Moderate Expected Utility

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Abstract

Individual choice data often violates strong stochastic transitivity (SST) while conforming to moderate stochastic transitivity (MST). We propose a slightly stronger version of the MST postulate, which we call MST+, and we show that MST and MST+ retain significantly more predictive power than weak stochastic transitivity (WST). Our first theorem shows that a binary choice rule satisfies MST+ if and only if it can be represented by a *moderate utility model* with two parameters: a utility function describes the value of each option, and a distance metric determines their the degree of comparability. Our second theorem introduces the *moderate expected utility model* and shows how utility and distance can be identified from choice data over lotteries.

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1 Introduction

Consider a decision maker who is most likely to choose option A in a binary comparison against B, and, in turn, most likely to choose option B in a binary comparison against C. Denoting by $\rho(A,B)$ the probability of choosing A over B and by $\rho(B,C)$ the probability of choosing B over C, we have

$$\rho(A,B) \geq 1/2 \text{ and } \rho(B,C) \geq 1/2. \tag{1}$$

A simple test of the transitivity of the decision maker's choices may require the decision maker to choose A most often in a binary comparison against C,

$$\text{If (1) holds, then } \rho(A,C) \geq 1/2. \tag{WST}$$

This basic postulate is known as *weak stochastic transitivity.* WST is the most permissive condition under which an analyst may obtain a coherent ranking over the choice options from binary choice data.

A second, more stringent transitivity criterion which is well-studied in the literature is *strong stochastic transitivity*:

$$\text{If (1) holds, then } \rho(A,C) \geq \max\{\rho(A,B), \rho(B,C)\}. \tag{SST}$$

Choice models that satisfy SST (such as the classic Logit model) are typically simple to analyze but fail to accommodate many empirically relevant phenomena.

In this paper, we consider a less studied, intermediate condition called *moderate stochastic transitivity*:

$$\text{If (1) holds, then } \rho(A,C) \geq \min\{\rho(A,B), \rho(B,C)\}. \tag{MST}$$

In Section 2, we show that MST allows for many empirically relevant patterns of choice behavior ruled out by SST, and yet has significantly more empirical bite than WST. Our main contribution is to characterize a family of parametric models of individual choice behavior that 'span' the space of choice datasets satisfying MST. This family can prove useful in applications where SST is violated, while at the same time offering more predictive power than WST models.

Our main results are two representation theorems for choice behavior that exhibits a moderate degree of transitivity. First, we introduce a slight strengthening of MST,

$$\text{If (1), then } \rho(A,C) > \min\{\rho(A,B), \rho(B,C)\} \text{ or } \rho(A,C) = \rho(A,B) = \rho(B,C) \quad \text{(MST+)}$$

which we call *moderate stochastic transitivity plus*, or MST+.

Theorem 1 shows that binary choice behavior over a finite set of alternatives satisfies MST+ if and only if it is a *moderate utility model* (MUM). A binary choice rule ρ is a MUM if there exists a utility function u and a distance metric d such that

$$\rho(A,B) \geq \rho(C,D) \iff \frac{u(A)-u(B)}{d(A,B)} \geq \frac{u(C)-u(D)}{d(C,D)}. \qquad \text{(MUM)}$$

In a MUM, the decision maker's ability to discriminate among any two options A and B depends both on the utility values of A and B and on their easiness of comparison given by the distance $d(A,B)$. Note the role of the distance metric: for a given difference in value $u(A)-u(B)$, higher values of the distance $d(A,B)$ drive choice probabilities closer to fifty-fifty, that is, more distant options are harder to compare. Specific functional forms of u and d yield several familiar models from the discrete choice literature as particular instances of MUM (Section 5).

Theorem 2 enriches the domain of choice options to include lotteries over the alternatives to obtain the identification of the utility and distance parameters. By imposing the additional assumptions of continuity, linearity, convexity and symmetry, in addition to MST+, our *moderate expected utility model* (MEM) characterization identifies (i) a unique von Neumann Morgenstern expected utility function over lotteries; and (ii) a norm on the relevant linear space that is unique up to multiplication by a positive scalar.

Section 5 relates our MUM and MEM representations to the existing literature. We show that some familiar models used to address failures of SST are particular instances of MUM. We also show that MUMs neither nest nor are nested in the set of binary random utility models.

2 Moderate stochastic transitivity

Let Z be a finite set of choice options. A (binary, stochastic) *choice rule* on Z is a function $\rho : Z \times Z \to [0,1]$ such that $\rho(x,y) + \rho(y,x) = 1$ for every pair $x, y \in Z$. The number $\rho(x,y)$ denotes the probability that the decision maker selects option x in a binary comparison against y.

Let $\wedge$ and $\vee$ denote the min and max operators, respectively, so that $a \wedge b = \min\{a,b\}$ and $a \vee b = \max\{a,b\}$. The two most commonly studied notions of transitivity for binary choice data are weak stochastic transitivity (WST) and strong stochastic transitivity (SST):

(WST) $\rho(x,y) \wedge \rho(y,z) \geq 1/2 \Longrightarrow \rho(x,z) \geq 1/2$

(SST) $\rho(x,y) \wedge \rho(y,z) \geq 1/2 \Longrightarrow \rho(x,z) \geq \rho(x,y) \vee \rho(y,z)$

In this paper we focus on a less studied, intermediate form of transitivity called moderate stochastic transitivity (MST):

(MST) $\rho(x,y) \wedge \rho(y,z) \geq 1/2 \Longrightarrow \rho(x,z) \geq \rho(x,y) \wedge \rho(y,z)$

The definitions clearly imply that SST $\Rightarrow$ MST $\Rightarrow$ WST. Our main results characterize the set of choice rules that satisfy a slight stronger version of this postulate, namely

(MST+) $\rho(x,y) \wedge \rho(y,z) \geq 1/2 \Longrightarrow \rho(x,z) > \rho(x,y) \wedge \rho(y,z)$ or $\rho(x,z) = \rho(x,y) = \rho(y,z)$.

The only difference between MST and MST+ is that the case in which

$$\rho(x,y) \vee \rho(y,z) > \rho(x,z) = \rho(x,y) \wedge \rho(y,z)$$

is allowed by MST but ruled out by MST+.

Choice models that satisfy MST+ are convenient for two reasons: first, the MST+ condition holds in many applications in which the more restrictive SST condition is violated. Hence, a choice model that satisfies MST+ but allows for violations of SST may provide the flexibility that is needed to accommodate empirically relevant choice phenomena. Second, MST+ is significantly more restrictive than WST, and restricting the analysis to models that conform to MST+ results in greater predictive power.

We provide four examples to illustrate how the flexibility provided by MST+ is useful compared to the more stringent SST. The classic Example 1 suggests that violations of SST must be expected when some pairs of alternatives are easier to compare than others. Examples 2–4 show violations of SST in individual choice experiments with human and non-human subjects alike.

Example 1 (attributed to J. Savage, adapted from Tversky (1972))**.** *An individual has a difficult time comparing a trip to Paris, denoted P and a trip to Rome, denoted R, so that he is equally likely to pick either option $\rho(P,R) = 1/2$. The individual still has trouble deciding if the trip to Paris is enhanced by a €5 bonus, denoted by P^+. In other words, $\rho(P^+,R)$ is still approximately $1/2$. But when pressed to decide between the two Paris trip options, the individual clearly prefers the bonus, so that $\rho(P^+,P)$ is close to 1. SST requires that $\rho(P^+,R) \geq \rho(P^+,P)$ which is intuitively violated in this case, while MST+ only requires the more plausible inequality $\rho(P^+,R) > \rho(P,R)$.*

Example 2 (Animal studies)**.** *Lea and Ryan (2015) recorded hundreds of mating decisions* *by female túngara frogs using three male options A, B and C. In the binary choice data, option B is chosen in 63% of the trials against A; option A is chosen in 84% of the trials against C; and option B is chosen in 69% of the trials against C. Choices therefore satisfy MST+ but violate SST.*

Example 3 (Perceptual choice data)**.** *Tversky and Russo (1969) collected choice data from 168 inmates in a prison in Michingan. Correct choices in perceptual tasks were incentivized using packs of cigarettes. One task involved identifying the largest rectangle in pairwise comparisons. A tall rectangle t and a wide rectangle w of identical sizes were compared against larger and smaller rectangles of varying proportions. For any fixed difference in rectangle size, subjects made less mistakes comparing more similar shapes. This demonstrates a systematic violation of SST. For example, a larger tall rectangle t^+ was chosen 90% of the time against t, and 77% of the time against w. Conversely, a smaller tall rectangle t^- was chosen 7% of the time against t, and 40% of the time against w. If $\rho(w,t) \geq 1/2$ then pairwise choices involving w, t and t^- violate SST. On the other hand, if $\rho(w,t) \leq 1/2$ then we have a violation of SST involving t^+, t and w. A priori, one expects exactly $\rho(w,t) = 1/2$ since s and w have identical size. Their experimental data supports this hypothesis, and MST+ holds for any value of $\rho(w,t)$ in the range between and 23% and 60%.*

Example 4 (Choice data over lotteries)**.** *Soltani, De Martino and Camerer (2012) recorded thousands of choices by 21 male Caltech undergraduates using simple lotteries (p, m) that pay m dollars with probability p in the lab. Two lotteries h and ℓ were fine-tuned to each individual to be approximately indifferent, (i.e., equally likely to be chosen in a binary comparison). Slightly perturbed versions of h and ℓ were then offered for comparison against several types of 'decoy' lotteries. Decoy lottery d_4 dominates ℓ and was chosen 95% of the time against ℓ but only 78% of the time against h. Decoy lottery d_6, on the other hand, is dominated by ℓ and was chosen 4% of the time against ℓ and 33% of the time against h. The same reasoning as in Example 3 shows these choices satisfy MST+ but violate SST.*

Relaxing SST to MST+ allows the analyst to address the range of empirical phenomena illustrated by the examples above. At the same time, MST+ retains significantly more empirical bite than WST. To see this, suppose the choice rule ρ on Z satisfies WST. Enumerate the n options in $Z = \{x^1, x^2, \ldots, x^n\}$ in such a way that $\rho(x^i, x^j) \geq 1/2$ whenever $i \leq j$. For the sake of simplicity, let us assume that choice probabilities differ whenever possible, so that the set $\{\rho(x, y) \in [0, 1] : x \neq y\}$ has maximum cardinality with $n(n-1)$ elements.

When $Z = \{x^1, x^2, x^3\}$ has three alternatives, WST allows ρ to have six strict orderings:

$$\rho(x^1, x^3) > \rho(x^1, x^2) > \rho(x^2, x^3)$$

$$\rho(x^1, x^3) > \rho(x^2, x^3) > \rho(x^1, x^3)$$

$$\rho(x^1, x^2) > \rho(x^1, x^3) > \rho(x^2, x^3)$$

$$\rho(x^2, x^3) > \rho(x^1, x^3) > \rho(x^1, x^2)$$

$$\rho(x^1, x^2) > \rho(x^2, x^3) > \rho(x^1, x^3)$$

$$\rho(x^2, x^3) > \rho(x^1, x^2) > \rho(x^1, x^3)$$

MST+ rules out the last two of the six strict orderings, where $\rho(x^1, x^3) < \rho(x^2, x^3) \wedge \rho(x^1, x^2)$. Let $\#WST(n) = [n(n-1)/2]!$ denote the number of strict orderings allowed by WST when Z has n options, and likewise, let $\#MST(n)$ denote the number of strict orderings allowed by MST+. The ratio $\#MST(n)/\#WST(n)$ can be interpreted as a measure of the restriction imposed on observable choice data by MST+ compared to WST. In the case $n = 3$ we just showed the ratio $\#MST(3)/\#WST(3)$ is equal to 2/3. This ratio decreases to less than 1/4 when $n = 4$ and less than 1/17 when $n = 5$. In the Appendix, we prove the ratio is arbitrarily small when n large:

Proposition 1. $\lim_{n\to\infty} \#MST(n)/\#WST(n) = 0$.

3 Moderate utility model

A choice rule ρ on a finite set Z is a *moderate utility model (MUM)* if there is a utility function $u : Z \to \mathbb{R}$ and a distance metric $d : Z \times Z \to \mathbb{R}_+$ such that, for all $w \neq x$ and $y \neq z$,

$$\rho(w,x) \geq \rho(y,z) \iff \frac{u(w)-u(x)}{d(w,x)} \geq \frac{u(y)-u(z)}{d(y,z)} \tag{2}$$

In particular, by taking $w = z \neq x = y$ above, it is easy to see that in a MUM we have $\rho(x,y) \geq 1/2$ if and only if $u(x) \geq u(x)$ for any $x, y \in Z$.

Taking the distance d in (2) to be the special case of the discrete metric $d(x,y) = 1$ if $x \neq y$ and $d(x,x) = 0$ for all x, we obtain Debreu's (1958) cardinal utility representation:

$$\rho(w,x) \geq \rho(y,z) \iff u(w) - u(x) \geq u(y) - u(z) \tag{3}$$

The role of a non-trivial distance metric d in a MUM is to make the choice probabilities of options that are harder to compare closer to 1/2.

A non-trivial distance metric d gives MUMs the flexibility that is needed to accommodate empirical violation of SST. For example, in the choice of a trip in Example 1, the three options have comparable utility values, the two Paris trips are close according to the metric d (which makes them easy to compare) while the trip to Rome option is more distant to each of the Paris options and therefore hard to compare. In Section 5, we show that several binary discrete choice models used in the literature (including probit, Bayesian probit, and elimination-by-aspects) are particular cases of MUMs.

It has been shown that all MUMs satisfy the MST condition (Halff, 1976). In our first characterization result, we show that MUMs also satisfy the stronger MST+ condition. In fact, we show that MST+ is both necessary and sufficient for a choice rule to be a MUM.

Theorem 1. *A choice rule ρ on a finite Z is a MUM if and only if it satisfies MST+.*

4 Moderate expected utility

We continue to let Z be a finite set of objects and we extend the domain of choice alternatives to the set of all lotteries over Z, denoted by Δ. We identify Δ with the $n-1$ dimensional simplex $\{x \in [0,1]^n : x_1 + \cdots + x_n = 1\}$. The function $U : \Delta \to [0,1]$ is an *expected utility function* if it is linear and onto. A choice rule $\rho : \Delta \times \Delta \to [0,1]$ is a *moderate expected utility model* (MEM) if there exist an expected utility function U and a norm $\|\cdot\|$ in $\mathbb{R}^n$ induced by an inner product, such that, for any four lotteries $w \neq x$ and $y \neq z$ in Δ,

$$\rho(x,y) \geq \rho(w,z) \iff \frac{U(x)-U(y)}{\|x-y\|} \geq \frac{U(w)-U(z)}{\|w-z\|}. \tag{4}$$

Every MEM satisfies MST+. This can be shown by repeating the argument for the MUM representation in the proof of Theorem 1. Compared to the MUM representation, however, the MEM representation is defined in the richer domain of lotteries contained in a linear vector space; it imposes linearity on the utility function U; and it requires the distance metric to be induced by an inner product. These assumptions carry additional testable implications beyond MST+.

First, every MEM is *continuous* at every point in the domain except along the diagonal $\{(x,x) \in \Delta \times \Delta : x \in \Delta\}$. Second, every MEM is *linear*, that is, for all $0 < \alpha < 1$ and any lotteries $x, y, z \in \Delta$ we have $\rho(x,y) = \rho(\alpha x + (1-\alpha)z, \alpha y + (1-\alpha)z)$. Third, every MEM ρ is *convex*, that is, whenever $\rho(x,y) = 1/2$ and $\rho(x,z) = \rho(y,z) \geq 1/2$, we have $\rho(\alpha x + (1-\alpha)y, z) \leq \rho(x/2 + y/2, z)$ for all $0 \leq \alpha \leq 1$. Fourth, every MEM is *symmetric*: let $\rho(x,z) \geq \rho(y,z) > 1/2$ and $\rho(x,y) = 1/2$. If z' is such that $\rho(z,z') \geq \rho(z,z'')$ for any $z'' \in \Delta$ then we must have $\rho(x,z') \geq \rho(y,z')$. And finally, the requirement that U is onto $[0,1]$ means that a MEM cannot be constant, that is $\rho \neq 1/2$.

Theorem 2. *ρ is a MEM iff $\rho \neq 1/2$ is linear, continuous, convex, symmetric and MST+.*

A MEM has two parameters $(U, \|\cdot\|)$ where U is a linear function and $\|\cdot\|$ is a norm induced by an inner product. Let $\mathbf{1}$ denote the linear function $\mathbf{1}(x) = x_1 + \cdots + x_n$ and note that for every pair of lotteries x, y we have $\mathbf{1}(x - y) = 0$, that is, $x - y \in \ker(\mathbf{1})$. The next proposition shows that U is uniquely identified, and that $\|\cdot\|$ is identified on the null space of U up to multiplication by a positive scalar.

Proposition 2. *If $(U_1, \|\cdot\|_1)$ and $(U_2, \|\cdot\|_2)$ are MEM representations of ρ, then $U_1 = U_2$ and there exists a constant $A > 0$ such that $\|x\|_1 = A\|x\|_2$ for all $x \in \ker(U) \cap \ker(\mathbf{1})$.*

5 Related literature

Halff (1976) proposed the original definition of a moderate utility model, which required the existence of a utility function u and a distance metric d, and, in addition, the existence of a strictly increasing real function F such that

$$\rho(x, y) = F\left(\frac{u(w) - u(x)}{d(w, x)}\right) \tag{5}$$

where $F(t) = 1 - F(1 - t)$ for all t. Since F is strictly increasing the parameters u and d in (5) must always satisfy our definition given in (2). Conversely, for any choice rule ρ satisfying our definition as in (2) for some parameters u and d, it is straightforward to construct F such that u, d, F satisfy (5). Hence, our definition of MUM is equivalent to Halff's.

Halff (1976) proved that every MUM satisfies MST, and left open the question of sufficiency. Our Theorem 1 answers the question posed by Halff by showing that, while MST is not sufficient for a choice rule to be a MUM, the slightly stronger $MST+$ condition is both necessary and sufficient.

Examples 5 and 6, below, show that two familiar discrete choice models used to address violations of SST in the literature are particular instances of MUM.

Example 5 (binary probit)**.** *The following model was first proposed by Thurstone (1927). The choice rule ρ on a finite Z is a* probit *rule if there exist a collection of random variables $(X_i)_{i \in Z}$ which are joint Gaussian distributed and such that $\rho(x, y) = \mathbb{P}\{X_x > X_y\}$ for all $x, y \in Z$. To see that every binary probit is a moderate utility model, let $u(x) = \mathbb{E}[X_x]$ for all $x \in Z$ and let $d(x, y) = \sqrt{Var(X_x - X_y)}$, which is a distance metric when we exclude the possibility of perfectly correlated random variables. Then,*

$$\rho(x, y) = \mathbb{P}\left\{\frac{X_x - X_y - \mathbb{E}[X_x - X_y]}{\sqrt{Var(X_x - X_y)}} > \frac{\mathbb{E}[X_x - X_y]}{\sqrt{Var(X_x - X_y)}}\right\} = \Phi\left(\frac{u(x) - u(y)}{d(x, y)}\right)$$

where Φ is the strictly increasing cdf of the standard Gaussian distribution.

Example 6 (Tversky's EBA)**.** *The choice rule ρ on a finite Z is an* elimination-by-aspects (EBA) *rule if there exist a mapping A that takes each option $x \in Z$ to a set of aspects $A(x)$ that x possesses, and a measure m over the set of all aspects such that*

$$\rho(x, y) = \frac{m\left[A(x)\right] - m\left[A(y)\right]}{m\left[A(x) \setminus A(y)\right] + m\left[A(y) \setminus A(x)\right]}.$$

To see that every EBA is a moderate utility model, let $u(x) = m[A(x)]$ for all $x \in Z$, let $d(x, y) = m\left[A(x) \setminus A(y)\right] + m\left[A(y) \setminus A(x)\right]$ and let f be the strictly increasing function $f(t) = 1/2 + t/2$. Then it is easy to verify that

$$\rho(x, y) = f\left(\frac{u(x) - u(y)}{d(x, y)}\right).$$

Probit and EBA are also instances of the *random utility model* (RUM). A choice rule ρ on a finite Z is a RUM if there exists a probability measure μ over the strict orderings on Z such that $\rho(x, y)$ equals the probability under μ of the event in which x beats y. Block and Marschak (1959) and Falmagne (1978) characterize the set of RUMs in an abstract setting of choice options when choice data for all finite menus is available. Gul and Pesendorfer (2006) impose linearity and provide a characterization of RUM in the richer setting of lotteries. A review of the literature that tackles the characterization of RUMs using binary choice data is provided by Fishburn (1992). Next, we show that neither MUM nor RUM nest each other.

Example 7. *We slightly modify an example given in de Souza (1983) to obtain a choice rule that satisfies MST+ but is not a RUM. Let* $Z = \{1, 2, 3, 4, 5, 6\}$ *and let the choice rule* ρ *on* Z *be given by*

$$\begin{aligned} \rho(4,5) = \rho(4,6) = \rho(2,5) = \rho(2,3) = \rho(1,6) = \rho(1,3) &= 1 \\ \rho(2,6) = \rho(1,5) &= \frac{1}{2} + \varepsilon \\ \rho(2,4) = \rho(1,4) = \rho(3,5) = \rho(3,6) &= \frac{1}{2} + \frac{\varepsilon}{2} \\ \rho(3,4) = \rho(1,2) = \rho(5,6) &= \frac{1}{2} + \frac{\varepsilon}{3} \end{aligned}$$

where $0 < \varepsilon < 1/2$*. It is straightforward to verify that* ρ *satisfies MST+. Now suppose* ρ *is a RUM generated by the probability* μ *on the set of strict orderings over* Z*. Since* $\rho(2,3) = \rho(4,6) = 1$*, for any strict ordering in the support of* μ *in which* $3 \succ 4$ *we also have* $2 \succ 3 \succ 4 \succ 6$ *and therefore* $2 \succ 6$*. This shows* μ *must assign zero probability to the intersection of events* $3 \succ 4$ *and* $6 \succ 2$*. By the same reasoning,* μ *must assign zero probability to the intersection of events* $3 \succ 4$ *and* $5 \succ 1$*; and* μ *must also assign zero probability to the intersection of events* $6 \succ 2$ *and* $5 \succ 1$*. Since* μ *is a probability measure, this implies* $\rho(3,4) + \rho(5,1) + \rho(6,2) \leq 1$*. But instead we have* $\rho(3,4) + \rho(5,1) + \rho(6,2) = 3/2 - 2\varepsilon/3 > 1$ *and therefore* ρ *cannot be a RUM.*

A converse, well-known example shows that RUM models can violate MST+. Let μ assign equal probability to three strict orderings $x \succ y \succ z$, $y \succ z \succ x$ and $z \succ x \succ y$ over the options x, y and z. Then the binary choice rule ρ generated by μ has $\rho(x, y) = \rho(y, z) = \rho(z, x) = 2/3$ which violates WST.

A Appendix: proofs

Proof of Proposition 1

Let Z be a finite set with n alternatives enumerated $x^1, x^2, \ldots, x^n$. Consider the set of choice rules ρ on Z which satisfy WST with $\rho(x^i, x^j) \geq 1/2$ whenever $i \leq j$ and for which the set $\{\rho(x, y) \in [0, 1] : x \neq y\}$ has maximum cardinality with $n(n-1)$ elements. Each such ρ induces a strict ordering $\succ_\rho$ of the $n(n+1)/2$ pairs $P_n := \{(x^i, x^j) : n \geq i > j \geq 1\}$ given by $(x^i, x^j) \succ_\rho (x^k, x^\ell)$ if and only if $\rho(x^i, x^j) > \rho(x^k, x^\ell)$. This set of choice rules ρ induces $\#WST(n) = [n(n-1)/2]!$ different strict orderings $\succ_\rho$ on P_n.

MST and MST+ allow the same number of different strict orderings over P_n which we denote $\#MST(n)$. Now consider the addition of alternative x^{n+1} to the set Z.

Lemma A.1. $\#MST(n+1) \leq [n(n-1)/2 + 1]^n \, \#MST(n)$

Proof. Take a single strict ordering over P_n compatible with MST. There are multiple ways to extend this strict ordering to incorporate the new pairs $(x^1, x^{n+1}), (x^2, x^{n+1}), \ldots, (x^n, x^{n+1})$ and obtain a strict ordering over P_{n+1} that is still compatible with MST. Since the original ordering has $n(n-1)/2$ pairs, there are $n(n-1)/2+1$ different positions to include (x^n, x^{n+1}). In this way we obtain $n(n-1)/2 + 1$ different strict orderings, all of which respect MST. The total number of strict orderings over $P_n \cup \{(x^n, x^{n+1})\}$ that satisfy MST is therefore $[n(n-1)/2+1] \, \#MST(n)$. Now we take one such strict ordering and extend it to incorporate a second pair (x^{n-1}, x^{n+1}). This pair can in principle be added into $n(n-1)/2 + 2$ different positions, but placing it in the very last position would violate MST, since MST requires $\rho(x^{n-1}, x^{n+1}) > \min\{\rho(x^{n-1}, x^n), \rho(x^n, x^{n+1})\}$. The total number of strict orderings over $P_n \cup \{(x^n, x^{n+1}), (x^{n-1}, x^{n+1})\}$ which satisfy MST must therefore be smaller or equal to $[n(n-1)/2 + 1]^2 \, \#MST(n)$. A simple inductive argument completes the proof. □

Lemma A.2. $\lim_{n\to\infty} \left[\prod_{k=1}^{n} \frac{n(n-1)/2+k}{n(n-1)/2+1}\right] = e$

Proof. The result can be shown by verifying that, for each n,

$$\left(1+\frac{1}{n}\right)^{n-1} \leq \left[\prod_{k=1}^{n} \frac{n(n-1)/2+k}{n(n-1)/2+1}\right] \leq \left(1+\frac{1}{n}\right)^{n}$$

and taking the limit as $n \to \infty$. We leave the details to the reader. □

Lemma A.1 implies that

$$\begin{aligned}\frac{\#MST(n+1)}{\#WST(n+1)} &\leq \frac{\#MST(n)}{\#WST(n)} \frac{[n(n-1)/2]!}{[n(n+1)/2]!}[n(n-1)/2+1]^n \\ &= \frac{\#MST(n)}{\#WST(n)} \left[\prod_{k=1}^{n} \frac{n(n-1)/2+1}{n(n-1)/2+i}\right]\end{aligned}$$

and by Lemma A.2 the last expression in brackets goes to $1/e$ when n goes to infinity, where $e \approx 2.718$ is the base of the natural logarithm. Hence for all n sufficiently large the ratio $\#MST(n+1)/\#WST(n+1)$ is less than half of the ratio $\#MST(n)/\#WST(n)$, which completes the proof. □

Proof of Theorem 1

For necessity, assume there exist u and d satisfying (2), and assume $\rho(x,y) \geq 1/2$ and $\rho(y,z) \geq 1/2$. If it were the case that $\rho(x,z) < \min\{\rho(x,y), \rho(y,z)\}$, then by (2) and the triangle inequality property of d it would follow that

$$\begin{aligned}u(x)-u(z) &< d(x,z)\min\left\{\frac{u(x)-u(y)}{d(x,y)}, \frac{u(y)-u(z)}{d(y,z)}\right\} \\ &\leq [d(x,y)+d(y,z)]\min\left\{\frac{u(x)-u(y)}{d(x,y)}, \frac{u(y)-u(z)}{d(y,z)}\right\} \\ &\leq d(x,y)\frac{u(x)-u(y)}{d(x,y)} + d(y,z)\frac{u(y)-u(z)}{d(y,z)} \\ &= u(x)-u(z)\end{aligned}$$

which is a contradiction. Hence, it must be the case that $\rho(x,z) \geq \min\{\rho(x,y), \rho(y,z)\}$.

Now suppose we have equality $\rho(x,z) = \min\{\rho(x,y), \rho(y,z)\}$. We consider the case $\min\{\rho(x,y), \rho(y,z)\} = \rho(x,y)$, while the remaining case is analogous and left to the reader. Representation (2) and the triangle inequality imply

$$\begin{aligned} u(x) - u(y) + u(y) - u(z) &= u(x) - u(z) \\ &= d(x,z)\left[\frac{u(x)-u(y)}{d(x,y)}\right] \\ &\leq [d(x,y) + d(y,z)]\left[\frac{u(x)-u(y)}{d(x,y)}\right] \\ &= u(x) - u(y) + d(y,z)\left[\frac{u(x)-u(y)}{d(x,y)}\right]. \end{aligned}$$

Subtracting $u(x) - u(y)$ from both sides we obtain

$$\frac{u(y)-u(z)}{d(y,z)} \leq \frac{u(x)-u(y)}{d(x,y)}$$

and therefore (2) yields $\rho(x,y) = \rho(y,z) = \rho(x,z)$ as desired.

For sufficiency, suppose ρ satisfies MST+. In particular, ρ satisfies WST, and hence, by letting $x \succcurlyeq y$ if and only if $\rho(x,y) \geq 1/2$, we obtain a complete and transitive relation $\succcurlyeq$ over the finite set of options Z. The relation $\succcurlyeq$ induced by ρ divides the n alternatives in Z into $k \leq n$ indifference classes. Therefore, there exists a utility function $u : Z \to \{1, \ldots, k\}$ that is onto and represents $\succcurlyeq$, that is, $u(x) \geq u(x)$ if and only if $x \succcurlyeq y$ if and only if $\rho(x,y) \geq 1/2$.

Let $Y := \{\{x,y\} \subset Z : \rho(x,y) \neq 1/2\}$, and let m be the cardinality of the set $\{|\rho(x,y) - 1/2| : \{x,y\} \in Y\}$. Partition the set Y into m disjoint sets $Y_1 \cup Y_2 \cup \cdots \cup Y_m = Y$ such that for any two pairs $\{w,x\}$ and $\{y,z\}$ in Y we have $\{w,x\} \in Y_i$ and $\{y,z\} \in Y_j$ with $i \geq j$ if and only if $|\rho(w,x) - 1/2| \leq |\rho(y,z) - 1/2|$. Thus, the pairs in Y_1 have the highest value of $|\rho(x,y) - 1/2|$, while the pairs in Y_m have the lowest value of $|\rho(x,y) - 1/2|$ among the pairs in Y.

The result is trivial when Z has $n \leq 2$ alternatives so suppose $n \geq 3$. Define a constant $C = (n-1)^{[n(n-1)/2+1]} > 0$ and define the sequence $D_1, D_2, \ldots, D_m$ by:

$$D_1 = 0; D_j = (n-1)^{j-2} \text{ for } j = 2, \ldots, m.$$

Let $d : Z \times Z \to [0, \infty)$ be defined as follows:

$$d(x,y) = \begin{cases} 0, & \text{if } x = y \\ C, & \text{if } x \neq y \text{ and } \rho(x,y) = 1/2 \\ (C/2 + D_j)\left|u(x) - u(y)\right|, & \text{if } \{x,y\} \in Y_j \end{cases} \tag{6}$$

From the definition (6) it is immediate that d satisfies (i) $d(x,y) \geq 0$; (ii) $d(x,y) = 0$ if and only if $x = y$; and (iii) $d(x,y) = d(y,x)$ for all $x, y \in Z$. To show that d is a metric, it remains to verify the triangle inequality: $d(x,z) \leq d(x,y) + d(y,z)$. The inequality trivially holds when any two options among x, y, z are equal. Consider three distinct options $x, y, z \in Z$.

Case 1: $u(x) = u(y) = u(z)$. By the definition of u we have $\rho(x,y) = \rho(y,z) = \rho(x,z) = 1/2$. By the definition of d we have $d(x,z) = C < 2C = d(x,y) + d(y,z)$.

Case 2: $u(x) \neq u(y) = u(z)$. The definitions of u and d imply

$$\begin{aligned} d(x,y) + d(y,z) - d(x,z) &= (C/2 + D_i)\left|u(x) - u(y)\right| + C - (C/2 + D_j)\left|u(x) - u(z)\right| \\ &= (D_i - D_j)\left|u(x) - u(z)\right| + C \\ &\geq -(n-1)^{m-2}(n-1) + C \\ &= (n-1)^{[n(n-1)/2+1]} - (n-1)^{m-1} \\ &> 0 \end{aligned}$$

where the last inequality follows from the fact that we defined m to be the cardinality of $\{|\rho(x, y) - 1/2| : \{x, y\} \in Y\}$ which is smaller or equal to $n(n-1)/2$.

Case 3: $u(y) \neq u(x) = u(z)$. The definitions of u and d imply

$$\begin{aligned} d(x,y) + d(y,z) - d(x,z) &= (C/2 + D_i)\,|u(x) - u(y)| + (C/2 + D_j)\,|u(y) - u(z)| - C \\ &= (C + D_i + D_j)\,|u(y) - u(z)| - C \\ &\geq 0. \end{aligned}$$

Case 4: $u(z) \neq u(x) = u(y)$. The inequality follows from the same argument as in Case 2.

Case 5: $u(x) > u(y) > u(z)$. By the definition of u we have $\{x, y\} \in Y_i$, $\{y, z\} \in Y_j$, and $\{x, z\} \in Y_\ell$, for some i, j, ℓ. The definition of d implies

$$\begin{aligned} d(x,y) + d(y,z) - d(x,z) &= (C/2 + D_i)\,|u(x) - u(y)| + (C/2 + D_j)\,|u(y) - u(z)| \\ &\quad - (C/2 + D_\ell)\,|u(x) - u(y) + u(y) - u(z)| \\ &= (D_i - D_\ell)\,|u(x) - u(y)| + (D_j - D_\ell)\,|u(y) - u(z)| \end{aligned}$$

The definition of u implies $\rho(x, y) > 1/2$ and $\rho(y, z) > 1/2$. By MST+ we have either $\rho(x, y) = \rho(y, z) = \rho(x, z)$ or $\rho(x, z) > \min\{\rho(x, y), \rho(y, z)\}$. The first case implies $D_i = D_j = D_\ell$ above and therefore $d(x, y) + d(y, z) - d(x, z) = 0$. The second case implies $D_\ell < \max\{D_i, D_j\}$. If $D_\ell \leq \min\{D_i, D_j\}$ then both $(D_i - D_\ell)$ and $(D_j - D_\ell)$ above are positive and the desired inequality holds. It remains to show the inequality holds when $\min\{D_i, D_j\} < D_\ell < \max\{D_i, D_j\}$, which implies

$$\begin{aligned} d(x,y) + d(y,z) - d(x,z) &\geq (\max\{D_i, D_j\} - D_\ell)\,1 + (\min\{D_i, D_j\} - D_\ell)\,(n-2) \\ &\geq (n-1)^{\ell-1} - (n-1)^{\ell-2} + [0 - (n-1)^{\ell-2}](n-2) \\ &= 0. \end{aligned}$$

Case 6: $u(x) > u(z) > u(y)$. By the definition of u we have $\{x, y\} \in Y_i$, $\{y, z\} \in Y_j$, and $\{x, z\} \in Y_\ell$, for some i, j, ℓ. The definition of d implies

$$\begin{aligned}
d(x,y) + d(y,z) - d(x,z) &= (C/2 + D_i)\,[u(x) - u(z) + u(z) - u(y)] \\
&\quad + (C/2 + D_j)\,[u(z) - u(y)] - (C/2 + D_\ell)\,[u(x) - u(z)] \\
&= (D_i - D_\ell)\,[u(x) - u(z)] + (C + D_i + D_j)\,[u(z) - u(y)] \\
&\geq \left(0 - (n-1)^{m-2}\right)(n-2) + (C + 0 + 0)\,1 \\
&= -(n-1)^{m-1} + (n-1)^{m-2} + (n-1)^{n(n-1)/2+1} \\
&> 0.
\end{aligned}$$

Case 7: $u(y) > u(x) > u(z)$. By the definition of u we have $\{x, y\} \in Y_i$, $\{y, z\} \in Y_j$, and $\{x, z\} \in Y_\ell$, for some i, j, ℓ. The definition of d implies

$$\begin{aligned}
d(x,y) + d(y,z) - d(x,z) &= (C/2 + D_i)\,[u(y) - u(x)] \\
&\quad + (C/2 + D_j)\,[u(y) - u(x) + u(x) - u(z)] \\
&\quad - (C/2 + D_\ell)\,[u(x) - u(z)] \\
&= (C + D_i + D_j)\,[u(y) - u(x)] + (D_j - D_\ell)\,[u(x) - u(z)] \\
&> 0.
\end{aligned}$$

Case 8: $u(y) > u(z) > u(x)$. Similarly to Case 7, we have

$$\begin{aligned}
d(x,y) + d(y,z) - d(x,z) &= (C + D_i + D_j)\,[u(y) - u(z)] + (D_i - D_\ell)\,[u(z) - u(x)] \\
&> 0.
\end{aligned}$$

Case 9: $u(z) > u(x) > u(y)$. Similarly to Cases 7 and 8, we have

$$\begin{aligned} d(x,y) + d(y,z) - d(x,z) &= (C + D_i + D_j)\,[u(x) - u(y)] + (D_j - D_\ell)\,[u(z) - u(x)] \\ &> 0. \end{aligned}$$

Case 10: $u(z) > u(y) > u(x)$. Since $d(x,y) + d(y,z) \leq d(x,z)$ if and only if $d(y,x) + d(z,y) \leq d(z,x)$, the inequality follows from the same argument as in Case 5.

By Cases 1 to 10 above, d satisfies the triangle inequality and is therefore a metric. Now, we verify that the utility u and the metric d constructed above provide a representation for ρ as in (2). First, $\rho(w,x) \geq \rho(y,z) > 1/2$ if and only if $\rho(w,x) > 1/2$, $\rho(y,z) > 1/2$, and $|\rho(w,x) - 1/2| \geq |\rho(y,z) - 1/2|$, if and only if $u(w) > u(x)$, $u(y) > u(z)$, $d(w,x) = (C/2 + D_i)[u(w) - u(x)]$, $d(y,z) = (C/2 + D_j)[u(y) - u(z)]$, and $i \leq j$, if and only if

$$\frac{u(w) - u(x)}{d(w,x)} = \frac{1}{C/2 + D_i} \geq \frac{1}{C/2 + D_j} = \frac{u(y) - u(z)}{d(y,z)} > 0.$$

Second, $\rho(w,x) \geq 1/2 \geq \rho(y,z)$ if and only if $u(w) - u(x) \geq 0 \geq u(y) - u(z)$ if and only if

$$\frac{u(w) - u(x)}{d(w,x)} \geq 0 \geq \frac{u(y) - u(z)}{d(y,z)}.$$

And, finally, $1/2 > \rho(w,x) \geq \rho(y,z)$ if and only if $\rho(w,x) < 1/2$, $\rho(y,z) < 1/2$, and $|\rho(w,x) - 1/2| \leq |\rho(y,z) - 1/2|$, if and only if $u(w) < u(x)$, $u(y) < u(z)$, $d(w,x) = (C/2 + D_i)[u(x) - u(w)]$, $d(y,z) = (C/2 + D_j)[u(z) - u(y)]$, and $i \geq j$, if and only if

$$0 > \frac{u(w) - u(x)}{d(w,x)} = -\frac{1}{C/2 + D_i} \geq -\frac{1}{C/2 + D_j} = \frac{u(y) - u(z)}{d(y,z)}$$

and we are done. $\square$

Proof of Theorem 2

Let the non-constant choice rule ρ on Δ be linear, continuous (outside the diagonal), convex, symmetric and satisfy MST+. First, we show that ρ has a unique linear extension to the $n-1$ dimensional hyperplane H that contains Δ.

Lemma A.3. *ρ has a unique linear extension to $H = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1 + \dots + x_n = 1\}$.*

Proof. Let ρ' and ρ'' be two linear extensions of ρ and let $x, y \in \mathbb{R}^n$ with $x_1 + \dots + x_n = y_1 + \dots + y_n = 1$. Let $z = (1/n, \dots, 1/n) \in \Delta$. Take $0 < \alpha < 1$ sufficiently small such that $0 < \alpha x_i + (1-\alpha)/n < 1$ and $0 < \alpha y_i + (1-\alpha)/n < 1$ for each i. Then $\alpha x + (1-\alpha)z \in \Delta$, $\alpha y + (1-\alpha)z \in \Delta$ and, by linearity,

$$\begin{aligned}
\rho'(x, y) &= \rho'(\alpha x + (1-\alpha)z, \alpha y + (1-\alpha)z) \\
&= \rho(\alpha x + (1-\alpha)z, \alpha y + (1-\alpha)z) \\
&= \rho''(\alpha x + (1-\alpha)z, \alpha y + (1-\alpha)z) \\
&= \rho''(x, y)
\end{aligned}$$

hence ρ' and ρ'' must be equal. $\square$

From this point on, we identify ρ with its unique linear extension. Define the relation $\succcurlyeq \subset \Delta \times \Delta$ by $x \succcurlyeq y$ if and only if $\rho(x, y) \geq 1/2$. Since ρ satisfies MST+, this $\succcurlyeq$ is complete and transitive. By linearity and continuity, $\succcurlyeq$ satisfies all the vNM axioms and admits an expected utility representation. Since ρ is non-constant, there is a unique linear function $U : \mathbb{R}^n \to \mathbb{R}$ which represents $\succcurlyeq$ with $U(\Delta) = [0, 1]$.

For each lottery x, let $I(x) := \{y \in H : \rho(x, y) = 1/2\}$ denote the set of lotteries that are stochastically indifferent to x. Note that $I(x)$ is an affine subspace of dimension $n-2$. Since ρ is not constant, there exist $\bar{x}, \bar{y} \in \Delta$ with $\rho(\bar{x}, \bar{y}) > 1/2$. By linearity, ρ is entirely determined by the values of the mapping $x \mapsto \rho(x, \bar{y})$ for $x \in I(\bar{x})$. For each $1/2 < p < 1$ let

$B(p) := \{x \in I(\bar{x}) : \rho(x, \bar{y}) \geq p\}$ be the upper contour set of elements that are stochastically indifferent to $\bar{x}$ and that are chosen over $\bar{y}$ with probability greater or equal to p.

Lemma A.4. *$B(p)$ is convex for all $p \in (1/2, 1)$.*

Proof. Let $x, x' \in B(p)$ and let $0 < \alpha < 1$. Since $I(\bar{x})$ is an affine subspace, $\alpha x + (1-\alpha)x' \in I(\bar{x})$. Linearity implies $\rho(\alpha x + (1-\alpha)x', \alpha\bar{y} + (1-\alpha)x') = \rho(x, \bar{y}) \geq p$. Linearity also implies $\rho(\alpha\bar{y} + (1-\alpha)x', \bar{y}) = \rho(x', \bar{y}) \geq p$. Then, MST+ implies $\rho(\alpha x + (1-\alpha)x', \bar{y}) \geq p$ and therefore $\alpha x + (1-\alpha)x' \in B(p)$. □

Lemma A.5. *$B(p)$ is compact for all $p \in (1/2, 1)$.*

Proof. $B(p)$ is closed by continuity. Let $|\cdot|$ denote the standard Euclidean metric, not necessarily equal to the metric we are going to construct for the representation. If $B(p)$ were not bounded, there would exist a sequence $x(k)$ in $B(p)$ with $|x(k) - \bar{y}| \geq k$ for all $k \in \mathbb{N}$. For each k, by linearity $\rho(\bar{y} + (x(k) - \bar{y})/|x(k) - \bar{y}|, \bar{y}) = \rho(x(k), \bar{y}) \geq p$. By Bolzano-Weierstrass the sequence $\bar{y} + (x(k) - \bar{y})/|x(k) - \bar{y}|$ would have a subsequence converging to some $z \neq \bar{y}$. By the linearity of U we would have $U(z) = U(\bar{y})$ and $\rho(z, \bar{y}) = 1/2$, contradicting continuity. Hence $B(p)$ must be bounded and therefore compact. □

Lemma A.6. *The mapping $x \mapsto \rho(x, \bar{y})$ has a unique maximizer $\hat{x}$ on $I(\bar{x})$.*

Proof. Since $\rho(\bar{x}, \bar{y}) > 1/2$ we have $B(p) \neq \emptyset$ for some $p > 1/2$. Since ρ is continuous outside the diagonal, the mapping $x \mapsto \rho(x, \bar{y})$ is continuous on $I(\bar{x})$. $B(p)$ is compact by Lemma A.5, hence the maximum $\rho(\hat{x}, \bar{y}) = \bar{p}$ is attained at some $\hat{x} \in B(p)$. Hence $B(\bar{p})$ is not empty, and by the previous lemmas it is compact and convex. It suffices to show that $B(\bar{p})$ is a singleton. If $B(\bar{p})$ were not a singleton there would exist a line in $I(\bar{x})$ whose intersection with the convex, compact $B(\bar{p})$ is the closed segment $[\hat{x}, \hat{x}']$ with $\hat{x} \neq \hat{x}'$. Let $x = 2\hat{x} - \hat{x}'$ and $x' = 2\hat{x}' - \hat{x}$ and note that $x, x' \in I(\bar{x})$. Since $\hat{x}$ and $\hat{x}'$ are on the boundary of $B(\bar{p})$ we must have both $\rho(x, \bar{y}) < \bar{p}$ and $\rho(x', \bar{y}) < \bar{p}$. Suppose $\rho(x', \bar{y}) \geq \rho(x, \bar{y})$ (the argument is entirely analogous with the reverse inequality). Let $z = 2\bar{y} - \hat{x}'$ so that $\bar{y} = z/2 + \hat{x}'/2$. By linearity $\rho(x, z) = \rho(x/2 + \hat{x}'/2, z/2 + \hat{x}'/2) = \rho(\hat{x}, \bar{y}) = \bar{p}$. Since $\rho(x, x') = 1/2$, $\rho(x', \bar{y}) \geq \rho(x, \bar{y}) > 1/2$ and $\rho(\bar{y}, z) = \bar{p}$, the symmetry of ρ implies $\rho(x', z) = \bar{p}$. But then by linearity $\rho(\hat{x}'/2 + x'/2, \bar{y}) = \rho(\hat{x}'/2 + x'/2, z/2 + \hat{x}'/2) = \rho(x', z) = \bar{p}$. But then the point $\hat{x}'/2 + x'/2$, which lies in the same line as $\hat{x}$ and $\hat{x}'$, but lies outside the segment $[\hat{x}, \hat{x}']$, would be in $B(\bar{p})$, a contradiction. □

For the rest of the proof, we denote by $\hat{x}$ the unique maximizer of $x \mapsto \rho(x, \bar{y})$ on $I(\bar{x})$.

Lemma A.7. *$x \in I(\bar{x})$ and $\rho(x, \bar{y}) = p$ implies $\rho(2\hat{x} - x, \bar{y}) = p$.*

Proof. The statement trivially holds if $x = \hat{x}$, so suppose $x \neq \hat{x}$. First note $2\hat{x} - x = \hat{x} + (\hat{x} - x) \in I(\bar{x})$. If $\rho(2\hat{x} - x, \bar{y}) < p$, since $x \mapsto \rho(x, \bar{y})$ is continuous in the segment $[\hat{x}, \hat{x} + (\hat{x} - x)]$, by the intermediate value theorem we have $\rho(x', \bar{y}) = p$ for some x' in the open segment $(\hat{x}, 2\hat{x} - x)$. But then since $\hat{x}$ is the unique maximizer in $I(\bar{x})$ it is also the unique maximizer in the segment $[x, x']$. Since $\hat{x} \neq x/2 + x'/2$ this contradicts the fact that ρ is convex. Hence we must have $\rho(2\hat{x} - x, \bar{y}) \geq p$. The same argument shows that $\rho(2\hat{x} - x, \bar{y}) \leq p$. □

Recall that $\hat{x}$ is the unique maximizer $\rho(\hat{x}, \bar{y}) = \bar{p}$ on $I(\bar{x})$. Let $B = B(p) - \hat{x}$ for some fixed $p \in (1/2, \bar{p})$. We first define an auxiliary norm $\|\cdot\|_B$ on the $n-2$ dimensional subspace $I(\bar{x}) - \hat{x}$ using B as the unit ball.

Lemma A.8. *$\|x\|_B := \inf\{\lambda \geq 0 : x \in \lambda B\}$ is a norm on $I(\bar{x}) - \hat{x}$.*

Proof. The *Minkowski functional* $\|\cdot\|_B$ defined above is a norm when B is a symmetric, convex set such that each line through zero meets B in a non-trivial, closed, bounded segment (Thompson, 1996). By definition $\|x\|_B \geq 0$ for all x. Moreover, if $\|x\|_B = 0$ then $x \in \lambda B$ for all $\lambda > 0$ and therefore $x = 0$. Now for each $\alpha \geq 0$ we have $x \in \lambda B$ if and only if $\alpha x \in \alpha\lambda B$ and therefore $\alpha\|x\|_B = \|\alpha x\|_B$. Lemma A.7 implies $x \in \lambda B$ if and only if $-x \in \lambda B$ and therefore $\|x\|_B = \|-x\|_B$. To verify the triangle inequality, note that B is closed by Lemma A.5, and therefore $x/\|x\|_B \in B$ for all x. B is also convex by Lemma A.4, and therefore

$$\frac{x+x'}{\|x\|_B + \|x'\|_B} = \left(\frac{\|x\|_B}{\|x\|_B + \|x'\|_B}\right)\frac{x}{\|x\|_B} + \left(\frac{\|x'\|_B}{\|x\|_B + \|x'\|_B}\right)\frac{x'}{\|x'\|_B} \in B.$$

Thus,

$$\left\|\frac{x+x'}{\|x\|_B + \|x'\|_B}\right\|_B \leq 1$$

and the triangle inequality $\|x + x'\|_B \leq \|x\|_B + \|x'\|_B$ holds. □

Lemma A.9. *If $\bar{p} \geq p \geq q > 1/2$ then $B(p) = \hat{x} + \lambda\,[B(q) - \hat{x}]$ for some $0 \leq \lambda \leq 1$.*

Proof. MST+ implies that, for any $x \neq \hat{x}$ in $B(p)$, the function $\alpha \mapsto \rho(\alpha\hat{x} + (1-\alpha)x, \bar{y})$ is strictly increasing for $0 \leq \alpha \leq 1$. It suffices to show that if $\rho(x, \bar{y}) = \rho(x', \bar{y})$ for $x, x' \in I(\bar{x})$ and $0 < \alpha < 1$, then $\rho(\alpha x + (1-\alpha)\hat{x}, \bar{y}) = \rho(\alpha x' + (1-\alpha)\hat{x}, \bar{y})$. Take the point $z = [1/\alpha]\bar{y} + [1-(1/\alpha)]\hat{x}$ and note that $\bar{y} = \alpha z + (1-\alpha)\hat{x}$. By linearity $\rho(\bar{y}, z) = \rho(\hat{x}, \bar{y}) = \bar{p}$. Since ρ is symmetric, $\rho(x, z) = \rho(x', z)$. Finally by linearity,

$$\begin{aligned}
\rho(\alpha x + (1-\alpha)\hat{x}, \bar{y}) &= \rho(\alpha x + (1-\alpha)\hat{x}, \alpha z + (1-\alpha)\hat{x}) \\
&= \rho(x, z) \\
&= \rho(x', z) \\
&= \rho(\alpha x' + (1-\alpha)\hat{x}, \alpha z + (1-\alpha)\hat{x}) \\
&= \rho(\alpha x' + (1-\alpha)\hat{x}, \bar{y})
\end{aligned}$$

as we wanted to show. □

Lemma A.10. *$\|\cdot\|_B$ is Euclidean, i.e., $\|x\|_B = \sqrt{\langle x, x\rangle_B}$ where $\langle\cdot,\cdot\rangle_B$ is an inner product.*

Proof. We use a characterization of inner product spaces by Gurari and Sozonov (1970), who showed that a normed linear space is an inner product space if and only if

$$\left\|\frac{1}{2}x+\frac{1}{2}y\right\| \leq \|\alpha x+(1-\alpha)y\| \quad \text{whenever } \|x\|=\|y\|=1 \text{ and } 0\leq\alpha\leq 1. \tag{7}$$

If $\|x\|_B = \|y\|_B = 1$ then x, y are on the boundary of B, hence $\rho(x+\hat{x},\bar{y}) = \rho(y+\hat{x},\bar{y}) = p > 1/2$ and $\rho(x+\hat{x}, y+\hat{x}) = 1/2$. Since ρ is convex, for each $0\leq\alpha\leq 1$ we must have

$$\rho(\alpha x+(1-\alpha)y+\hat{x},\bar{y}) \leq \rho(x/2+y/2+\hat{x},\bar{y})$$

thus $\alpha x+(1-\alpha)y$ is on the boundary of $B(q)-\hat{x}$ and $x/2+y/2$ is on the boundary of $B(q')-\hat{x}$ for some $q\leq q'$. By Lemma A.9 $\|x/2+y/2\|_B \leq \|\alpha x+(1-\alpha)y\|_B$ and therefore the norm $\|\cdot\|_B$ satisfies (7). □

Now we extend the inner product $\langle\cdot,\cdot\rangle_B$ on the $n-2$ dimensional subspace $I(\bar{x})-\hat{x}$ obtained in the last Lemma to an inner product $\langle\cdot,\cdot\rangle$ on the $n-1$ dimensional subspace $H-\hat{x}$. Let $v_1,\ldots,v_{n-2}$ be an orthonormal base for the subspace $I(\bar{x})-\hat{x}$ endowed with $\langle\cdot,\cdot\rangle_B$. Let $v_{n-1} := \hat{x}-\bar{y}$ and for every $1\leq i,j\leq n-1$ let $\langle v_i,v_j\rangle = 0$ if $i\neq j$ and $\langle v_i,v_j\rangle = 1$ if $i=j$. We let the norm be induced by this inner product $\|x\| := \sqrt{\langle x,x\rangle}$ for all $x\in H-\hat{x}$.

Lemma A.11. *U and $\|\cdot\|$ are a MEM representation of ρ as in (4).*

Proof. First, suppose $\rho(w,x)\geq\rho(y,z)>1/2$. Then $w\succ x$, $y\succ z$ and since U represents $\succcurlyeq$ we have $U(w)>U(x)$ and $U(y)>U(z)$. Let

$$\begin{aligned} w' &= \bar{y}+\frac{U(\hat{x})-U(\bar{y})}{U(w)-U(x)}(w-x) \\ y' &= \bar{y}+\frac{U(\hat{x})-U(\bar{y})}{U(y)-U(z)}(y-z) \end{aligned}$$

and note that $w',y'\in H$. Since U is linear, $U(w')=U(y')=U(\bar{x})$ and hence $w',y'\in I(\bar{x})$. By the linearity of ρ, $\rho(w',\bar{y})=\rho(w,x)\geq\rho(y,z)=\rho(y',\bar{y})$. Hence $\|w'-\hat{x}\|_B\leq\|y'-\hat{x}\|_B$.

By construction, $\hat{x} - \bar{y}$ is orthogonal to $I(\bar{x}) - \hat{x}$, and therefore

$$\begin{aligned}\|w' - \bar{y}\|^2 &= \|w' - \hat{x} + \hat{x} - \bar{y}\|^2 \\ &= \|w' - \hat{x}\|^2 + \|\hat{x} - \bar{y}\|^2 \\ &\leq \|y' - \hat{x}\|^2 + \|\hat{x} - \bar{y}\|^2 \\ &= \|y' - \bar{y}\|^2\end{aligned}$$

Thus

$$\left\| \frac{U(\hat{x}) - U(\bar{y})}{U(w) - U(x)}(w - x) \right\| = \|w' - \bar{y}\| \leq \|y' - \bar{y}\| = \left\| \frac{U(\hat{x}) - U(\bar{y})}{U(y) - U(z)}(y - z) \right\|$$

which implies

$$\frac{U(w) - U(x)}{\|w - x\|} \geq \frac{U(y) - U(z)}{\|y - z\|}.$$

Next, suppose $\rho(w, x) \geq 1/2 \geq \rho(y, z)$ with $w \neq x$ and $y \neq z$. Then $U(w) \geq U(x)$ and $U(z) \geq U(y)$ which implies

$$\frac{U(w) - U(x)}{\|w - x\|} \geq 0 \geq \frac{U(y) - U(z)}{\|y - z\|}.$$

Finally, suppose $1/2 > \rho(w, x) \geq \rho(y, z)$. Then $\rho(z, y) \geq \rho(x, w) > 1/2$ and the desired inequality follows from the first step.

Reversing the argument to show that

$$\frac{U(w) - U(x)}{\|w - x\|} \geq \frac{U(y) - U(z)}{\|y - z\|} \implies \rho(w, x) \geq \rho(y, z)$$

is straightforward and left to the reader. □

Proof of Proposition 2

Let $\bar{x}, \bar{y}, \hat{x}$ be defined exactly as in the proof of Theorem 2. Let $(U, \|\cdot\|)$ be any MEM representation of ρ as in (4), and let $\langle \cdot, \cdot \rangle$ be the inner product that induces the norm.

Lemma A.12. *$\langle x - \hat{x}, \hat{x} - \bar{y} \rangle = 0$ for all x with $\rho(x, \hat{x}) = 1/2$.*

Proof. This holds by construction for the particular representation obtained in the proof of Theorem 2, and now we show it must hold for any representation. If $x = \hat{x}$ the statement trivially holds. Suppose $x \neq \hat{x}$. By Lemma A.7 $\rho(x, \bar{y}) = \rho(2\hat{x} - x, \bar{y})$. By the representation (4) it must be $\|x - \bar{y}\| = \|2\hat{x} - x - \bar{y}\|$. Hence

$$\begin{aligned} \|x - \hat{x}\|^2 + 2\langle x - \hat{x}, \hat{x} - \bar{y} \rangle + \|\hat{x} - \bar{y}\|^2 &= \langle x - \bar{y}, x - \bar{y} \rangle \\ &= \langle 2\hat{x} - x - \bar{y}, 2\hat{x} - x - \bar{y} \rangle \\ &= \|x - \hat{x}\|^2 + 2\langle x - \hat{x}, \bar{y} - \hat{x} \rangle + \|\hat{x} - \bar{y}\|^2 \end{aligned}$$

which implies $4\langle x - \hat{x}, \hat{x} - \bar{y} \rangle = 0$ and we are done. □

Lemma A.13. *$\rho(x, \hat{x}) = \rho(x', \hat{x}) = 1/2$ and $\rho(x, \bar{y}) = \rho(x', \bar{y})$ implies $\|x - \hat{x}\| = \|x' - \hat{x}\|$.*

Proof. By the representation (4) we must have $\|x - \bar{y}\| = \|x' - \bar{y}\|$. By Lemma A.12, $\langle x - \hat{x}, \hat{x} - \bar{y} \rangle = \langle x' - \hat{x}, \hat{x} - \bar{y} \rangle = 0$. Thus,

$$\|x - \hat{x}\|^2 + \|\hat{x} - \bar{y}\|^2 = \|x - \bar{y}\|^2 = \|x' - \bar{y}\|^2 = \|x' - \hat{x}\|^2 + \|\hat{x} - \bar{y}\|^2$$

and therefore $\|x - \hat{x}\| = \|x' - \hat{x}\|$ as desired. □

The expected utility function U is unique by the requirement that it is linear and that $U(\Delta) = [0, 1]$. Now suppose both $\|x\|_1 = \sqrt{\langle x, x \rangle_1}$ and $\|x\|_2 = \sqrt{\langle x, x \rangle_2}$ are norms induced by their respective inner products and that each norm together with U represents ρ. Fix a lottery $z \neq \hat{x}$ with $\rho(z, \hat{x}) = 1/2$ and let $A := \|z - \hat{x}\|_1 / \|z - \hat{x}\|_2 > 0$. Now take any x with $U(x) = U(\hat{x})$. To show that $\|\cdot\|_1 = A\|\cdot\|_2$ on the null space of U, it suffices to show that $\|x - \hat{x}\|_1 = A\|x - \hat{x}\|_2$. This clearly holds if $x = \hat{x}$ so suppose $x \neq \hat{x}$. Let $q = \rho(x, \bar{y})$ and $p = \rho(z, \bar{y})$. Lemma A.6 implies $p, q < \bar{p} = \rho(\hat{x}, \bar{y})$. Suppose wlog $p \geq q$. By Lemma A.9 $B(p) = \hat{x} + \lambda[B(q) - \hat{x}]$ for some $0 \leq \lambda \leq 1$. Since $p < \bar{p}$ it must be $0 < \lambda \leq 1$. Then $\rho(\lambda x + (1 - \lambda)\hat{x}, \bar{y}) = p = \rho(z, \bar{y})$. By Lemma A.13 we have

$$\|\hat{x} + \lambda(x - \hat{x}) - \hat{x}\|_1 = \|z - \hat{x}\|_1 = A\|z - \hat{x}\|_2 = A\|\hat{x} + \lambda(x - \hat{x}) - \hat{x}\|_2$$

hence $\lambda\|x - \hat{x}\|_1 = \lambda A\|x - \hat{x}\|_2$ and since $\lambda > 0$ we obtain $\|x - \hat{x}\|_1 = A\|x - \hat{x}\|_2$ as desired.

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