On Preference for Flexibility

Paulo Natenzon

October 28, 2009

Motivation and Outline

- Keep options open
- Unforeseen contingencies
- Large literature from Koopmans 1964, Kreps 1979

Outline

- Explore a representation: $\min_{s\in S}\max_{z\in x} u(s,z)$.
- 3 propositions
- some illustrative examples

Simple Setup: finite sets

- $Z \neq \emptyset$ finite set of final consequences
- $X = 2^Z \setminus \{\emptyset\}$ menus
- $\succsim \subset X \times X$ preference

Postulates on the preference $\succsim$:

- A1 Weak order: $\succsim$ is complete and transitive.
- A2 Set Monotonicity: $x \supset x'$ implies $x \succsim x'$.
- A3 Ordinal submodularity:
  $x \sim x \cup x'$ implies $x \cup x'' \sim x \cup x' \cup x''$ for all $x'' \in X$.

Theorem (Kreps 1979)

$\succsim$ satisfies A1, A2 and A3 if and only if it can be represented by

$$U(x) = \sum_{s \in S} \max_{z \in x} u(z, s) \tag{1}$$

where

1. *S is a finite set of subjective states,*
2. *$u : Z \times S \rightarrow \mathbb{R}$ is a state dependent utility index.*

A3 Ordinal submodularity:
$x \sim x \cup x'$ implies $x \cup x'' \sim x \cup x' \cup x''$ for all $x'' \in X$.

Example (Ergin 2003)

Consider the decision problem of an individual who needs to buy a car. The alternatives are Mercedes, Lexus and Toyota, and accordingly we write $Z = \{m, l, t\}$. We observe her preferences over car dealerships, given by

$$\{m, l, t\} \succ \{l, t\} \succ \{m, t\} \sim \{t\} \succ \{m, l\} \sim \{l\} \succ \{m\}$$

which satisfy A1 and A2 but not A3.

Postulates on the preference $\succsim$:

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Theorem (Ergin 2003)

$\succsim$ satisfies A1, A2 if and only if can be represented by

$$W(x) = \max_{\pi \in \mathcal{I}} \left[\sum_{E \in \pi} \max_{z \in x} \sum_{s \in E} u(z, s) - c(\pi) \right] \qquad (2)$$

where

1. *S is a finite set of subjective states,*
2. *$u : Z \times S \to \mathbb{R}$ is a state dependent utility index,*
3. *$\mathcal{I}$ is a set of partitions of S that contain $\{S\}$,*
4. *$c : \mathcal{I} \to \mathbb{R}_+$ is a cost function such that $c(\{S\}) = 0$.*

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Theorem (Kreps 1979)

$\succsim$ satisfies A1, A2 and A3 if and only if it can be represented by

$$U(x) = \sum_{s \in S} \max_{z \in x} u(z, s) \tag{3}$$

where

1. *S is a finite set of subjective states,*
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Postulates on the preference $\succsim$:

- A1 Weak order: $\succsim$ is complete and transitive.
- A2 Set Monotonicity: $x \supset x'$ implies $x \succsim x'$.
- A3 ~~Ordinal submodularity:~~
  ~~$x \sim x \cup x'$ implies $x \cup x'' \sim x \cup x' \cup x''$ for all $x'' \in X$.~~

Proposition

$\succsim$ satisfies A1, A2 if and only if it can be represented by

$$V(x) = \min_{s \in S} \max_{z \in x} u(z, s) \tag{4}$$

where

1. *S is a finite set of subjective states,*
2. *$u : Z \times S \rightarrow \mathbb{R}$ is a state dependent utility index.*

Proof of Proposition 1

1. Sufficiency: easy.
2. Necessity: note that X is finite and can be partitioned in equivalence classes of $\sim$ labeled $I_0, I_{-1}, I_{-2}, \ldots, I_{-n}$ such that for any $x \in I_j$ and $x' \in I_k$ we have $x \succsim x'$ if and only if $j \geq k$. Hence the map $x \mapsto j(x)$ where for each $x \in X$ we take $j(x) \in \mathbb{R}$ such that $x \in I_{j(x)}$ is well defined and represents $\succsim$.
3. Take
   - $S = X$
   - For each $s \in S$ let $\mathbb{I}_s : Z \to \mathbb{R}$ be the indicator function defined by $\mathbb{I}_s(z) = 1$ if $z \in s$ and $\mathbb{I}_s(z) = 0$ otherwise.
   - Let $u : Z \times S \to \mathbb{R}$ by $u(z, s) = j(s)\mathbb{I}_s(z)$
   - Define $V : X \to \mathbb{R}$ by

$$V(x) = min_{s \in S} max_{z \in x} u(z, s).$$

Proof of Proposition 1 (continued)

3. It suffices to show that $V = j$, since j represents $\succsim$. For a given $s \in S$ we have $\max_{z \in x} u(z, s)$ equal to $j(s)$ whenever $x \subset s$ and equal to zero otherwise. Also for all $s \supset x$ by A2 we have $s \succsim x$ therefore $j(s) \geq j(x)$. Thus for all $x \in X$ we have

$$\begin{aligned} V(x) &= min_{s \in S} max_{z \in x} u(z, s) \\ &= min_{s \supset x} max_{z \in x} u(z, s) \\ &= min_{s \supset x} j(s) \\ &= j(x) \end{aligned}$$

□

Example (continued)

Preference over car dealerships

$$\{m,l,t\} \succ \{l,t\} \succ \{m,t\} \sim \{t\} \succ \{m,l\} \sim \{l\} \succ \{m\}$$

In this case an additive representation is not possible, but it is easy to find a representation as in Proposition 1.
Define $S = \{s_1, s_2, s_3, s_4\}$ and let $u : Z \times S \to \mathbb{R}$ be given by

	s_1	s_2	s_3	s_4
$u(m,\cdot)$	0	−2	−3	−4
$u(l,\cdot)$	−1	0	−3	0
$u(t,\cdot)$	−1	−2	0	0

Proposition 2

$\succsim$ *satisfies A1, A2 and A3 if and only if there exist a finite set of subjective states* S *and a state dependent utility function* $u : Z \times S \to \mathbb{R}$ *such that* $\succsim$ *is represented* **at the same time** *both by*

$$U(x) = \sum_{s \in S} \max_{z \in x} u(z, s)$$

and by

$$V(x) = \min_{s \in S} \max_{z \in x} u(z, s)$$

Proof of Proposition 2 (sketch)

1. Use Kreps 1979 to construct S and $u : Z \times S \to \mathbb{R}$. Recall that in Kreps' construction $S \subset X$ and for each $s \in S$ we have

$$u(z,s) = \begin{cases} a(s) < 0 & \text{if } z \in s \\ 0 & \text{otherwise} \end{cases}$$

   Without loss of generality, we can pick all $a(s)$ to be strictly negative integers. We obtain an additive representation.

2. Splitting states: the key to the proof is to note that if we replace any state $s_0 \in S$ with n new states $s_1, s_2, \ldots, s_n$ and for each new state s_i and each $z \in Z$ we define $u(z, s_i) = u(z, s_0)/n$, the value of $U(x)$ for any $x \in X$ remains unchanged:

$$\max_{z \in x} u(z, s_0) = \max_{z \in x} \frac{1}{2} u(z, s_0) + \max_{z \in x} \frac{1}{2} u(z, s_0)$$

Proof of Proposition 2 (sketch)

3. Now partition X into indifference classes $I_1, I_2, \ldots, I_k$, where for all $x \in I_i$ and $x' \in I_k$ we have $x \succsim x'$ if and only if $i \geq k$. The mapping $x \mapsto j(x)$ where $x \in I_j(x)$ represents $\succsim$. And so does $x \mapsto -1/j(x)$.
4. Now remember $S \subset X$ and go through the states $s \in S$ starting from the bottom indifference class to the top, applying the splitting procedure described in step 2. Split each state $s \in S$ into $|a(s)| \times j(s)$ pieces.
5. This results in $V(x) = \min_{s \in S} \max_{z \in x} u(z, s) = -1/j(x)$ so V represents $\succsim$. And since the splitting procedure left U unchanged, it also still represents $\succsim$. □

Example

Preference over meals

$$\{a,b,c\} \sim \{a,b\} \sim \{b,c\} \succ \{a,c\} \sim \{a\} \succ \{b\} \succ \{c\}$$

First construct the representation as in Kreps 1979, then...

	s_1	s_2	s_3
$u(a,\cdot)$	-1	0	0
$u(b,\cdot)$	0	-2	0
$u(c,\cdot)$	-1	0	-2

Example

Preference over meals

$$\{a, b, c\} \sim \{a, b\} \sim \{b, c\} \succ \{a, c\} \sim \{a\} \succ \{b\} \succ \{c\}$$

First construct the representation as in Kreps 1979, then...

	s_1	s_2	s_{31}	s_{32}
$u(a,\cdot)$	-1	0	0	0
$u(b,\cdot)$	0	-2	0	0
$u(c,\cdot)$	-1	0	-1	-1

Example

Preference over meals

$$\{a, b, c\} \sim \{a, b\} \sim \{b, c\} \succ \{a, c\} \sim \{a\} \succ \{b\} \succ \{c\}$$

First construct the representation as in Kreps 1979, then...

	s_1	s_{21}	s_{22}	s_{23}	s_{24}	s_{31}	s_{32}
$u(a,\cdot)$	-1	0	0	0	0	0	0
$u(b,\cdot)$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0
$u(c,\cdot)$	-1	0	0	0	0	-1	-1

Example

Preference over meals

$$\{a,b,c\} \sim \{a,b\} \sim \{b,c\} \succ \{a,c\} \sim \{a\} \succ \{b\} \succ \{c\}$$

First construct the representation as in Kreps 1979, then...

	s_{11}	s_{12}	s_{13}	s_{14}	s_{21}	s_{22}	s_{23}	s_{24}	s_{31}	s_{32}
$u(a,\cdot)$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	0	0	0	0	0	0
$u(b,\cdot)$	0	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0
$u(c,\cdot)$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	0	0	0	0	-1	-1

Consider the following general setup. There is a possibly infinite underlying set of objects Z and a set $X \subset 2^Z \setminus \emptyset$ of non-empty menus.

Proposition 3

Assume that the preference $\succsim \subset X \times X$ *has a numerical representation* $U : X \to \mathbb{R}$ *and satisfies set-monotonicity: for any* $x, x' \in X$ *if* $x \supset x'$ *then* $x \succsim x'$. *We claim it can also be represented by*

$$V(x) = \min_{s \in S} \max_{z \in x} u(z, s)$$

for some set of subjective states S *and some state-dependent utility index* $u : Z \times S \to \mathbb{R}$.

Proof of Proposition 3

1. Note that if $U : X \to \mathbb{R}$ represents $\succsim$ then U must be bounded above, because it attains it's maximum at X, the largest menu. So assume without loss of generality that $U(X) = 0$.
2. Take $S = X$ and for each menu $s \in S$ let $\mathbb{I}_s : Z \to \mathbb{R}$ be the indicator function defined by $\mathbb{I}_s(z) = 1$ if $z \in s$ and $\mathbb{I}_s(z) = 0$ otherwise. Let $u : Z \times S \to \mathbb{R}$ by $u(z, s) = U(s)\mathbb{I}_s(z)$ and define $V : X \to \mathbb{R}$ by

$$V(x) = \min_{s \in S} \max_{z \in x} u(z, s)$$

Proof of Proposition 3 (continued)

3. Note V is well defined. For each $s \in S$ we have

$$\max_{z \in x} u(z,s) = \max_{z \in x} U(s)\mathbb{I}_s(z)$$
$$= \begin{cases} U(s) & \text{if } s \supset x \\ 0 & \text{otherwise} \end{cases}$$

And for each $x \in X$ the set

$$\{max_{z \in x} u(z,s) : s \in S\}$$

attains its minimum at $s = x$.

Proof of Proposition 3 (continued)

4. Finally

$$\begin{aligned} V(x) &= min_{s \in S} max_{z \in x} u(z, s) \\ &= min_{s \supset x} max_{z \in x} u(z, s) \\ &= min_{s \supset x} U(s) \\ &= U(x) \end{aligned}$$

□

Thank you