of our paper is the consideration of congestible LPGs. Most previous studies focus on the non-exclusive nature of LPGs (e.g., see Fujita, 1986; Sakashita, 1987; Peng, 1996), ignoring the possibility that public goods may generate local congestion as a function of the locationally dependent number of users. Local congestability is important, as in the case of parks, public schools and libraries, where aggregate usage affects the quality of services. It is interesting to study how the presence of local congestability may affect the welfare properties of allocations in the spatial context.

The third distinctive feature is that our paper is the only one based on the concept of Pareto optimality in determining the optimal public facility configuration. We construct a concept of "equal-treatment" identical-provision optimality that has all households achieving the same level of utility.[4] Fujita and Sakashita employ representative consumer utility maximization, which may be reinterpreted as an equally weighted social welfare maximum of households with concave utility functions. In Sakashita, land rents exit the system and thus his command optimal outcomes need not be consistent with Pareto optimality. While our equal-treatment identical-provision optimality concept is well-defined, the feature of equal treatment allows us to compare our results with those in previous studies.

[3] The discrete model is the original urban economic framework developed by Alonso (1964).

[4] We will show that with identical provision of LPGs, our concept of optimality generally forces a second-best allocation since non-identical provision of LPGs may be Pareto dominant.

In contrast with most previous studies in the area of LPGs and spatial economics, where models with a continuum households are used, our paper employs a discrete-household framework.[5] This is an important departure because in conventional location models with a continuum of agents, one can prove that $0=1$ (cf. Berliant, 1985). Moreover, it is typical in the congestible LPG literature that a finite number of consumers patronize a given facility.[6] It is unclear how to represent this in a spatial context with a continuum of consumers.[7] To circumvent theoretical shortcomings in continuum models, we adopt the discrete framework for conducting mathematically rigorous general-equilibrium analysis.

The main findings of this paper are as follows. First, we prove the existence of an equal-treatment identical-provision optimum and establish a sufficient condition for the level of LPG provision to be independent of the public facility configuration and other endogenous variables. Second, given a single public facility, the equal-treatment optimal location of the public facility may not be at the geographic center and such an optimal location need not be unique. Third, with the number of public facilities endogenously determined, a sufficiently high degree of both congestability and commuting cost and a sufficiently low household valuation of the LPG imply that the equal-treatment identical-provision optimal public facility configuration is always dispersed. These main findings are then shown to be robust to the number of households. Finally, an equal-treatment identical-provision optimum is generally Pareto inferior to a non-identical provision feasible allocation, suggesting that the notion of optimality used frequently in the literature yields second-best allocations.

## 2. The economy

Consider a community accommodating $N$ identical households that must locate in $X \equiv \{1,2,\ldots,N\}$, where $N < \infty$. Each residential region has a unit length in a one-dimensional interval $L \equiv [0, N]$, and is of homogeneous quality (except for the aspect of LPGs). Each household is required to reside in a location. In the absence of vacant land or multiple occupancy, land allocation in our spatial economy becomes a simple assignment problem in which each lot is occupied by only one household.[8] Households are homogenous in taste. Thus, household $i$ consumes the interval $(i-1, i]$ and is conveniently labeled by the "front-door" location of its residence. The aggregate endowment of composite private good is $E > 0$.

[5] A drawback to our discrete model is inelastic demand for land. Nonetheless, it is a normative model of public facility location with a fixed stock of housing. Our model structure can be regarded as an extension of the discrete model of Berliant and Fujita (1992) to incorporate congestible LPGs with the level of provision, the number and the locations of facility sites endogenously determined.

[6] For example, see Conley and Wooders (2001) and Ellickson et al. (in press).

[7] It has been suggested that the number of public facilities be allowed to take any nonnegative real value instead of nonnegative integer values.

[8] This is similar to the Koopmans and Beckmann (1957) assignment problem, meaning that it is infeasible to have more than one household residing in the same lot.

Consumers enjoy consuming both composite private good and the LPG provided by the community at certain assigned *public facility* locations.[9] At a given location, a household has the nonnegative orthant of $\mathbb{R}^3, \mathbb{R}^3_+$, as its consumption set. There are three types of goods: (i) a composite consumption good, (ii) a LPG, whose facility service is valued by each household, and (iii) land. To simplify the analysis, we assume that the demand for land is perfectly inelastic—i.e., each household consumes a fixed, unit quantity of land. Thus, the consumption set reduces to $\mathbb{R}^2_+$.

The community government uses one unit of the composite private good to produce one unit of the LPG. Let $K$ denote both the composite consumption good used in production and the total supply of the LPG. This total supply is divided among the $n$ facilities, where $n$ is determined endogenously. The LPG may be provided at one (*concentrated*) or more than one (*dispersed*) location and (ex ante) more than one facility is allowed to be at the same location. Denote the set of public facility locations as $H \subset$ L. When there is a single public facility site at $\eta \in L$, we have $H \equiv \{\eta\}$. All households patronize this facility. When there are $n$ sites, we have $H \equiv \{\eta_1, \eta_2, \ldots, \eta_n\}$ where $\eta_j \in L$ with a corresponding level of the LPG denoted by $K_j$, and $M_j$ where is the number of households patronizing site j ($j = 1, \ldots, n$) (obviously, $\sum_{j=1}^{n} K_j = K$ and $\sum_{j=1}^{n} M_j = N$). Thus, by denoting the cardinality of a discrete set as $|\cdot|$, we have $n = |H|$ and the pair $(n, H)$ summarizes the configuration of public facilities, i.e., both the number and locations of local public facility sites. Examples of such LPGs include schools, libraries, museums, parks, theater halls, public pools, exercise fields and other recreational facilities.

Each household must incur a commuting cost to travel to the assigned public facility locations.[10] Assume that the amount of land required by the facility is negligible.[11] We consider throughout the paper *identical provision* (henceforth called IP) of the LPG in the sense that when there are multiple public facility sites, the same level of LPG is provided at each site and the same number of households use each of them. Thus, when there are $n$ sites, the identical provision of the LPG implies that each site is equipped with a LPG of size $K_j = K/n$ and shared by a population of households of $M_j = N/n$ (for all $j = 1, \ldots, n$). While the feature of IP allows us to compare our results with those in previous studies, we note that in general, a non-identical provision of the LPG may be Pareto dominant (see Section 5 below).

Let $Y_i \subset H$ be a subset of public facilities to which household $i$ travels and denote $y_i \in Y_i$ as a travel destination of household $i$. Household i may be assigned to randomize, with probability $\theta(i, \eta_j) \in [0,1]$ and $\sum_{\eta_j \in H} \theta(i, \eta_j) = 1$, to travel to a particular destination $\eta_j$ where $\theta(i, \eta_j) > 0$ for $\eta_j \in Y_i$ with $|Y_i| \leq n$ and $\theta(i, \eta_j) = 0$ for $\eta_j \notin Y_i$.[12]

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[9] We follow the conventional wisdom assigning one resident to each location. To permit each region $x_i$ to be occupied by a finite number of identical households would not alter our main results.

[10] Fujita (1986) denoted this as a "traveled-for" public good.

[11] Fujita (1986), Sakashita (1987) and Peng (1996) make a similar assumption. The model can easily be modified to allow public facilities to use land, provided that the land requirement for the provision of $K$ units of LPG is fixed. In this case, the division of a larger public facility into several small units would not change total land usage.

[12] Consider the case of $N = 3$ and $n = 2$ where the two facility sites are at $\eta_1 = 1$ and $\eta_2 = 2$. A possible travel pattern is: households 1 and 3 commute to $y_1 = \eta_1$ and $y_3 = \eta_2$, respectively, whereas household 2 randomizes by commuting to $y_2 \in Y_i = \{\eta_1, \eta_2\}$ with $\theta(2, \eta_1) = \theta(2, \eta_2) = 1/2$.

Obviously, $n$ is at most $N$ under this setup and hence $n \in X$. This randomization setup is designed to be consistent with the consideration of identical LPG provision.[13]

The commuting cost (measured in terms of the composite good), $T_i(y_i)$, varies with distance from the LPG site. Since any single point has zero measure, we do not have to differentiate between open and closed intervals and can utilize the closest-point distance, denoted $d$, to measure the distance between a single point and a half open interval.[14] Specifically, the distance between household $i$'s residential region in $L$ and a public facility location $y_i$, denoted $d(i, y_i)$, is defined as: (i) $(y_i - i)$, if $y_i > i$ [rightward traveling pattern]; (ii) 0, if $i-1 \leq y_i \leq i$ [non-traveling]; (iii) $(i-1)-y_i$, if $y_i < i-1$ [leftward traveling pattern], respectively. We assume that the commuting cost schedule satisfies:

**A1** (*Commuting cost*). $T_i = T(d(i,y_i))$ where $T$ is twice continuously differentiable, $T(0)=0$, $T' > 0$ and $T'' \geq 0$.

While (A1) is imposed in obtaining general theorems, we consider a simple linear commuting cost schedule frequently used in the literature (e.g., see Alonso, 1964; Fujita, 1986) to characterize optimal public facility configurations (in Sections 4 and 5 below):

$$T_i = td(i, y_i), \qquad t > 0. \tag{1}$$

Denote the total commuting cost as $C(n,H) \equiv \sum_{i=1}^{N} \sum_{y_i \in Y_i} \theta(i,y_i) T(d(i,y_i))$ where $Y_i \subset H \subset L$ for all $i=1,\ldots,N$. For given $H=\{\eta_1,\ldots,\eta_n\}$, $n=|H|$ and $K_j = K/n$, a set of random travel assignments $\Theta = \{\theta(i,\eta_j)\}_{\eta_j \in H;\ i=1,\ldots,N;\ j=1,\ldots,n}$ is called *cost-minimizing* if $Y_i \equiv \operatorname{argmin}_{\eta_j \in H} d(i, \eta_j)$ for all $i \in X$ (i.e., every household is assigned to the closest public facility sites). For given $n$ and $K_j = K/n$, a public facility configuration associated with a set of public facility sites $H$ and a set of random travel assignments $\Theta = \{\theta(i, \eta_j)\}_{\eta_j \in H;\ i=1,\ldots,N;\ j=1,\ldots,n}$ is called *aggregate-cost-minimizing* if for all $H' \subset$ L and $\Theta' = \left\{\theta'(i, \eta_j)\right\}_{\eta_j \in H'; i=1,..N; j=1,..,n}$, $C(n,H) \equiv \sum_{i=1}^{N} \sum_{y_i \in Y_i \subset H} \theta(i,y_i) T(d(i,y_i)) \leq \sum_{i=1}^{N} \sum_{y_i' \in Y_i' \subset H'} \theta'(i,y_i') T(d(i,y_i'))$.

Denote $z$ and $G$ as composite good and LPG consumption, respectively, and $\alpha$ as the *degree of congestability* of the LPG. The amount of public service embodying congestion is specified as[15]:

**A2** (*Consumption of the local public good*). $G(K_j, M_j) = K_j (M_j)^{-\alpha}$, $\alpha \in [0,1]$.

Under identical provision, the LPG allocated to each site is $K_j = K/n$ and each site is patronized by $M_j = N/n$ households. We can thus rewrite: $G = (K/n)/(N/n)^{\alpha}$. Notably,

[13] An alternative is to follow Lucas (1990), assuming a continuum of household members in each household with the household head making a joint decision for all members. In this case, for the example in Footnote 11, we simply assume that half the members of household 2 travel to one site and the remaining half to the other.

[14] The standard Euclidean metric is not applicable in this case. For further discussion, the reader is referred to Klein and Thompson (1984, p. 39) and Berliant and Wang (1993, pp. 130–131). In Section 4.2 below, we will discuss the consequences of using mid-point distance in place of closest point distance.

[15] A LPG subject to the external effect of congestion is defined as a congestible LPG. For detailed discussion of congestible LPGs, see Starrett (1991).

$\alpha=0$ implies a pure public good, whereas $\alpha=1$ implies a private good. In a special case where all households share one concentrated LPG at a single facility of size $K$ (i.e., $n=1$), our setup reduces to that considered by Hochman (1982), Thisse and Wildasin (1992) and Peng (1996): $G=K(N)^{-\alpha}$. Notably, considering our normative analysis (Pareto optimality), whether a household or the local government bears the commuting costs is irrelevant. Thus, the type of LPGs considered in this paper can include police or fire protection where the services are taken to the consumers. However, our congestible LPG does not permit either positive externalities (where one may enjoy consuming the good with others) or the possibility of a variable degree of congestability (which may depend on the number of users).

**A3** (*Well-behaved preferences of households*). $u(z, G)$ is twice continuously differentiable, $\partial u/\partial z>0$, $\partial u/\partial G>0$, and is strictly concave.[16]

An *allocation* is defined as a list of (i) nonnegative quantities of composite good $z_i \in \mathbb{R}_+$ consumed by each household $i$ ($i=1,\ldots,N$), (ii) a configuration of public facilities described by a set of public facility sites $H \equiv \{\eta_1,\eta_2,\ldots,\eta_n\} \subset L$ with the number of sites $n=|H| \in X$, (iii) a nonnegative level of provision of the *LPG* $K_j \in \mathbb{R}_+$ at each location $\eta_j \in L (j=1,\ldots,n$ and $\sum_{j=1}^{n} K_j = K)$, with the number of households patronizing each public facility site $M_j \in \mathbb{R}_+ (j = 1,\ldots,n$ and $\sum_{j=1}^{n} M_j = N)$, (iv) a subset of location indexes $Y_i \subset H$ ($i=1,\ldots,N$) containing the public facility sites that household $i$ patronizes, and (v) a randomization probability $\theta(i,\eta_j) \in [0,1]$ assigning the probability with which household $i$ travels to a particular destination $\eta_j \in H$ with $\sum_{\eta_j \in H} \theta(i, \eta_j) = 1$, $\sum_{i=1}^{N} \theta(i, \eta_j) = M_j$, and $\theta(i,\eta_j)=0$ for $\eta_j \notin Y_i$. In short, an allocation can be written as: $a=(\{z_i,Y_i\}_{i=1,\ldots,N}, \{K_j,M_j\}_{j=1,\ldots,n}, \{\theta(i,\eta_j)\}_{i=1,\ldots,N;\ j=1,\ldots,n}, n, H)$.

Given $y_i \in Y_i \subset H$ (i=1,…,$N$), the *material balance condition* for the composite good is:

$$\sum_{i=1}^{N} \left[ z_i + \sum_{y_i \in Y_i} \theta(i, y_i) T(d(i, y_i)) \right] = E - K. \tag{2}$$

We now define a *feasible allocation* as an allocation $a=(\{z_i,Y_i\}_{i=1,\ldots,N}, \{K_j,M_j\}_{j=1,\ldots,n}, \{\theta(i,\eta_j)\}_{i=1,\ldots,N;\ j=1,\ldots,n}, n, H)$ satisfying the material balance condition (2).

Next we restrict our attention to the case of identical provision of the LPG. An *identical-provision* (IP) *feasible allocation* is a feasible allocation such that $K_j=K/n$ and $M_j=N/n$ for all $j=1,\ldots,n$. An *identical-provision optimum* is an IP feasible allocation such that there is no other IP feasible allocation making no household worse-off and at least one better-off. Under (A2), household randomization and identical provision, the LPG consumed inclusive of the congestion factor becomes:

$$G(n, K) = K/\left(N^{\alpha} n^{1-\alpha}\right), \tag{3}$$

[16] It is necessary to assume strict concavity (rather than quasi-concavity) in order to ensure the sufficiency of the Lindahl–Samuelson condition to be derived in Eq. (5) below.

which is constant across all households. The term $N^{\alpha}n^{1-\alpha}$ is a "net congestion factor," increasing in the population and the number of public facility sites. Since all consumers are identical, the assignment of consumers to parcels is arbitrary and thus land allocation is trivial. Identical LPG provision enables us to write $K_j = K/n$ and $M_j = N/n$ for all $j$, so the IP optimality problem is simply given by:

$$\max_{\{z_i, y_i, \theta(i, y_i)\}_{i=1,\ldots,N}, K, n, H} \quad u(z_1, G(n,K)), \qquad (4)$$
$$\text{s.t. (i) } u(z_i, G(n,K)) \geq v_i, i = 2, 3, \ldots, N$$

$$\text{(ii) Eq. (2); } z_i \geq 0, K \geq 0, n \in X, Y_i \subset H, i = 1, \ldots, N; \sum_{i=1}^{N} \theta(i, \eta_j) = \frac{N}{n} \forall \eta_j \in H.$$

This states the IP optimality problem as one to maximize the well-being of household 1 subject to required utility levels for others (constraint (i)) and feasibility ((2)) as well as the IP constraint in (ii).

In the existing literature, an "optimal" configuration of public facilities is based on the concept of "command optimality" in which the community government determines the public facility configuration subject to (i) the competitive determination of the land rents (that often exit the system without a landlord to formally close the model) and (ii) locational equilibrium (that equalizes households' utility). It is clear that such a concept is not well-defined in the Pareto sense—indeed, it is a mix of equilibrium and optimality concepts.[17] To be compatible with the literature, however, we modify the concept of IP optimality by considering *equal treatment* (ET) among all households. Specifically, an *equal treatment identical-provision optimum* (in short, an ETIP optimum) is an IP optimum such that all households reach the same utility, i.e., we have $u(z_i, G(n, K)) = v_o \geq 0$ for all $i = 1, \ldots, N$ in problem (4).

Three comments are now in order. First, ETIP allocations may be justified by the equal protection clause of the U.S. Constitution, federal law, or by state constitutions. For example, the judicial system uses this clause and federal law to force equal school spending within jurisdictions (see Inman and Rubinfeld, 1979, p. 1699 for application of the equal protection clause and pp. 1697–1701 for application of federal law) while state constitutions can require equal school spending across jurisdictions (see Inman and Rubinfeld, 1979, pp. 1705–1708). Second, an ETIP optimum is Pareto optimal provided that the provision of LPG at all facilities is the same in an equal utility Pareto optimum and the number of consumers served at each facility is the same at the Pareto optimum. That is, an ETIP optimum may fail to be first-best if it is welfare enhancing to alter the ETIP optimum to serve different numbers of consumers or provide different levels of LPGs at different sites. As discussed in Section 5 below, there may be an allocation with non-identical LPG provision that Pareto dominates the allocations restricted to identical provision. Third, our definition of ETIP optimality basically determines the optimal distribution to consumers for given aggregate

[17] This problem has been pointed out by Berliant and ten Raa (1994, p. 638): "Most of the models [of public facility location] appear to use a mix of positive and normative concepts, equilibria and optima. At best, they can be interpreted as partial welfare analysis."

endowments of the composite good and land ($E$ and $L$, respectively), as well as optimal public facility configurations in terms of the number and locations of sites. Thus, it does not involve an equilibrium concept, contrasting with the command optimality concept in previous studies.

## 3. Existence of an equal-treatment identical provision optimum

In this section we establish theorems concerning the existence of an ETIP optimum and its general properties. An ETIP optimum can be obtained by solving the problem specified in (4) with $v_i=u(z_1,G(n,K))=v_o$ for all $i=2,\ldots,N$ and (4) (i) holding with equality.

To begin, we determine the *Lindahl–Samuelson condition* that gives the optimal allocation between the public and the private composite good: $\sum_{i=1}^{N}[(\partial u_i/\partial G)/(\partial u_i/\partial z_i)]\partial G/\partial K=1$. With regard to ETIP optimality, we define $\Lambda(K,n,z)\equiv(1/N)(\partial u/\partial z)/[(\partial u/\partial G)(\partial G/\partial K)]$ and show that any ETIP optimal provision of the LPG satisfies the following necessary condition (see Appendix A):

$$\Lambda(K,n,z)=1 \tag{5}$$

Denote $\mathrm{MRS}_{Gz}\equiv(\partial u/\partial G)/(\partial u/\partial z)$ as the marginal rate of substitution between the public and the private composite goods. It may be more intuitive to use (3) to rewrite (5) as: $\Lambda(K,n,z)=(N^{\alpha}n^{1-\alpha})/[N(\mathrm{MRS}_{Gz})]=1$. That is, the sum of the marginal willingness-to-pay (i.e., $N(\mathrm{MRS}_{Gz})$, under ETIP) is equal to the marginal cost of LPG provision (i.e., the net congestion factor $N^{\alpha}n^{1-\alpha}$). Notably, the optimal composite good allocation, the optimal number of public facility sites and the optimal level of the LPG are all jointly determined, making it difficult to characterize the ETIP optimum. We thus take the method adopted by Bergstrom and Cornes (1983), seeking a condition such that the optimal level of LPG provision is not only uniquely determined but also *independent* of the public facilities configuration and the distribution of composite good. By examining (5), we impose:

**Condition *K*** : $\partial\Lambda/\partial n=\partial\Lambda/\partial z=0, \quad \partial\Lambda/\partial K>0.$

Condition $K$ requires that ex ante the sum of the marginal willingness-to-pay by all households is *not* affected by the number of public facilities or the allocation of the composite good, though individual valuation of the LPG is allowed to depend on the public facility configuration. The independence of the aggregate marginal willingness-to-pay with respect to the allocation of the composite good eliminates the income effect concerning the entire community as a whole, whereas that with respect to the number of public facility sites removes the spatial-distribution effect in aggregation. Condition $K$ makes the system recursive (see Lemma 2 below), thus greatly simplifying the analysis with respect to travel assignments.

Condition $K$ is sufficient for (5) alone to pin down the optimal level of K, independent of $n$ and $z$. An example satisfying Condition $K$ will be provided shortly.

**Lemma 1** *Under (A2) and (A3) and Condition K, an ETIP optimum, if it exists, is associated with a uniquely determined level of public good $K_0$ that is independent of the public facility configuration.*

**Proof** All proofs are relegated to Appendix A.

To guarantee a nonempty feasible set, we assume the aggregate endowment of composite private good $E$ is sufficient to cover the spending on providing the LPG at the optimal level $K_0$[18]:

**Condition *E*** : $K_0<E$.

An example is now in order. Consider $u(z,G)=\gamma z+\ln(G)$ where $\gamma>0$. Then we have $\Lambda=\gamma K/N$ and hence (5) implies $K=K_0\equiv N/\gamma$, independent of any endogenous variables. It is clear that Condition $E$ is met if $\gamma(E/N)>1$, meaning that households' valuation of the average composite good endowment be greater than one. Notably, due to the endogenous number of public facilities, Bergstrom and Cornes' (1983) condition of a separable utility function that is quasi-linear in the composite good is insufficient in our model to guarantee an optimal level of aggregate public good provision independent of other endogenous variables. The imposition of Condition $K$ simplifies our analysis greatly.

Under (A1) (A2) (A3) and Condition $K$ and given the optimal number of facilities, we obtain another important necessary condition governing optimal travel assignments and public facility locations:

**Lemma 2** *Under (A1) (A2) (A3) and Condition K, an ETIP optimum with an optimal number of public facilities n*, if it exists, is associated with an aggregate-cost-minimizing travel assignment among all ETIP feasible travel assignments and public facility configurations with* $n=n^*$. *That is, let* $(\{z_i,Y_i\}_{i=1,\ldots,N}, \{K_j,M_j\}_{j=1,\ldots,n^*}, \{\theta(i,\eta_j)\}_{i=1,\ldots,N;\, j=1,\ldots,n^*}, n^*, H)$ *be an ETIP optimum. Then* $\forall H'$, $|H'|=n^*$, $\forall\theta'(i,y_i')$, $y_i'\in Y_i'\subset H'$ $(i=1,\ldots,N)$,

$$\sum_{i=1}^{N}\sum_{y_i\in Y_i\subset H}\theta(i,y_i)T(d(i,y_i))\leq\sum_{i=1}^{N}\sum_{y_i'\in Y_i'\subset H'}\theta'(i,y_i')T(d(i,y_i')) \tag{6a}$$

*Moreover, the number of households patronizing a particular public facility site* $j=1,\ldots,n^*$ *satisfies:*

$$M_j=\sum_{k=1}^{n}\frac{1}{k}\left|\left\{i:\eta_j\in Y_i\subset H,|Y_i|=k\right\}\right|=\frac{N}{n}. \tag{6b}$$

Lemma 2 says that given the optimal number of public facilities, an optimal set of sites must be such that the aggregate travel cost is minimized. This property is crucial for determining households' travel patterns and the optimal public facility configuration. Should Condition $K$ be violated, the travel assignment generally interacts with the level of

[18] Condition $E$ is not necessary—if it is not satisfied, the level of private good consumption is zero, implying a corner solution for the optimum.

LPG provision, making the characterization of the optimal number and locations of public facilities intractable even when focusing on ETIP feasible allocations.

At an ETIP optimum, LPG $G$ is uniform across households (as specified in Eq. (3)) and thus ET implies that the composite good consumption must be identical. Clearly, the total commuting cost $C(n,H)$ is non-increasing in $n$. The ETIP optimal composite good allocation to each household is then:

$$z = (E - K - C(n, H))/N. \tag{7}$$

Since $C$ is a function of $(n,H)$, $G$ is a function of $(n,K)$ and, from Lemma 1, $K=K_0$ (an exogenous constant), households' indirect utility must be a function of $(n,H)$ alone. We denote this value by $v(n,H)$.

**Theorem** *(Existence of ETIP Optimum). Under (A1) (A2) (A3) and Conditions K and E, there exists an equal-treatment identical-provision optimum.*

It is important to note that from the proof of the theorem, the Maximum Value Theorem can also be applied to the set of feasible allocations A (as A is nonempty and compact). Thus, the existence of a Pareto optimum (that is not necessarily ETIP or IP) can easily be established.

All we need to establish the theorem are the continuity of the households' utility function and the commuting cost function, and that $G$ is non-increasing in $n$ (to ensure that $n$ is finite, bounded by $N$), which follows from the assumption that $u$ is increasing in $G$ and $T$ is increasing in $d(i,y_i)$. The assumptions regarding differentiability, specific functional forms, or interior solutions imposed in (A1) (A2) (A3) are not needed here, but are made for a complete characterization of the ETIP optimum in Section 4.

Moreover, $H=\{\eta_1,\ldots,\eta_n\}$ and $n$ are inter-related. For each $n$, define $H^*(n)=\text{argmin}_H\, C(n,H)$. While the total commuting cost $C(n,H^*(n))$ is non-increasing in $n$, the net congestion factor $N^{\alpha}n^{1-\alpha}$ is strictly increasing in $n$ (for $\alpha<1$). Thus, under (A2), (A3) and Condition $K$, Eqs. (3), (7) and $K=K_0$ imply that per capita composite good consumption $z$ is non-decreasing in $n$ and $G$ is strictly decreasing in $n$. An immediate consequence of these conflicting effects on $u(z,G)$ is that an ETIP optimum may be associated with an intermediate value of $n\in\{2,\ldots,N-1\}$, i.e., an ETIP optimal public facility configuration need not be completely concentrated ($n=1$) or completely dispersed ($n=N$). Furthermore, there are in general a continuum of ETIP optimal public facility configurations, even when $n$ is fixed. For example, consider the case of $n=1$ and $N$ as an odd number. Denote $[D]$ as the Gauss operator of $D\in\mathbb{R}_+$, defined as the largest integer that is less than or equal to $D$. We will show in Section 4 that under (A1) (A2) (A3) and Conditions $K$ and $E$, any $\eta\in[[N/2],[N/2]+1]$ (i.e., the central region) is ETIP optimal.

## 4. Characterization of optimal public facility configurations

We characterize ETIP optimal public facility configurations explicitly given a linear commuting cost schedule specified as in (1). We focus on three issues: (i) whether an ETIP optimal configuration is uniquely determined, (ii) whether public facilities should be concentrated (one site) or dispersed (more than one site), and (iii) whether a concentrated public facility should be geographically centralized. The main purpose of this paper is to perform a *normative* analysis of local public facility configuration, which contrasts sharply with the existing literature based on a *positive* analysis of command optimality.

### *4.1. Optimal location of a public facility: the case of $n=1$*

First we fix the number of facilities exogenously at 1.

**Proposition 1.** *(ETIP optimal location of a public facility with $n=1$). Assume (A2), (A3), Conditions K and E, and the linear commuting cost schedule as specified in (1). Given $n=1$, the set of ETIP optimal public facility configurations satisfies the following properties:*

- **(i)** *for even numbers $N$, it is unique and geographically centralized;*
- **(ii)** *for odd numbers $N>1$, it is a continuum consisting of any location in the central interval.*

The results in Proposition 1 contrast with findings in the framework with a continuum of households of Fujita (1986) and Sakashita (1987) in two important dimensions. First, in our model, an ETIP optimal location of the public facility need not be at the geographic center. Second, we find that multiple ETIP optimal configurations are possible in the sense that the optimal location of the public facility need not be unique. These discrepancies are mainly due to the difference between the finite versus continuum of households setups and the concept of equal-treatment identical-provision optimality versus command optimality. It may be noted that as $N$ goes to infinity (by expanding $L$ to the entire extended real line), this multiplicity property is still robust.

### *4.2. Optimal number and locations of public facilities: the case of $N=3$*

We next turn to endogenizing the number of public facilities. When $N=1$ or $N=2$, the ETIP optimal facility location problem is trivial. When $N\geq 3$, it is possible that the ETIP optimal number of public facilities is no longer one. For illustrative purposes, it suffices to consider the case of $N=3$.

**Lemma 3** *Assume (A2) with $\alpha<1$, (A3), Conditions K and E, and the linear commuting cost schedule specified as in Eq. (1). Given $N=3$, the ETIP optimal number of public facilities $n$ can only be 1 or 2.*

Lemma 3 rules out the possibility of $n=3$. Hence, there are three candidate ETIP optimal public facility configurations remaining to be examined with $N=3$: (i) centrally concentrated (type-C), (ii) non-centrally concentrated (type-N) and (iii) dispersed (type-D). Notably, Proposition 1 implies that an ETIP optimum with $n=1$ (concentrated) must have the public facility of size $K_0$ in the central interval. In this case, there are two possibilities:

either $H=\{3/2\}$ (i.e., type-C) or $H=\{\eta\}$ with $\eta \in [1,2] \backslash \{3/2\}$ (i.e., type-N). With regard to the dispersed case ($n=2$), we conclude[19]:

**Lemma 4** *Assume (A2), (A3), Conditions K and E, and the linear commuting cost schedule specified as in Eq. (1). Given N= 3, if the type-D public facility configuration is ETIP optimal, there are two locationally symmetric and identically provided public facilities of size $K_0/2$ located at $\eta$= 1 and 2.*

We are now left to examine each candidate configuration and to derive the corresponding conditions to support each case. In order to compute a household's indirect utility, we further assume that the utility function of the households takes the following form (satisfying Condition $K$)[20]:

$$u(z,G) = \gamma z + \ln(G) \tag{8}$$

where $\gamma > 0$. From Lemma 1, we solve the optimal provision of the LPG as $K_0 = N/\gamma$. Using Lemmas 2–4 and Eqs. (3), (7) and (8), we get the household value for each type of public facility configuration:

$$v^C = v^N = \gamma(E - K_0)/3 + \ln(K_0/3) + (1-\alpha)\ln 3 - \gamma t/3 \tag{9a}$$

$$v^D = \gamma(E - K_0)/3 + \ln(K_0/3) + (1-\alpha)\ln 3 - (1-\alpha)\ln 2 \tag{9b}$$

where the superscripts denote the types of public facility configurations and the corresponding aggregate commuting costs are: $C^C = C^N = t$ and $C^D = 0$.

To compare different types of public facility configurations, we need to check if they are Pareto rankable. This is easily done by comparing households' value for each case, as given by (9a) and (9b), provided that $\gamma E > 3$, which is ensured by Condition $E$.

**Proposition 2.** *(ETIP optimal number and locations of public facilities with N= 3). Assume (A2) with $\alpha < 1$, Condition E, the linear commuting cost schedule specified as in (1), and the utility function specified as in (8). When N= 3, an ETIP optimal public facility configuration satisfies the following properties:*

- **(i)** *if $t > 3$ (ln2) $(1-\alpha)/\gamma$, then an ETIP optimal public facility configuration is always dispersed;*
- **(ii)** *if $t < 3$ (ln2) $(1-\alpha)/\gamma$, then an ETIP optimal public facility configurations is always concentrated, either centralized or non-centralized.*

The determination of the optimal public facility configuration depends crucially on the commuting cost ($t$), the degree of non-exclusiveness ($1-\alpha$), and the household valuation of the LPG ($1/\gamma$).

[19] By using the mid-point distance measure (rather than the closest-point distance), the multiplicity of ETIP optima still emerges, although it now occurs when $N$ is an even number. Moreover, the marginal total commuting cost in this case is no longer constant, which will lead to greater complexity in characterizing the ETIP optimal configuration of public facilities in the general case.

[20] Those interested in the conventional Cobb–Douglas utility functional form are referred to Berliant et al. (2005). Since in that case Condition $K$ is not satisfied, we instead assume that $K$ is exogenously fixed. Under this assumption, the main conclusions remain valid.

**Proposition 3** *Assume (A2) with $\alpha<1$, Condition E, the linear commuting cost schedule specified as in (1), and the utility function specified as in (8). Given $N=3$, an ETIP optimal public facility configuration is dispersed if the degree of congestability is large, the household valuation of the public good is low, and the unit commuting cost is high; otherwise, an ETIP optimal public facility configuration is concentrated.*

Thus, building a large central park/library/school is not ETIP optimal in an economy with a high degree of LPG congestability or a high commuting cost. Local community parks/libraries/schools provided at dispersed facility sites with smaller size can result in Pareto improvements.

*4.3. Optimal number and locations of public facilities: the general case of $N>3$ and $n\geq 1$*

In Section 4.1, we considered the case $n=1$ for any given $N$, whereas Section 4.2 examines the case of $N=3$ for an endogenous determination of ETIP optimal $n$ that need not be equal to one.[21] We now characterize the general case of $N>3$ allowing for multiple public facility sites. We begin by showing:

**Lemma 5** *Assume (A2) with $\alpha<1$, Condition E, the linear commuting cost schedule specified as in (1), and the utility function specified as in (8). In an ETIP optimum, the following properties always hold:*

- **(i)** *for any even number $N$, there should be no more than $N/2$ public facilities;*
- **(ii)** *for any odd number $N$ that is a multiple of 3, there can be no more than $2N/3$ public facilities;*
- **(iii)** *for an odd number $N$ that is not a multiple of 3, there may be as many as $N-1$ public facilities.*

Part (iii) of Lemma 5 is due to the consideration of identical LPG provision that requires the number of consumers patronizing each site to be the same. Although the completely dispersed ($n=N$) public facility configuration can never be optimal, it is possible to have the maximum dispersed ($n=N-1$) public facility configuration as a candidate for ETIP optimality. In general, for any number $N>3$, a maximum dispersed public facility configuration can be ETIP optimal only if $N$ is not a multiple of 2 or 3.

Before establishing the general proposition for $N>3$ in Lemma 6 and Proposition 4 below, we note that with regard to concentrated ($n=1$) versus dispersed ($n>1$) public facility configurations, Proposition 3 can be applied to the general case of $N>3$. In particular, an ETIP optimal public facility configuration is always dispersed when there is a high degree of LPG congestability, it is costly to commute and household valuation of

[21] The case of $N=2$ is trivial: the ETIP optimal configuration must be $n=1$ and $\eta=1$ (concentrated and centralized). It is therefore omitted from our discussion.

the LPG is low. Otherwise, there are multiple ETIP optimal public facility configurations where the optimal configurations could be concentrated or dispersed.

More generally, we can characterize ETIP locations of public facilities for $N>3$ given the linear commuting cost schedule as in (1) and the utility function specification as in (8). For illustrative purposes, we focus on a set of parameters such that an ETIP optimum features $n=2$.

**Lemma 6** *Assume (A2) with* $\alpha<1$*, Condition E, the linear commuting cost schedule specified as in (1), and the utility function specified as in (8). Given* $N>3$*, the set of parameters resulting in a cost-minimizing ETIP optimal number of public facilities equal to two is nonempty.*

**Proposition 4** *Assume (A2) with* $\alpha<1$*, Condition E, the linear commuting cost schedule specified as in (1), and the utility function specified as in (8). Given* $N>3$ *and a set of parameters such that* $\arg\min_n C(n,H^*(n))=2$*, ETIP optimal locations of public facilities under cost-minimizing travel assignments possess the following properties:*

- **(i)** *when N is a multiple of 4,* $H=\{N/4,3N/4\}$*;*
- **(ii)** *when N is even but not a multiple of 4,* $H=\{\eta_1,\eta_2\}$ *with* $\eta_1\in[[N/4],[N/4]+1]$ *and* $\eta_2\in[N-[N/4]-1,N-[N/4]]$*;*
- **(iii)** *when N is an odd number,* $H=\{[(N+1)/4],N-[(N+1)/4]\}$*.*

## 5. On the non-identical provision of the local public good

Up to this point, we have restricted our attention to the case of identical LPG provision. In this section, we study the consequences of relaxing the IP assumption. Recall that in Section 3 we have verified the existence of a Pareto optimum that is not necessarily IP. Our focus here is on showing that a feasible allocation with non-identical provision of a LPG can Pareto dominate any IP feasible allocation.

**Proposition 5** *Assume (A2) with* $\alpha<1$*, Condition E, the linear commuting cost schedule specified as in (1), and the utility function specified as in (8). An ETIP optimum may be Pareto inferior to a non-identical provision feasible allocation.*

Since ET only involves reallocation of the composite good, Proposition 5 suggests that a non-IP feasible allocation can Pareto dominate any IP feasible allocation. By comparing the non-IP, ET allocation with the ETIP optimum in the Proof of Proposition 5, the number, the locations and the levels of LPG provision are identical. The lone difference is the travel assignment and hence the allocation of users of each public facility. Thus, an IP travel assignment is generally Pareto suboptimal.

To gain further insight towards understanding the intuition underlying the inferiority of IP feasible allocations, we note that for the example constructed in the Proof of Proposition 5, the inferiority of an IP feasible allocation is due to a higher aggregate commuting cost. However, this is not the only reason for inferiority. In Appendix A, we consider a case with $N=3$ in which the ETIP optimum is Pareto dominated by a non-IP, ET feasible allocation that does not lower aggregate commuting cost but generates a net per capita utility gain of $(1-\alpha)(5\ \ln 2-\ln 3)/3$ due to the reduced congestion externality—the net utility gain vanishes as the degree of congestability diminishes (i.e., $\alpha \rightarrow 1$).

In summary, a non-IP feasible allocation can Pareto dominate any IP feasible allocation. On the one hand, a non-IP feasible allocation can improve welfare by reducing aggregate commuting costs through more flexible travel assignments and/or non-identical provision of LPGs at different public facility sites. On the other, a non-IP feasible allocation can make individuals better-off by reallocating the users of public facilities to decrease the congestion externality associated with LPGs. This suggests that the command optimum considered in the literature is generally Pareto suboptimal and more research is needed to explore systematically non-IP feasible reallocations in the interest of the community's welfare. It also suggests that IP restrictions in law can reduce welfare.

## 6. Concluding remarks

We have examined a finite-household model with a congestible LPG where the number and locations of facility sites and the level of LPG provision are endogenously determined. Along these lines, there are at least three possible avenues of future research. First, one may explore the theoretical front, examining the nonemptiness of an ETIP core (thus refining the set of ETIP optima) and the validity of the first and second welfare theorems based on Lindahl or competitive spatial equilibrium. Second, one may relax the assumption of identical households to allow for, say, two types of households, with heterogeneous tastes. One may study whether an optimal public facility configuration is geographically biased toward the rich or a consumer who likes the LPG. Finally, it is possible to consider more than one LPG. When all LPGs are necessities, it is interesting to examine whether a concentrated public facility configuration may still emerge where different LPGs are provided at the same site.

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## Appendix A

This Appendix provides a sketch of the proofs with the details given in Berliant et al. (2005).

**Derivation of Eq. (5) and Proof of Lemmas 1 and 2** Manipulating the first-order conditions with respect to $z$ and $K$ yields $1/\Lambda(\cdot)=1$, or (5). The second-order condition, $-(d\Lambda_0/dK)/\Lambda_0^2<0$, is ensured by Condition $K$. The first-order condition with respect to $\theta$ implies (6a) whereas (6b) follows immediately under ETIP. By comparing $v(n,H)$ for each $n\in X$, the optimal $n^*$ is pinned down. □

**Proof of the Main Theorem** Denote the set of feasible allocations as $A=\{(\{z_i,Y_i\}_{i=1,\ldots,N}, \{K_j,M_j\}_{j=1,\ldots,n}, \{\theta(i,\eta_j)\}_{i=1,\ldots,N;\ j=1,\ldots,n}, n, H)$: $\{z_i,Y_i\}_{i=1,\ldots,N}$ and $\{K_j\}_{j=1,\ldots,n}$ satisfy (2), $Y_i\subset H\ \forall i=1,\ldots, N$, $\sum_{i=1}^{N}\theta(i,\eta_j)=M_j\ \forall\eta_j\in H$, $\sum_{j=1}^{n}M_j=N$, $n=|H|\in X$ and $H=\{\eta_1,\ldots,\eta_n\}\subset L\}$. Under (A1) and noting that $|X|=N$ is finite, it is obvious that $A$ is compact (closed and bounded). We then show that the set of IP feasible allocations is closed. Denote the set of IP feasible allocations as $P=\{(\{z_i,Y_i\}_{i=1,\ldots,N}, \{K_j,M_j\}_{j=1,\ldots,n}, \{\theta(i,\eta_j)\}_{i=1,\ldots,N;\ j=1,\ldots,n}, n, H)\in A$: each site $\eta_j\in H$ has a LPG quantity $K_j=K/n$ and $\{Y_i\}_{i=1,\ldots,N}$ and $\{\theta(i,\eta_j)\}_{i=1,\ldots,N;\ j=1,\ldots,n}$ are consistent with $\sum_{k=1}^{n}\theta(i,\eta_j)=M_j=\sum_{i=1}^{N}\frac{1}{k}\left|\{i:\ \eta_j\in Y_i, |Y_i|=k\}\right|=\frac{N}{n}\ \forall\eta_j\in H, i=1,\ldots,N, j=1,\ldots,n\}$. The IP restrictions only involve randomization probabilities $\theta(i,\eta_j)$, which takes real values from a compact set [0,1], and quantities $n$, $y_i\in Y_i$ $(i=1,\ldots,N)$ and $\eta_j\in H$ $(j=1,\ldots,n)$, which all take values from 0 to $N$. Note that all variables are defined in Euclidean spaces and that more than one facility is allowed to locate at a given point in $L$ when the set of feasible allocations $A$ is constructed. Thus, the closedness property is trivial and $P$ is compact as it is a closed subset of a compact set $A$. Further denote the set of ETIP feasible allocations as $F=\{(\{z_i,Y_i\}_{i=1,\ldots,N}, \{K_j,M_j\}_{j=1,\ldots,n}, \{\theta(i,\eta_j)\}_{i=1,\ldots,N;\ j=1,\ldots,n}, n, H)\in P$: $u(z_i,G(n,K))=v_o\geq 0$ for all $i=1,\ldots,N\}$. We claim that $F$ is closed. Under ETIP and Condition $K$, we have $K=K_0$ by Lemma 1 and aggregate-cost-minimizing travel assignments by Lemma 2. It is sufficient to focus on continuous quantities $Z=(z_1,\ldots,z_N, \theta(1,\eta_1),\ldots,\theta(N,\eta_1),\theta(1,\eta_2),\ldots,\theta(N-1,\eta_n),\theta(N,\eta_n))$. Consider now a sequence of ETIP allocations $a^{\ell}=(\{z_i^{\ell},Y_i\}_{i=1,\ldots,N}, \{K_0,M_j\}_{j=1,\ldots,n}, \{\theta(i,\eta_j)\}_{i=1,\ldots,N;\ j=1,\ldots,n}, n, H)\in F$ for all $\ell=0,1,2,\ldots$ and it has a limit point $a=(\{z_i,Y_i\}_{i=1,\ldots,N}, \{K_0,M_j\}_{j=1,\ldots,n}, \{\theta(i,\eta_j)\}_{i=1,\ldots,N;\ j=1,\ldots,n}, n, H)\in P$. By construction, $u(z_i^{\ell},G(n,K_0))=v_o\geq 0$ for all $i=1,\ldots,N$. Since $u$ is continuous by (A3), we have $u(z_i,G(n,K_0))=v_o$ and hence $a\in F$, which proves the assertion. As $F$ is a closed subset of a compact set $P$, $F$ is compact.

Next, we show that $F$ is nonempty. By constructing an IP public facility configuration $(n,H)$ with $n=N$, $y_i=\eta_i\in(i-1, i]$, $K_i=K_0/N$, $\theta(i,\eta_j)=1$ for all $\eta_i\in(i-1, i]$, $\theta(i,\eta_j)=0$ for all $\eta_i\notin(i-1, i]$, and $M_i=1$ $(i=1,\ldots,N)$, the corresponding total commuting cost is $C(n,H)=0$ and $G(n,K_0)=K_0/N$. Under Condition $E$, $E-K_0-C(n,H)>0$ and by choosing $z_i=(E-K_0)/N$, such allocations are ETIP feasible, implying $F\neq\phi$.

Because $n$, $y_i\in Y_i$ (i=1,…,$N$) and $\eta_j\in H$ $(j=1,\ldots,n)$ are discrete and finite and $u$ is continuous for each fixed $(\{Y_i\}_{i=1,\ldots,N}, n, H)$, the ETIP optimality problem has a continuous objective over a compact and nonempty domain $F$. By straightforward application of the Maximum Value Theorem implied by the Bolzano–Weierstrauss Theorem (Bartle, 1976, pp. 154–155), an ETIP optimum exists. □

**Proof of Proposition 1** With $n=1,|Y_i|=1$ for all $i \in X$. To prove part (i), we use Fig. 1 to show that the allocation $(z_i,K,n,H)=(E/N-K_0/2,K_0,1,\{1\})$ is ETIP optimal. To prove part (ii), we use Fig. 2(a) to show that any allocation with $H=\{\eta\}$ where $\eta \in [[N/2], [N/2]+1]$ is ETIP optimal. □

**Proof of Lemma 3** This lemma can easily be proved by contradiction. □

**Proof of Lemma 4** By utilizing Fig. 2(b), we show that public facilities must be located at 1 and 2. □

**Proof of Lemma 5** Parts (i) and (ii) can easily be proved by contradiction. For part (iii), it is sufficient to consider the case of $N=5$, where there are five IP optimal candidates with $n=1$, 2, 3, 4 and 5 and $H^*(1) \in [2,3]$, $H^*(2)=\{1,4\}$, $H^*(3)=\{1,5/2,4\}$, $H^*(4)=\{1,2,3,4\}$ and $H^*(5)=\{\eta_j\}_{j=1,\ldots,5}$ with $\eta_j \in [j-1, j]$. Since $C$ is minimized at $C(4, H^*(4))=C(5,H^*(5))=0$, the ETIP optimum must be associated with $n=4$ or 5 provided that $K_0<E<K_0+t/3$. These arguments together with Lemma 3 imply that for an odd number that is not a multiple of 3, it is possible to have an ETIP optimal configuration with $n=N-1$. □

**Proof of Lemma 6** Substituting (3), (7) and $K=K_0$ into (8) and equating utility values for $n=1$ and 2, given $N>3$, yield: $C(1,H^*(1))-C(2,H^*(2))=(\ln 2)\ (1-\alpha)/\gamma$. For $N=4$, $C(1,H^*(1))-C(2,H^*(2))=2t$. Define $I_N \equiv \left\{2+\sum_{J=5}^{N}\left(\left[\frac{J-3}{2}\right]-\left[\frac{J-5}{4}\right]\right)\right\}$. By induction, $C(1,H^*(1))-C(2,H^*(2))=tI_N$ for all $N \geq 5$. Let $\hat{t}=(\ln 2)(1-\alpha)/(\gamma I_N)$. For any $t$ with $t-\hat{t} \to 0$ from above, $n=2$ is ETIP optimal. □

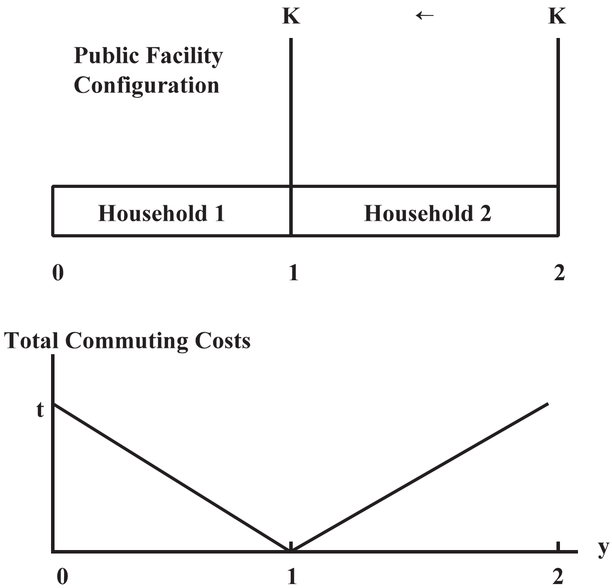


Fig. 1. Public facility configuration and total commuting costs with $N=2$.

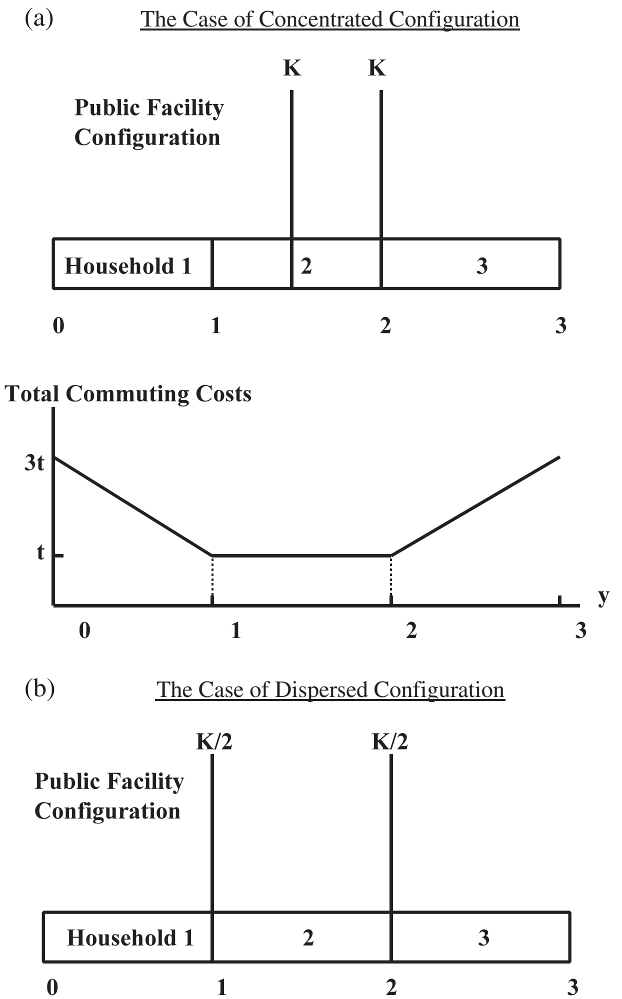


Fig. 2. Public facility configuration and total commuting costs with $N=3$.

**Proof of Proposition 4** Case (i) is trivial. For case (ii), divide the entire region evenly into two where each half has a public facility site and since $N$ is not a multiple of 4, an odd number of households resides in each half. By symmetry, we can restrict attention to the left half. From Proposition 1(ii), the ETIP optimal site must be in $[[N/4], [N/4]+1]$ where an odd number of households reside. For case (iii), we begin by removing the household residing in the central region (denoted $i_m$) and noting that $[0, [N/2]]$ must contain one of the two sites. If $[N/2]$ is even, by Proposition 1(i), the ETIP optimal public facility location is at $\eta=[N/2]/2$ and adding back $i_m$ would not change the result. If $[N/2]$ is odd, case (ii) above implies the optimal location is in $[[N/4], [N/4]+1]$, but by adding back $i_m$, the optimal location becomes $\eta=[N/4]+1$. Combining these two subcases, we obtain the ETIP optimal location as $\eta=[(N+1)/4]$. □

**Proof of Proposition 5** It is sufficient to examine the special case of $N=5$ where there are four possible IP feasible allocations with $n=1$, 2, 4 and 5 consistent with cost-minimizing travel assignments and where an ETIP allocation with non-cost-minimizing travel assignments in the case of $n=3$ is dominated by a non-IP feasible allocation. Specifically, the IP feasible allocation with $n=3$ involving lowest $C$ is $H=\{1,5/2,4\}$, $K_j=K_0/3$ ($K_0=5/\gamma$, $j=1,2,3$), $\theta(1,1)=\theta(5,4)=\theta(3,5/2)=1$, $\theta(2,1)=\theta(4,4)=2/3$, $\theta(2,5/2)=\theta(4,5/2)=1/3$, and $z=[E-K_0-t/3]/5$. If $\max\{(\frac{3\alpha}{\gamma})[4\ln 2 - 5\ln(\frac{5}{3})], \frac{15(1-\alpha)}{2\gamma}\ln(\frac{3}{2})\} \le t \le \frac{15(1-\alpha)}{\gamma}\ln(\frac{4}{3})$, then this allocation weakly dominates other ETIP feasible allocations with $n\neq 3$ but is inferior to a non-IP, ET feasible allocation with $H=\{1,5/2,4\}$, $K_j=K_0/3$, $\theta(1,1)=\theta(2,1)=\theta(3,5/2)=\theta(4,4)=\theta(5,4)=1$, $z_3=[E-K_0-4(\alpha/\gamma)\ln 2]/5$, and $z_i=[E-K_0+(\alpha/\gamma)\ln 2]/5$ ($i=1,2,4,5$), because this alternative allocation results in $C(n,H)=0$ with a net per capita utility gain of $\{\gamma t/3-\alpha[4\ln 2-5\ln(5/3)]\}/5$. □

**Alternative example to illustrate the inferiority of an IP feasible allocation** Consider the case of $N=3$ with $t\geq\frac{3(1-\alpha)}{\gamma}\ln 2$, so that the ETIP optimum is $H=\{1,2\}$, $K_j=K_0/2$ (with $K_0=3/\gamma$ and $j=1,2,3$), $\theta(1,1)=\theta(3,2)=1$, $\theta(2,1)=\theta(2,2)=1/2$, and $z=(E-K_0)/3$ with $C(2,H)=0$. We can construct a Pareto superior, non-IP, ET feasible allocation with $H=\{1/2,2\}$, $\{K_0/3,2K_0/3\}$, $\theta(1,1)=\theta(2,2)=\theta(3,2)=1$, $z_1=\{E-K_0+2[(1-\alpha)/\gamma]\ \ln 2\}/3$ and $z_i=\{E-K_0-[(1-\alpha)/\gamma]\ln 2\}/3$ ($i=2,3$), which maintains zero aggregate commuting cost but generates a net per capita utility gain of $(1-\alpha)\ (5\ \ln 2-3\ln 3)/3$. □

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