Complementarity Revisited

Jonathan Weinstein

July 11, 2017: Workshop in Honor of Ehud Kalai

First Pass

- Let preferences $\succeq$ on $\mathbb{R}^n$ (bundle space) be represented by a smooth function $u : \mathbb{R}^n \to \mathbb{R}$. Denote partial derivatives by u_i, u_{ij}, etc.
- Naively, we might try classifying goods i and j as complements or substitutes according to the sign of u_{ij}.
- But this doesn't work, because it is sensitive to the choice of representation: if $u_i u_j \neq 0$, we can make the sign of u_{ij} whatever we want by replacing u with $f \circ u$ for smooth increasing f.
- Interestingly, if(f) $u_i u_j = 0$, then $\operatorname{sgn}(u_{ij})$ *is* invariant to representation. More on this later.

Standard Notions

- Gross Complements: Negative uncompensated cross-price effect
- Hicksian Complements: Negative compensated cross-price effect

Discontents with Standard Notions

- Gross complementarity may be asymmetric, i.e. $\partial x_i / \partial p_j$ and $\partial x_j / \partial p_i$ may have different signs due to income effects.
- Hicksian complementarity is trivial in the two-good case (never holds).
- Both have meaning only when preferences are (locally) convex; tacit assumption is that we never see the full preference, only responses to optimization under a single linear constraint.
- Both depend on the complete list of available goods:

Deep Discontent: Basis-sensitivity of cross-price effects

- Restaurant M: Three goods: drinks, fries, burgers. Quantities $x = (x_1, x_2, x_3)$, prices $p = (p_1, p_2, p_3)$.
- Restauarant M': Three goods: drinks, fries, "meal deal". Quantities $z = (x_1 - x_3, x_2 - x_3, x_3)$, prices $q = (p_1, p_2, p_1 + p_2 + p_3)$. Identical menus, represented differently.

Deep Discontent: Basis-sensitivity of cross-price effects

- Restaurant M: Three goods: drinks, fries, burgers. Quantities $x = (x_1, x_2, x_3)$, prices $p = (p_1, p_2, p_3)$.
- Restauarant M': Three goods: drinks, fries, "meal deal". Quantities $z = (x_1 - x_3, x_2 - x_3, x_3)$, prices $q = (p_1, p_2, p_1 + p_2 + p_3)$. Identical menus, represented differently.
- Cross-price effects on drinks-fries differ (compensated or uncompensated) at restaurants M and M':

$$\frac{\partial z_2}{\partial q_1} = \frac{\partial z_2}{\partial p_1} - \frac{\partial z_2}{\partial p_3}$$

$$= \frac{\partial x_2}{\partial p_1} - \frac{\partial x_2}{\partial p_3} - \frac{\partial x_3}{\partial p_1} + \frac{\partial x_3}{\partial p_3} \neq \frac{\partial x_2}{\partial p_1}$$

- Changing q_1 has different meaning from changing p_1 because different things are fixed.
- Similarly, "Effect on z_2" has different meaning from "Effect on x_2".

Deep Discontent: What's going on?

- Recall that cross-price effects are also second derivatives of the expenditure function:

$$\frac{\partial x_2}{\partial p_1} = \frac{\partial x_1}{\partial p_2} = \frac{\partial^2 E}{\partial p_1 \partial p_2}$$

  where $E(p, u)$ is the minimum expenditure to achieve u at prices p.
- Crucially, price vectors do not lie in bundle space; they lie in its *dual*, i.e. price vector is a linear functional from bundles to $\mathbb{R}$
- Standard complementarity *really* looks at complementarity between *dual* vectors (in their effect on E), then relies on an isomorphism between a vector space and its dual...but this isomorphism is *non-canonical*, i.e. basis-dependent.

Deep Discontent: What's going on?

- Intuitively "Increase the price of fries by 1¢" does not have definite meaning, because you need to specify what you hold fixed (the basis).
- Even more obviously, "increase the price of a meal deal" is completely unclear as to what's held fixed. But complementarity should have definite meaning for "composite goods" as well.
- NB the basis-dependence here is not mere dependence on what goods are available (the span of all goods); it is dependence on how available goods are *expressed*. This is *ugly*.
- On the other hand, "I'll have another fry" has *basis-free* meaning. To give basis-free meaning to complementarity of a marginal fry with a marginal drink, we must be careful to work in bundle-space, not its dual, price-space.

Common Ground – The Three-Good Quasilinear Case

Let

$$u(x_1, x_2, x_3) = f(x_1, x_2) + x_3$$

Then at each point where preferences are locally convex, these are equivalent:

- Gross complementarity of Goods 1 and 2
- Hicksian complementarity of Goods 1 and 2
- $u_{12} > 0$

Intuitively, the numeraire good gives cardinal meaning to utility, hence to u_{12}. There is a "distinguished" representation of preferences.

Common Ground starts to shake

Let

$$u(x_1, x_2, x_3, x_4) = f(x_1, x_2, x_3) + x_4$$

- Gross complementarity (of any pair of Goods 1, 2 and 3) is equivalent to Hicksian complementarity
- But these are *not* equivalent to $u_{ij} > 0$
- Rather, $u_{12} > 0$ is equivalent to: if we fix x_3 (remove a market), are 1 and 2 gross/Hicksian complements?
- There are examples where $u_{12} < 0$, $u_{13}, u_{23} > 0$ and 1,2 are Hicksian complements
- Such cases are still plagued by basis-dependence, i.e. sensitive to replacing Good 3 with a meal deal, while u_{12} is not

Calculus on Ordinal Functions

- Let $u : V \to \mathbb{R}$ be a C^∞ function on a finite-dimensional real vector space V
- At each $x \in V$, we have $Du(x) \in V^*$, i.e. a linear map $Du(x) : V \to \mathbb{R}$, where $Du(x)(v)$ is the first-order approximation of $u(x+v) - u(x)$
- Write $u \sim \hat{u}$ if $\hat{u} = f \circ u$ for a C^∞ function $f : \mathbb{R} \to \mathbb{R}$ with $f' > 0$ everywhere. Write $[u]$ for the associated equivalence class (an "ordinal C^∞ function").
- Chain rule says $D\hat{u}(x) = f'(u(x))Du(x)$. So

  $$D[u](x) = \{\alpha Du(x) : \alpha \in \mathbb{R}^+\} \in V^*/\mathbb{R}^+$$

  i.e. the derivative is defined up to positive scalar.
- $D[u](x)$ corresponds, canonically, to a (signed) hyperplane in V – the "indifference plane," $I = \mathrm{Ker}(Du(x))$

Calculus on Ordinal Functions – Second Derivative

- At each $x \in V$, $D^2u(x)$ is a (symmetric) bilinear form $D^2u(x) : V \times V \to \mathbb{R}$
- Equivalently, $D^2u(x) \in (V \otimes V)^*$
- Again, let $\hat{u} = f \circ u$, then, suppressing x,

$$D^2\hat{u}(v, w) = f'(u(x))D^2u(v, w) + f''(u(x))Du(v)Du(w)$$

$$D^2[u] = \{\alpha D^2u + \beta(Du \otimes Du) : \alpha \in \mathbb{R}^+, \beta \in \mathbb{R}\}$$

Calculus on Ordinal Functions – Second Derivative

- At each $x \in V$, $D^2u(x)$ is a (symmetric) bilinear form $D^2u(x) : V \times V \to \mathbb{R}$
- Equivalently, $D^2u(x) \in (V \otimes V)^*$
- Again, let $\hat{u} = f \circ u$, then, suppressing x,

$$D^2\hat{u}(v, w) = f'(u(x))D^2u(v, w) + f''(u(x))Du(v)Du(w)$$

$$D^2[u] = \{\alpha D^2u + \beta(Du \otimes Du) : \alpha \in \mathbb{R}^+, \beta \in \mathbb{R}\}$$

- Define the "first-order-indifferent tensors"

$$I^2 := \mathrm{Ker}(Du \otimes Du) = \mathrm{Span}(I \otimes V \cup V \otimes I) \subseteq (V \otimes V)$$

- On I^2, $D^2[u]$ is well-defined up to a positive scalar
- So $D^2[u]$ corresponds, canonically, to a (signed) hyperplane in I^2, namely $N := I^2 \cap \mathrm{Ker}(D^2u)$

Calculus on Ordinal Functions: Summarizing First and Second-Order Information

- First-order: $D[u]$ labels each $v \in V$ as good, indifferent, or bad. It can be summarized by the (signed) indifference plane $I \in V$.
- $D[u]$ also defines "indifferent tensors" $I^2 \subseteq (V \otimes V)$

Calculus on Ordinal Functions: Summarizing First and Second-Order Information

- First-order: $D[u]$ labels each $v \in V$ as good, indifferent, or bad. It can be summarized by the (signed) indifference plane $I \in V$.
- $D[u]$ also defines "indifferent tensors" $I^2 \subseteq (V \otimes V)$
- Second-order: $D^2[u]$ labels each tensor in I^2 as complementary, neutral, or substitutive. $D^2[u]$ can be summarized by the (oriented) neutral plane $N \subseteq I^2$.
- This is *all* the first and second-order information preserved by equivalence

Complementarity of Neutrals

$\mathrm{sgn}(D^2[u](x)(v_1, v_2))$ is well-defined $\Leftrightarrow (Du(x)(v_1))(Du(x)(v_2) = 0$

- That is, the sign of a cross-partial is well-defined iff one of the "goods" is actually a neutral

Complementarity of Neutrals

$$\mathrm{sgn}(D^2[u](x)(v_1, v_2)) \text{ is well-defined} \Leftrightarrow (Du(x)(v_1))(Du(x)(v_2) = 0$$

- That is, the sign of a cross-partial is well-defined iff one of the "goods" is actually a neutral
- Intuition: Taking $Du(x)(v_1) = 0$, $D^2u(x)(v_1, v_2) > 0$ means that heading in direction v_2 converts v_1 from a neutral to a good.
- Locally, preferences are convex if $D^2u(x)(v, v) < 0$ for all $v \in I$, i.e. D^2u is negative-definite on I. Then $-D^2u$ is an inner product on I, unique up to scalar, and elements of I are substitutes if the "angle" between them is less than $\pi/2$, complements otherwise

Hicksian Complements and Neutrals

- In a classic three-good problem with basis goods v_1, v_2, v_3, Hicksian complementarity of v_1, v_2 is determined by complementarity of the neutrals

$$\left(\frac{v_1}{Du(v_1)} - \frac{v_3}{Du(v_3)}, \frac{v_2}{Du(v_2)} - \frac{v_3}{Du(v_3)}\right)$$

in I.
- In words, it asks: If I substitute v_1 for v_3 while staying on the indifference plane, does the relative value of v_2 to v_3 increase (complements) or decrease (substitutes)? The dependence on v_3 is obvious.
- (For $n \geq 4$ the expression is more complicated, involving the inverse of D^2u restricted to I.)

An alternate summary of $D^2[u](x)$

In generic cases, $D^2[u](x)$ can be represented as

1. A bilinear form on I, defined up to positive scalar, together with
2. A vector $v_x^* \notin I$, the "numeraire" or "income effect," defined up to scalar, satisfying $D^2u(x)(v_x^*, w) = 0$ for all $w \in I$

Movement in the v_x^* direction has no first-order effect on MRSs, i.e. leaves $D[u](x)$ unchanged up to a scalar

Link Strength of Goods

- We don't have to give up when working with non-neutrals:
- For $v_1, v_2 \notin I$, define

$$L_{v_1,v_2}[u,x] = \frac{D^2 u(x)(v_1, v_2)}{(Du(x)(v_1))(Du(x)(v_2))}$$

- This expression measures how good v_1 affects the marginal utility from v_2, on a per-util basis. Units are inverse utils.
- The definition is basis-free. It is "a bit" sensitive to choice of u, but...

Link Strength of Goods

- We don't have to give up when working with non-neutrals:
- For $v_1, v_2 \notin I$, define

$$L_{v_1,v_2}[u,x] = \frac{D^2u(x)(v_1,v_2)}{(Du(x)(v_1))(Du(x)(v_2))}$$

- This expression measures how good v_1 affects the marginal utility from v_2, on a per-util basis. Units are inverse utils.
- The definition is basis-free. It is "a bit" sensitive to choice of u, but...
- It creates an *ordering* on vector-pairs which is well-defined on $[u]$ (Ls can be ranked)
  Proof: Let

$$t = \frac{v_1 \otimes v_2}{(Du(v_1))(Du(v_2))} - \frac{v_3 \otimes v_4}{(Du(v_3))(Du(v_4))}$$

  Then $(Du \otimes Du)(t) = 1 - 1 = 0$, so $t \in I^2$, so $\mathrm{sgn} D^2[u](t)$ is well-defined and is the sign of $L_{v_1,v_2} - L_{v_3,v_4}$

Relative Complementarity

Let $M_{v,w} = Du(v)/Du(w)$ be the marginal rate of substitution of good w for good v. Then

$$D(\ln M_{v,w})(z) = Du(z)(L_{v,z} - L_{w,z})$$

I suggest reading $L_{v,z} > L_{w,z}$ as "z complements v better than it complements w"; equivalently, an increase in z increases $M_{v,w}$, the relative value of v to w.
Moral: the ranking of Ls is a useful way to think about *relative* complementarity. We'll get to "absolute" complementarity.

Hicksian Complements and Link Strength

- In a classic three-good problem with basis goods v_1, v_2, v_3, Hicksian complementarity of v_1, v_2 is determined by

$$\mathrm{sgn}(L_{v_1,v_2} - L_{v_1,v_3} - L_{v_2,v_3} + L_{v_3,v_3})$$

Back to the 3-Good Quasilinear Case

Let

$$u(x_1, x_2, x_3) = f(x_1, x_2) + x_3$$

- Here, gross or Hicksian complementarity between goods 1 and 2 is equivalent to $L_{12} > 0 = L_{13} = L_{23} = L_{33}$.
- That is, 1 and 2 are gross/Hicksian complements if 1 complements 2 better than 1 (or 2) complements 3, i.e. if adding Good 1 increases the value of Good 2 relative to Good 3. This notion is symmetric because $L_{13} = L_{23}$
- A natural numeraire makes it reasonable to convert the relative notion into an absolute one. What makes Good 3 a natural numeraire?

Back to the 3-Good Quasilinear Case

Let

$$u(x_1, x_2, x_3) = f(x_1, x_2) + x_3$$

- Here, gross or Hicksian complementarity between goods 1 and 2 is equivalent to $L_{12} > 0 = L_{13} = L_{23} = L_{33}$.
- That is, 1 and 2 are gross/Hicksian complements if 1 complements 2 better than 1 (or 2) complements 3, i.e. if adding Good 1 increases the value of Good 2 relative to Good 3. This notion is symmetric because $L_{13} = L_{23}$
- A natural numeraire makes it reasonable to convert the relative notion into an absolute one. What makes Good 3 a natural numeraire?
  - In the standard representation, utils are in units of Good 3.
  - Avoiding representations: increases in Good 3 do not affect MRS between any two goods. Equivalently, $L_{13} = L_{23} = L_{33}$.
  - Is there always a good with this property? Locally, yes, if we're not tied to a basis...

Direct Complements

1. Recall: At each point x, there is a v_x^* (the income effect) such that motion in the v_x^* direction does not change any MRS
2. This is equivalent to $L_{v_x^*,v'}[u,x] = L_{v_x^*,v''}[u,x]$ for all v', v''
3. There is a "locally quasilinear" representation u^q such that $D^2u^q(v_x^*, v) = 0$ for all v
4. By analogy with the quasilinear case, we call w, z *direct complements* at x if

$$D^2u^q(x)(w,z) > 0 \Leftrightarrow L_{w,z}[u,x] > L_{w,v_x^*}[u,x]$$

Direct Complements

1. Recall: At each point x, there is a v_x^* (the income effect) such that motion in the v_x^* direction does not change any MRS
2. This is equivalent to $L_{v_x^*,v'}[u,x] = L_{v_x^*,v''}[u,x]$ for all v', v''
3. There is a "locally quasilinear" representation u^q such that $D^2 u^q(v_x^*, v) = 0$ for all v
4. By analogy with the quasilinear case, we call w, z *direct complements* at x if

$$D^2 u^q(x)(w,z) > 0 \Leftrightarrow L_{w,z}[u,x] > L_{w,v_x^*}[u,x]$$

5. Special case: For $n = 2$, goods are direct complements if both normal, direct substitutes if one is inferior

Direct Complements: More Equivalent Definitions

- Any bundle w can be decomposed as

$$w = \lambda_w v_x^* + w^n$$

  where v_x^* is the numeraire and $w^n \in I$ is a neutral.
- Bundles are composed of **nutrients** (utility-rich at first-order, second-order-neutral) and *flavor* (first-order-neutral, with second-order impact).
- Direct complementarity of (w, z) is equivalent to complementarity of (w^n, z^n) (or of (w^n, z) or (w, z^n)). It is the flavors which are complements (or substitutes.)
- It asks: As we acquire good w, does relative value of z to v_x^* increase (then w, z are complements) or decrease (substitutes)?

Some Properties

Write $v_1 C_{x,u} v_2$ to mean v_1 directly complements v_2 at x under preferences represented by u.

1. (Symmetry) $v_1 C_{x,u} v_2 \Leftrightarrow v_2 C_{x,u} v_1$
2. (Representation-Invariance) If $u, \hat{u}$ represent the same preferences, $v_1 C_{x,u} v_2 \Leftrightarrow v_1 C_{x,\hat{u}} v_2$
3. (Locality) If there is a neighborhood U of x such that $u = \hat{u}$ on U, then $v_1 C_{x,u} v_2 \Leftrightarrow v_1 C_{x,\hat{u}} v_2$
4. (Translation-Invariance) $v_1 C_{x,u} v_2 \Leftrightarrow v_1 C_{0,u \circ T_x} v_2$ where $T_x(x') \equiv x + x'$
5. (Basis-Free) For any invertible linear transformation $A : \mathbb{R}^n \to \mathbb{R}^n$, $(Av_1) C_{0,u \circ A^{-1}} (Av_2) \Leftrightarrow v_1 C_{0,u} v_2$

Hicksian complementarity satisfies all *but* the last.

Relationship to Price Effects

- Recall $D^2[u]$ induces a bilinear form D_I^2 on I, defined up to positive scalar.
- View this form as a map $I \to I^*$. Its inverse $(D_I^2[u])^{-1} : I^* \to I$ describes compensated price effects up to positive scalar. It induces a bilinear form $D_{I^*}^2$ on I^*.
- Two prices changes p_1, p_2 are "complements" iff $D_{I^*}^2(p_1^I, p_2^I) < 0$, where p_i^I is the restriction of p_i to I.

The Last Word

“ The really important pieces of mathematics are those that can be reduced to at most a few pages. An idea that is more complicated than that will eventually be forgotten.” – Robert Aumann