Subjective Ambiguity and Preference for Flexibility

Paulo Natenzon[*]

January 2016

Abstract

A preference over menus is monotonic when every menu is at least as good as any of its subsets. We show that any numerical representation for a monotonic preference can be written as a minimax utility. A minimax utility suggests a decision maker who is uncertain about her own future tastes and who exhibits an extreme form of ambiguity aversion with respect to this uncertainty. We show the minimax representation relaxes the submodularity axiom of Kreps (1979) in a setting with a finite number of choice alternatives. Finally, we show that every monotonic preference admits a representation with a weakly increasing aggregator of indirect utilities, but submodular preferences (and only the submodular) admit a representation with a strictly increasing aggregator.

1 Preference for Flexibility

Let Z be an arbitraty set of choice objects and let $X \subset 2^Z \setminus \{\varnothing\}$ be a collection of non-empty menus containing Z. A preference over menus is given by a binary relation $\succcurlyeq \subset X \times X$. A preference over menus is *monotonic* if $x \supset x'$ implies $x \succcurlyeq x'$. A function $U : X \to \mathbb{R}$ is a numerical representation for $\succcurlyeq$ whenever for all $x, x' \in X$ we have $x \succcurlyeq x'$ if and only if $U(x) \geq U(x')$. Our main result is that we can rewrite any numerical representation for a monotonic preference over menus in minimax form over a set of subjective states.

Theorem 1. *Any representation U for a monotonic preference $\succcurlyeq$ can be rewritten as*

$$U(x) = \min_{s \in S} \max_{z \in x} u(z, s) \tag{1}$$

where S is a set of subjective states and $u : X \times S \to \mathbb{R}$ is a state-dependent utility.

Proof. Since U represents $\succsim$ it must be bounded above, attaining its maximum at Z, the largest menu. Assume without loss of generality that $U(Z) = 0$.

Take $S = X$ and for each menu $s \in S$ let $\mathbb{I}_s : Z \to \mathbb{R}$ be the indicator function defined by $\mathbb{I}_s(z) = 1$ if $z \in s$ and $\mathbb{I}_s(z) = 0$ otherwise. Define $u : Z \times S \to \mathbb{R}$ by $u(z, s) = U(s)\mathbb{I}_s(z)$ for each $z \in Z$ and $s \in S$.

Note that for any $s \in S$ we have $\max_{z \in x} u(z, s)$ equal to $U(s)$ whenever $x \subset s$ and equal to zero otherwise. Also whenever $s \supset x$ we have $s \succsim x$ and therefore $U(s) \geq U(x)$. Moreover, for each $x \in X$ the set $\{\max_{z \in x} u(z, s) : s \in S\}$ attains its minimum at $s = x$. Hence the right-hand side of (1) is a well-defined function of x and for all $x \in X$ we have

$$\min_{s \in S} \max_{z \in x} u(z, s) = \min_{s \supset x} \max_{z \in x} u(z, s)$$
$$= \min_{s \supset x} U(s)$$
$$= U(x) \qquad \qquad \square$$

The representation (1) has the following interpretation. The decision maker chooses a menu today as if she were unsure about what her ranking of the alternatives in Z will be tomorrow. She can foresee a few possibilities represented by the subjective state space S. Each element $s \in S$ renders a utility index $z \mapsto u(z, s)$ representing a possible ranking over the alternatives in Z. The decision maker exhibits ambiguity aversion with respect to this subjective uncertainty. In other words, she is extremely pessimistic when evaluating which subjective state will occur when she chooses a particular menu.

A parallel can be drawn between representation (1) and the multiple priors model of Gilboa and Schmeidler (1989). To see this formally, identify the choice of a menu $x \in X$ with an act $f_x : S \to Z \times S$ that delivers a payoff $f_x(s) = (\arg\max_{z \in x} u(z, s), s)$ which depends on the realization of the uncertain state $s \in S$. The decision maker exhibits an extreme form of ambiguity aversion and has a set of priors equal to the entire simplex $\Delta(S)$. She evaluates each menu $x \in X$ (represented by the act f_x) according to

$$U(x) = \min_{s \in S} \max_{z \in x} u(z, s)$$
$$= \min_{\mu \in \Delta(S)} \sum_{s \in S} \mu(s) \max_{z \in x} u(z, s)$$
$$= \min_{\mu \in \Delta(S)} \int_S u \circ f_x \, d\mu$$

which is the well-known functional form in the multiple priors model.

There is also one fundamental difference with the multiple priors model of of Gilboa and Schmeidler (1989). In the multiple priors model the state space is objectively described. In contrast, here the state space is subjective, unobserved and obtained as part of the representation.

Remark 1. Theorem 1 is silent on the existence of a numerical representation. These conditions are well-known. In particular, it follows immediately from Theorem 1 above and Theorem 3.1 in Fishburn (1970), that a preference relation $\succeq \subset X \times X$ admits a numerical representation $U : X \to \mathbb{R}$ given by (1) if and only if it is complete, transitive, monotonic, and has an order-dense countable subset.

2 Application to Discrete Environments

As an application, we explore the consequences of Theorem 1 in settings where the set of choice objects Z is finite. This is the setting of the preference for flexibility model of Kreps (1979) and the costly contemplation model of Ergin (2003). We obtain a representation that is behaviorally equivalent to the costly contemplation representation of Ergin (2003), but with a radically different interpretation. Our decision maker violates submodularity as a result of pessimism, even in the absence of costly contemplation.

Let $X = 2^Z \setminus \{\varnothing\}$ be the collection of all non-empty menus formed from objects in Z. Kreps (1979) imposes the following axioms on the preference $\succeq \subset X \times X$:

Axiom (A1). Weak order: $\succeq$ is complete and transitive.

Axiom (A2). Monotonicity: $x \supset x'$ implies $x \succeq x'$.

Axiom (A3). Submodularity: $x \sim x \cup x'$ implies $x \cup x'' \sim x \cup x' \cup x''$ for all $x'' \in X$.

A1 is standard and allows a numerical representation in this finite setting. A2 can be interpreted as the decision maker never valuing commitment, and allows preference for flexibility. A3 is also referred to as *additivity*. Kreps (1979) proved that $\succeq$ satisfies A1–A3 if and only if there exists a finite set of subjective states S and a state dependent utility index $u : Z \times S \to \mathbb{R}$ such that $\succeq$ is represented by

$$U(x) = \sum_{s \in S} \max_{z \in x} u(z, s). \tag{2}$$

The content of Kreps' representation is ordinal. The summation operator in (2) can be replaced with other strictly increasing aggregators of the vector of second period

indirect utilities. For example, (2) can be re-written as an expectation with respect to a probability measure over the space of subjective states S.[1] As the next result shows, having a strictly increasing aggregator in (2) is directly related to the submodularity axiom.

Proposition 2. $\succcurlyeq$ *satisfies A1 and A2 if and only if there exist a finite set of subjective states S and a state dependent utility index $u : Z \times S \to \mathbb{R}$ such that $\succcurlyeq$ is represented by*

$$V(x) = \min_{s \in S} \max_{z \in x} u(z, s). \tag{3}$$

Proof. Since Z is finite, X is finite and under A1 $\succcurlyeq$ admits a numerical representation $V : X \to \mathbb{R}$. Apply Theorem 1 to rewrite V as in (3). It is clear from the proof of Theorem 1 that S can be taken to be finite in this case. $\qquad\square$

Example 1. Consider the decision problem of an individual who needs to buy pasta to cook for dinner next Sunday (cf. the car dealership choice example in Ergin (2003)). The pasta alternatives are manicotti, linguine and tagliatelle, and accordingly we write $Z = \{m, \ell, t\}$. We observe her preferences over menus, given by

$$\{m, \ell, t\} \succ \{\ell, t\} \succ \{m, t\} \sim \{t\} \succ \{m, \ell\} \sim \{\ell\} \succ \{m\}$$

which satisfy A1 and A2 but not A3. For these particular preferences, an additive representation as in (2) is not possible, but it is easy to find a representation as in (3). Define $S = \{s_1, s_2, s_3, s_4\}$ and let $u : Z \times S \to \mathbb{R}$ be summarized by

	s_1	s_2	s_3	s_4
$u(m, \cdot)$	0	-2	-3	-4
$u(\ell, \cdot)$	-1	0	-3	0
$u(t, \cdot)$	-1	-2	0	0

it is easy to check that $V : X \to \mathbb{R}$ defined as in (3) with these choices of S and u represents $\succcurlyeq$. $\qquad\diamond$

The decision maker in Example 1 values flexibility, in the sense that having more varieties of pasta available never hurts. The representation provided in the Example

---

[1] In a finite choice environment it is impossible to pin down a unique probabilistic belief over the subjective states. Dekel et al. (2001) and Dekel et al. (2007) obtain uniqueness results by enriching the set choice of objects with a lottery structure and imposing a normalization on the subjective state space. Sadowski (2013) shows beliefs can be uniquely identified in a model with an exogenous state space of publicly observable information, when choice is assumed to be sufficiently dependent on information. Other recent advances in the identification of beliefs on a subjective space include Chatterjee and Krishna (2011), Ahn and Sarver (2013), and Schenone (2016).

suggests a decision maker who can foresee four possible subjective states when the time comes to choose a type of pasta to cook for dinner on Sunday. Each menu is evaluated according to the worst case scenario for the realization of the state. For example, adding the manicotti pasta to the menu that only has tagliatelle leaves the decision maker indifferent, since in both cases the worst case scenario is the realization of state s_2 and the maximum attainable utility in both cases is -2. But adding the same manicotti pasta to a menu that already contains linguine and tagliatelle leaves the decision maker strictly better off, since it offers full insurance. The full menu guarantees the decision maker can have her favorite option no matter what subjective state is realized.

2.1   Pessimism versus Costly Contemplation

Our decision maker is pessimistic: she evaluates each menu of alternatives using the worst case scenario for the realization of the subjective state. In Proposition 2 we applied our main result to the special case of a finite universe of choice alternatives and obtained a representation that is behaviorally indistinguishable from the costly contemplation model of Ergin (2003).

To compare the costly contemplation and the pessimism representations, consider how the costly contemplation model rationalizes the preferences in Example 1. Following Ergin (2003), let the state dependent utility index be given by

	s_1	s_2	s_3
$u(m,\cdot)$	-5	0	5
$u(\ell,\cdot)$	-2	1	4
$u(t,\cdot)$	0	0	0

and let the decision maker's prior beliefs over the three subjective states be given by $\mathbb{P}(\{s_1\}) = 0.6$, $\mathbb{P}(\{s_2\}) = 0.3$, and $\mathbb{P}(\{s_3\}) = 0.1$. In addition, this decision maker dislikes thinking about what kind of pasta is the best option for dinner on Sunday. This is modeled by a set of contemplation strategies and contemplation cost function. Suppose the decision maker has two contemplation strategies available: the first contemplation strategy is to not contemplate at all. The state of knowledge in this case is described by the trivial partition of the subjective state space $\{S\}$ and carries a contemplation cost $c(\{S\}) = 0$. The second contemplation strategy perfectly reveals the state. It corresponds to the partition $\{\{s_1\}, \{s_2\}, \{s_3\}\}$ and carries a contemplation cost $c(\{\{s_1\}, \{s_2\}, \{s_3\}\}) = 0.6$.

The costly contemplation decision maker chooses a menu that balances two goals: maximizing the expected value of final consumption, while keeping contemplation costs down. For instance, adding the manicotti pasta to the menu that contains the single linguine option is not enough of an incentive to engage in costly contemplation. Contemplating carries a utility cost of 0.6; the benefit of contemplating is cooking manicotti instead of linguine whenever the unlikely subjective state s_3 is realized, which only brings an expected utility gain of 0.1. We have $\{m, \ell\} \sim \{\ell\}$ because in both menus the decision maker optimally avoids contemplation and cooks the linguine.

Now adding the same manicotti pasta option to the menu that contains not only linguine but also tagliatelle is a different story. The presence of linguine and tagliatelle makes it worth it contemplating and fully revealing the subjective state. Contemplating allows the decision maker to cook the linguine whenever states s_2 and s_3 occur. This in turn brings an expected utility gain of 0.7, which more than compensates the utility cost of contemplation. Once contemplation is optimal, adding the manicotti option brings an additional expected utility gain while incurring no additional contemplation cost. Hence $\{m, \ell, t\} \succ \{\ell, t\}$, violating the submodularity axiom A3.

In contrast to the costly contemplation model, the decision maker of Proposition 2 has no trouble choosing the best pasta to cook once Sunday dinner time comes around. In other words, she does not find it costly to contemplate. Accordingly, the formula in Proposition 2 does not include a contemplation partition nor a contemplation cost function. The source of failures of submodularity for our decision maker is pessimism. When evaluating a menu of options, she always thinks of the worst case scenario for the realization of the subjective state.

Ergin (2003) shows that the costly contemplation decision maker satisfies the submodularity axiom A3 if and only if it is possible to find a costly contemplation representation with a zero contemplation cost function. In the next subsection, we show that submodularity holds for our decision maker if and only if it is possible to find a representation with a strictly increasing aggregator of indirect utilities.

2.2 Pessimism and Submodularity

When the preference $\succeq$ satisfies all three axioms, it of course still admits a representation as in (3). In fact, it is possible to find a pair of representations as in (2) and (3) that share the same finite subjective state space and the same state dependent utility function. We leave the proof of this result to the Appendix:

Proposition 3. $\succcurlyeq$ *satisfies A1, A2 and A3 if and only if there exist a finite set of subjective states S and a state dependent utility function $u : Z \times S \to \mathbb{R}$ such that $\succcurlyeq$ is represented at the same time both by U as defined in (2) and by V as defined in (3).*

Proposition 3 makes clear that any distinction for monotonic preferences that satisfy the submodularity axiom A3 cannot be based on any properties of the state space S nor the utility function u. The main takeaway is that Axiom A3 corresponds in the representation to a property of the aggregator of second period indirect utilities. Say that a function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is *increasing* if $a_i > b_i$ for all $i = 1, \ldots, n$ implies $\varphi(a) > \varphi(b)$. It is *strictly increasing* if $a \geq b$ and $a \neq b$ implies $\varphi(a) > \varphi(b)$. We summarize the discussion in the following result, which follows from Proposition 2 above and Theorem 1' in Kreps (1979).

Proposition 4. *If Z is finite, a binary relation on X satisfies A1 and A2 if and only if there exists a finite set S, a function $u : Z \times S \to \mathbb{R}$ and an increasing aggregator function $\varphi : \mathbb{R}^S \to \mathbb{R}$ such that if $w : X \to \mathbb{R}^S$ is defined by*

$$(w(x))(s) = \max_{z \in x} U(z, s),$$

then $\varphi \circ w$ represents $\succcurlyeq$. Moreover, $\succcurlyeq$ satisfies A3 if and only if the aggregator φ can be chosen to be strictly increasing.

3 Conclusion

We showed that whenever a preference $\succcurlyeq$ over menus of choice objects can be represented numerically and satisfies monotonicity, the numerical representation can be rewritten in minimax form. This result allowed us to generalize the model of preference for flexibility in Kreps (1979), by relaxing the submodularity assumption. Applying our main result to the special case of a finite set of alternatives, we obtained a model that is behaviorally equivalent to the costly contemplation model of Ergin (2003). This allowed us to interpret failures of submodularity as a result of pessimism, even when the decision maker does not find it costly to contemplate.

Appendix: Proof of Proposition 3

We proceed with the construction of S and $u : Z \times S \to \mathbb{R}$ as described in the proof of Theorem 1 in Kreps (1979). Recall that in Kreps' construction S is a subset of X and for each $s \in S$ we have $u(z, s) = a(s)$ if $z \in s$ and $u(z, s) = 0$ otherwise, where $a(s)$

are negative numbers. Without loss of generality, we can pick all $a(s)$ to be strictly negative integers. We obtain the representation given in equation (2).

The key to the proof is to note that if we replace any state $s_0 \in S$ with n new states $s_1, s_2, \ldots, s_n$ and for each new state s_i and each $z \in Z$ we define $u(z, s_i) = u(z, s_0)/n$, the value of $U(x)$ for any $x \in X$ in equation (2) remains unchanged.

Formally, given a state $s_0 \in S$ and and an integer $n \in \mathbb{N}$, for each $i = 1, \ldots, n$ define s_i to be the pair (s_0, i). Note that $s_i \neq s$ for all $s \in S$ and each i. Now let $\tilde{S} = \{s_1, \ldots, s_n\} \cup S \setminus \{s_0\}$ and define $\tilde{u} : Y \times \tilde{S} \to \mathbb{R}$ as follows. For each $z \in Y$ and each $s \in S \setminus \{s_0\}$, let $\tilde{u}(z, s) = u(z, s)$ and for each $i = 1, \ldots, n$ let $u(z, s_i) = u(z, s_0)/n$.

We thus have for any $x \in X$,

$$
\begin{aligned}
\sum_{s \in \tilde{S}} \max_{z \in x} \tilde{u}(z, s) &= \sum_{s \in S \setminus \{s_0\}} \max_{z \in x} u(z, s) + \sum_{i=1}^{n} \max_{z \in x} \tilde{u}(z, s_i) \\
&= \sum_{s \in S \setminus \{s_0\}} \max_{z \in x} u(z, s) + n \times \max_{z \in x} u(z, s_0)/n \\
&= \sum_{s \in S} \max_{z \in x} u(z, s) \\
&= U(x)
\end{aligned}
$$

so the representation still works with the new set of subjective states $\tilde{S}$ and state dependent utility $\tilde{u}$.

We can interpret this procedure as effectively 'splitting' a subjective state s_0 into n smaller pieces that, when added together, amount to the original state. In fact, we have

$$
\sum_{i=1}^{n} \max_{z \in x} \tilde{u}(z, s_i) = \max_{z \in x} u(z, s_0)
$$

and all we are doing is substituting the left hand side for the right hand side of this equation in the representation (2). Note that this 'splitting' procedure can be applied again to any subjective state in the new representation. The rest of the proof consists in applying this procedure repeatedly.

Since $S \subset X$ and X is finite we can partition S into $\sim$-equivalence classes $S_1, S_2, \ldots, S_m$ where for any $z \in S_j$ and $z' \in S_k$ we have $z \succsim z'$ if and only if $j \geq k$. Consider the mapping $s \mapsto j(s)$ where to each $s \in S$ we assign the number $j(s) \in \mathbb{N}$ such that $s \in S_{j(s)}$. We can extend the map j to all of X in the natural way so that j represents $\succsim$. Note that $j(x) \in \mathbb{N}$ for all $x \in X$ hence the map $x \mapsto -1/j(x)$ also represents $\succsim$.

We now proceed to apply the procedure repeatedly: begin with S_1, the $\sim$-equivalence class of $\succsim$-least preferred elements of S. For each of the original states $s \in S_1$ apply

8

the procedure above using $n = |a(s)|$. Note that the value of n will depend on the particular $s \in S_1$ and that for all newly created states s' and any $z \in Z$ the value of $u(z, s')$ will be either zero or -1.

Now proceed "upwards" through the $\sim$-equivalence classes, applying the splitting procedure to all original states in each equivalence class. Each time the 'splitting' procedure is applied to a state $s \in S_j$ choose $n = |a(s)| \times j$ so that the value of $u(z, s')$ for each newly created state s' is either zero or $-1/j$.

Since applying the 'splitting' procedure does not change the value of U, it still represents $\succsim$. Define V as in equation (3) and with an argument similar to the proof of Proposition 2 it is easy to show that $V(x) = -1/j(x)$ for all $x \in X$ and therefore V also represents $\succsim$. $\qquad \square$

Remark 2. Since in this last proof we effectively split each subjective state $s \in S$ into several new states, the number of subjective states in the representation of Proposition 3 is larger than that of Proposition 2. A modest modification of the proof above maintains the same number of subjective states. Instead of 'splitting' each state into smaller pieces, it consists of 'normalizing' state-dependent utilities so that $u(\cdot, s)$ equals either zero or $-1/j(s)$ for each $s \in S$ and adding probability weights to the representation in (2) so as to keep the value of $U(x)$ the same for every $x \in X$. We leave the details to the reader.

References

Ahn, David S and Todd Sarver, "Preference for flexibility and random choice," *Econometrica*, 2013, *81* (1), 341–361.

Chatterjee, Kalyan and R Vijay Krishna, "On preferences with infinitely many subjective states," *Economic Theory*, 2011, *46* (1), 85–98.

Dekel, Eddie, Barton L Lipman, Aldo Rustichini, and Todd Sarver, "Representing Preferences with a Unique Subjective State Space: A Corrigendum," *Econometrica*, March 2007, *75* (2), 591–600.

_ , Barton L. Lipman, and Aldo Rustichini, "Representing Preferences with a Unique Subjective State Space," *Econometrica*, July 2001, *69* (4), 891–934.

Ergin, Haluk, "Costly Contemplation," Working Paper, MIT December 2003.

Fishburn, Peter C., *Utility Theory for Decision-Making*, New York: John Wiley and Sons, 1970.

Gilboa, Itzhak and David Schmeidler, "Maxmin expected utility with non-unique prior," *Journal of Mathematical Economics*, 1989, *18* (2), 141–153.

Kreps, David M., "A Representation Theorem for "Preference for Flexibility"," *Econometrica*, 1979, *47* (3), 565–577.

Sadowski, Philipp, "Conditional Preference for Flexibility: Eliciting Beliefs from Behavior," *Theoretical Economics*, 2013, *8* (2), 503–534.

Schenone, Pablo, "Identifying subjective beliefs in subjective state space models," *Games and Economic Behavior*, 2016.