Pol Sci 5052 Mathematical Modeling

Fall 2016. Instructor: Randall Calvert

Basic pre-requisites, summer help session, and assessment

These problems are divided into five sections covering

1. **Algebra:** working with equalities and inequalities; giving particular attention to exponents and logarithms.
2. **Sets:** defining and operating on sets; understanding relationships between sets; properties of some important sets.
3. **Functions and Graphs:** the idea of a function; some important functions; exercises in analytic geometry; graphs.
4. **Statements and Proof:** mathematical statements; combining statements; establishing truth or falsehood; implications, quantifiers, logical operations, and contrapositives.
5. **Assessment:** a few basic questions on the major math topics we will be covering in the actual Fall-semester class.

The purpose of the first four sections is to review (and elicit and answer your questions about) basic math techniques before the semester begins. Once we get started with the Fall semester, I'll be assuming you can do these things. We'll hold meetings during the week before the semester officially begins so that you can ask questions about the exercises or any related ideas from the first four groups. You don't need to turn in your answers to these—use them for practice, or simply to generate questions. Feel free to write "don't know," or to skip material with which you are totally familiar.

I would, however, like to collect your answers to problems in the Assessment section. Its purpose is to help me assess your prior training on those topics. Accordingly, turn in your answers by our first regular class meeting on Monday Aug. 29.

Textbook coverage of the basic material

Most of the basic techniques for our pre-semester review session are touched upon in the early chapters of our main textbook, *Mathematics for Economists: An Introductory Textbook* (4th edition) by Pemberton and Rau. The explanatory notes below on each topic below give the relevant sections of the textbook. (Later in the course we will also make use of another text, *All of Statistics* by Wasserman, but you don't need that yet.)

I also suggest two optional, supplementary texts for the course, for those interested. (These are not, to my knowledge, carried by the campus store.) Those books also discuss these preliminary techniques, so if you are looking for some additional explanation you might turn there as well. Jeff Gill's *Essential Mathematics for Political and Social Research* includes the following:

- Algebra: Chapter 1, Sections 1.2, 1.3, 1.4, and 1.7 (pages 1-18, and 34-40).

- Sets: Chapter 7, Section 7.3 (pages 291-306).
- Functions and Graphs: Chapter 1, Section 1.5 (pages 18-34), and Chapter 2, Section 2.1 and 2.2 (pages 51-54).

The other suggested supplementary text, Moore and Siegel's *A Mathematics Course for Political and Social Research*, covers the following "Building Blocks" (Part I of the book):

- "Variables and Constants" and "Sets," sections 1.1 and 1.2
- Statements and proofs in section 1.7
- "Algebra Review," chapter 2
- Functions in section 3.1

Other material in Moore and Siegel's Part I will appear in our course during the semester. Their frequent asides on "Why Should I Care?" are unique and helpful.

1 Algebra

On quadratic equations, consult section 4.1 in the Pemberton and Rau textbook. On exponents (which they call "indices") see 4.2, and on logarithms, 4.3. On solving a system of simultaneous equations, consult section 1.3. On working with inequalities, consult section 2.1.

Unless otherwise noted, assume all variables are "real numbers" (that is, not complex or imaginary; not involving $\sqrt{-1}$).

(A.) Equations. Solve the following equations for x, or answer the question given:

1. $3x + 7 = 13$
2. $(3x - 5x + 8x)/2 = 24$
3. $3x - 5x + 8x/2 = 24$
4. $ax - bx + xy/c = z$
5. $\frac{5x+6}{2} = \frac{9+x}{5}$
6. $\frac{5x+6}{2} + \frac{3x-1}{7} = \frac{9+x}{5}$
7. $3 \cdot \sqrt{x+1} + 3 = 5$
8. $(x+3)^2 = 36$
9. $\sqrt[3]{4x-9} = 3$
10. $|x+4| = 5$
11. $(x-1)(x-2) = 0$
12. $x^2 - 3x + 2 = 0$
13. $2x^2 + 2x - 24 = 0$
14. $\frac{x^2}{2} = -8 - 4x$
15. Does $x^2 - 4x + 5 = 0$ have any solutions in real numbers? If so, give one. If not, explain why.
16. Solve for y: $x^2 + y^2 = r^2$
17. $10^x \cdot 10^{2x} = 1$ million
18. $\log_{10} x = 7$
19. $\log_{10}(xy) = a$

20. $\log_x(y) = a$ where $x > 0$

21. $y = A^x B^{1-x}$

22. Write each of the following expressions in the simplest possible form:

    (a) $(-x^4y^2)^2$

    (b) $\frac{x^4}{x^3}$

    (c) $y^7y^6y^5y^4$

    (d) $\frac{2a/7b}{11b/5a}$

    (e) x^0

    (f) $\frac{n!}{(n-1)!}$

23. (Exercise 1.3.1 in P&R) Solve simultaneously for x and y:

$$\begin{array}{rcrcr} x & + & 2y & = & 3 \\ 2x & - & 3y & = & 13 \end{array}$$

24. (Exercise 1.3.3 in P&R) Solve simultaneously for x and y:

$$\begin{array}{rcrcrcr} x & + & 2y & + & 2z & = & 1 \\ 2x & - & 2y & + & z & = & 2 \\ x & - & y & + & 3z & = & 3 \end{array}$$

25. (Exercise 1.3.2 in P&R) Show that the following system of equations in x and y, where c is a constant, has no solution unless $c = 1/2$. For the case $c = 1/2$, describe what values of x and y solve the system.

$$\begin{array}{rcrcr} 2x & - & 5y & = & c \\ 4x & - & 10y & = & 1 \end{array}$$

(Problems continue on next page.)

(B.) Inequalities. Transform each of the following inequalities into an equivalent inequality, or combination of inequalities, with x alone on one side:

1. $\frac{x}{-y^2} \geq 1$
2. $ax + b \geq c$
3. $x^2 > 4$
4. $x^2 \leq 0$
5. $xy \leq y$
6. $\frac{1}{x^2} < 100$
7. $\frac{1}{-x^2} < -y^2$
8. $x^2 + y^2 \leq 4$
9. $2x^2 + 2x - 24 < 0$

2 Sets

Additional basic material on sets is covered by Pemberton and Rau in sections 3.1 and 3.2.

Notation and terminology: In what follows, sets are denoted by italic capitals, as in the set A, and their members ("elements") by lower-case italics, x

- We write a set either by listing its elements, $\{1, 2, 3\}$, or by giving a rule describing it, generally in the form $\{\text{counting numbers } k : k \text{ is no greater than } 3\}$. We "list" a large or infinite set by using an ellipsis to indicate how it continues: $\{1, 2, 3, \ldots, 100\}$. (Note that this convention is loose; in the case of numbers, there is always more than one pattern that matches any finite beginning.)

- The *cardinality* of a finite set is the number of elements in that set. For our purposes in this course, the cardinality of any infinite set can just be taken to be "infinity," denoted ∞. Various symbols are used to denote the cardinality of a set; $\#A$ or $\#(A)$ is common.

- $\mathbb{R}$ denotes the set of all real numbers, that is, all (1) numbers with finitely many digits after the decimal point, (2) repeating decimals, and (3) infinitely-long non-repeating decimals. Round and square brackets are used to denote *intervals* of real numbers: if $x \leq y$ then

$$[x, y] = \{z \in \mathbb{R} : x \leq z \leq y\}$$
$$[x, y) = \{z \in \mathbb{R} : x \leq z < y\}$$
$$(x, y] = \{z \in \mathbb{R} : x < z \leq y\}$$
$$(x, y) = \{z \in \mathbb{R} : x < z < y\}$$

- $\varnothing$ denotes the empty set, also written $\{\}$.

- "$\in$" means "is a member (element) of the set", as in $x \in A$; "$\notin$" means "is not a member of the set", as in $\sqrt{-1} \notin \mathbb{R}$.

- The union of A and B, denoted $A \cup B$, is the set of elements belonging to A or to B or both; formally, $A \cup B = \{x : x \in A \text{ or } x \in B\}$.

- The intersection of A and B, denoted $A \cap B$, is the set of elements belonging both to A and to B; formally, $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

- The complement of A in B, denoted $B \smallsetminus A$, is the set of all elements of B that do not also belong to A; formally, $B \smallsetminus A = \{x \in B : x \notin A\}$.

- We use the following symbols to express relationships between sets:

  $A \subseteq B$ if A is a subset of B, that is, every element of A is also an element of B (includes the possibility $A = B$).

  $A \subset B$ if A is a *proper* subset of B, that is, every element of A is also an element of B but some element of B is not in A (thus excludes the possibility that $A = B$).

and of course $A = B$ if the following statement is true for all x: $x \in A$ if and only if $x \in B$

Problems:

1. Use set-theoretic notation to describe each of the following:

   (a) Some non-numerical sets:
   - the Three Stooges
   - the Baltic States
   - U.S. Presidents elected in the 1800s after Reconstruction

   (b) possible totals from the roll of two dice

   (c) possible outcomes of the roll of two dice (that is, pairs of numbers)

   (d) all positive even numbers

   (e) all positive proper fractions in simplest terms

   (f) the real numbers between -1 and 1, inclusive

   (g) the positive real numbers no greater than 7

   (h) the set of all subsets of $\{1, 2, 3\}$

2. For each of the pairs of sets A and B defined below, describe $B \setminus A$:

   (a) $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $B = \{6, 7, 9, 11, 12\}$

   (b) $A = \{1, 2, 3, \ldots, 100\}$, $B = \{1, 3, 5, \ldots, 99\}$

   (c) $A = \{x \in \mathbb{R} : x > 4\}$, $B = \{x \in \mathbb{R} : x < 10\}$

   (d) $A = \{x \in \mathbb{R} : x > 4\}$, $B = \{x \in \mathbb{R} : x > 10\}$

   (e) $A = [0, 2]$, $B = [1, 2]$

   (f) $A = (0, 1)$, $B = [0, 1]$

3. Let $A = \{1, 2, 3, 4, 5\}$, $B = \{0, 1, 4, 8\}$, and $C = \{2, 5, 7, 9, 11, 13, 17\}$. Compute each of the following.

   (a) $A \cup B$

   (b) $A \cap B$

   (c) $A \setminus (B \cup C)$

   (d) $A \cap (B \cap C)$

   (e) $(A \cup B) \cap (A \cup C)$

   (f) $(A \cap C) \setminus (B \cap C)$

4. Find the cardinality of each set:

   (a) $\{1, 2, 3, 4, \ldots, n\}$
   (b) $\{1, 1, 2, 2, 3, 3, 2, 2, 1, 1\}$
   (c) $[0, 1]$
   (d) $[0, 1/2]$
   (e) the set of all subsets of $\{1, 2, 3\}$

5. For each $n \in \{1, 2, 3, \ldots\}$, define $A_n = (0, 1 + 1/n)$. Give the simplest description you can of each of the following sets:

   (a) A_2
   (b) $A_1 \cap A_2$
   (c) $A_1 \cap A_2 \cap A_3$
   (d) $A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n$
   (e) $A_1 \cap A_2 \cap A_3 \cap \ldots$

6. Let A be the set of all integer multiples of 5 and B be the set of all even integers. Compute:

   (a) $A \cup B$
   (b) $A \cap B$
   (c) $A \setminus B$
   (d) $B \setminus A$

7. Explain why each of the following is true or false.

   (a) $[0, 100] = \{0, 1, 2, \ldots, 100\}$
   (b) $\{-1, -2, -3, 1, 2, 3, 0\} = \{-3, -2, -1, 0, 1, 2, 3\}$
   (c) $\{1, 1, 2, 2, 3, 3, 2, 2, 1, 1\} = \{1, 2, 3\}$
   (d) $\{7, 8, 9\} \subseteq \{0, 1, 2, \ldots, 100\}$
   (e) $[7, 9] \subseteq \{0, 1, 2, \ldots, 100\}$
   (f) $[1, 2] \subseteq [0, 2)$

3 Functions and Graphs

Additional basic material on functions and graphs is covered by Pemberton and Rau in sections 1.1 (formula for, and graph of, a straight line) and 3.3-3.4 (functions and "mappings").

Notation and Terminology:

- For any two sets A and B, $A \times B$ denotes their Cartesian product: the set of all pairs of elements, the first of which is drawn from A and the second from B. Formally,

  $$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

  We denote the set of all pairs with both elements drawn from A by $A \times A$ or A^2.

- Thus $\mathbb{R}^2$ denotes the set of all pairs of real numbers—geometrically, the Euclidean plane, that is, the set of all points in a limitless plane with a Cartesian system of coordinates imposed upon it.

- For any two non-empty sets A and B, $f : A \to B$ denotes a function whose domain is A and whose range is in B. A function associates with each $a \in A$ some element in B, denoted $f(a)$. The range of f, often written as$f(A)$, is the set $\{b \in B : b = f(a) \text{ for some } a \in A\}$. Not all members of B need be in the range of f.

- An *inverse function* of a function $f : A \to B$ is a function $g : f(A) \to A$ defined by assigning $g(b)$ the value of a for which $f(a) = b$. Not every function has an inverse function; for example, if $f(a) = f(a') = b$ then the definition of $g(b)$ would not be well defined. Usually we write the inverse function of f as f^{-1}.

- For any function $f : \mathbb{R} \to \mathbb{R}$, the graph of f is the set $\{(x, y) \in \mathbb{R}^2 : y = f(x)\}$. Thus when you "graph a function," you are drawing a picture of this set.

Problems:

1. Make a drawing of the following sets in $\mathbb{R}$ or $\mathbb{R}^2$ as appropriate:
   (a) $[-1, 1) \cup (2, 3) \cup (3, 4)$
   (b) $\{(x, y) \in \mathbb{R}^2 : x = t, y^2 = t\}$ for $t \in \{-2, -1, 0, 1, 2\}$
2. For the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x + 2$,
   (a) draw the graph of f.
   (b) What is the slope of the function?
   (c) Does f have an inverse function? If not, why not? If so, what is it?
   (d) Depict the graph of $f(x) = -ax - b$, where a and b are positive constants.

3. For the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$,

   (a) draw the graph of f.

   (b) Where is function steepest? Least steep?

   (c) Does f have an inverse function? If not, why not? If so, what is it?

   (d) Do the same exercises for $f(x) = x^3$.

   (e) Do the same exercises for $f(x) = \log(|x|)$.

4. Graph the following functions for $x \in \mathbb{R}$:

   (a) $y = x$

   (b) $x + y = a$ for some constant a

   (c) $y = 1/x$

   (d) $y = \sqrt{x}$

   (e) $y = \begin{cases} x+1 & \text{if } -1 < x \leq 0 \\ 1-x & \text{if } \quad 0 < x \leq 1 \\ 3x & \text{otherwise} \end{cases}$

5. For a legislative system using plurality-rule (that is, "first past the post") single-member districts with two parties, the Cube Law holds that the ratio between the two parties' seat shares is approximately equal to the cube of the ratio between their vote shares. That is, if S is the majority party's seat share and V its vote share,

$$\frac{S}{1-S} = \left(\frac{V}{1-V}\right)^3.$$

   (a) Derive a formula for the majority seat share as a function of the majority vote share.

   (b) Draw the corresponding graph.

   (c) Derive and describe in words the inverse of the function you found in (a).

4 Statements and Proof

Pemberton and Rau present elementary material on statements and proof in their section 31.1 (“Rigour”), pp. 700-702.

Notation and terminology:

- A “statement” is any mathematical expression that states a definition or establishes a condition or relationship. Some examples of statements are:

  $x = y$

  $x \in A$

  $\{0, 1, 2, \ldots\} \subset \mathbb{R}$

  $1 + 1 = 3$

  Note that a statement can be either true or false. The following expressions, on the other hand, are not statements:

  x

  $a + b$

  42

  $\leq$

- In mathematical statements, the following symbols, called “quantifiers,” are sometimes used to express where or how often a statement is true: $\forall$, meaning “for every;” $\exists$, meaning “there exists at least one;” and $\nexists$, meaning “there is no.” Examples of true statements using these symbols are given in the following exercises.

- Statements are often modified or combined using the operators AND (or $\wedge$), OR (or $\vee$), NOT (or $\neg$), IF...THEN (equivalently IMPLIES or $\Rightarrow$), and IF AND ONLY IF (or IFF or $\Leftrightarrow$). When the grouping of elements in a complicated statement is ambiguous, brackets and parentheses can be used to clarify. For example, the formal definition of the intersection of two sets is equivalent to the following statement:

  $\forall A, B \; [x \in A \cap B \Leftrightarrow (\; x \in A \text{ AND } x \in B \;)\;].$

(Problems follow on the next page.)

1. **Proofs using algebra.** Let x, y, a, and b be positive real numbers. Using the basic definition of a logarithm—namely that $\log_b(x)$, or $\log_b x$, is the number y such that $b^y = x$—and the rules of exponentiation, prove that the following statements are true. If no base is indicated for the logarithm, simply "log," then the base doesn't matter, but you can insert it if you need to. Hint: Proving these statements is just like solving the indicated equation or simplifying the indicated expression.

   (a) $b^{\log_b(x)} = x$

   (b) $\frac{2a/7b}{11b/5a} = \frac{10a^2}{77b^2}$

   (c) $y = a^x b^{1-x} \Rightarrow x = \frac{\log(y/b)}{\log(a/b)}$

2. **Extra "credit:" the quadratic formula.** Prove that the equation $ax^2 + bx + c = 0$, where a, b, and c are constants, is satisfied always and only by the following values of x:

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

   and

$$x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

3. **Statements using symbols.** Explain in words what each of the following statements is telling you about the symbols and terminology defined above:

   (a) $A \subseteq B \Leftrightarrow [x \in A \Rightarrow x \in B]$.

   (b) $\forall (x \in \mathbb{R})$ [$\exists (y \in \mathbb{R})$ such that $x - y = 0$].

   (c) $\nexists (x \in \mathbb{R})$ such that $x = \sqrt{-1}$.

   (d) if $x > y$ is false, then [NOT $x > y$] is true.

   (e) $\forall (x, y \in \mathbb{R}), x < y \Rightarrow$ [$\exists (z \in \mathbb{R})$ such that $x < z < y$]

   (f) $x \leq y \Leftrightarrow$ NOT $x > y$.

4. Let A and B be sets contained in a universal set U. Which (one or more) of the following statements is necessarily true if we assume $A \subset B$? Use a Venn diagram to show why.

   (a) $\forall x \in U$, if $x \in A$ then $x \in B$

   (b) $\forall x \in U$, if $x \notin A$ then $x \notin B$

   (c) $\forall x \in U$, if $x \in B$ then $x \in A$

   (d) $\forall x \in U$, if $x \notin B$ then $x \notin A$

(Problems continue on the next page.)

5. When all possible elements lie in some universal set U, then for any set $S \subseteq U$ we write S^c for the *complement* of S in U: $S^c = U \setminus S = \{x \in U : x \notin S\}$. For the following statements, suppose all elements lie in some universal set. For each statement, explain how you can be absolutely sure the statement is true; or else, give an instance in which it is false. Make use of the formal definitions given above (both in this section and the previous section on Sets) as needed.

   (a) $A \cap B = B \cap A$
   (b) $A \cup (B \cup C) = (A \cup B) \cup C$
   (c) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
   (d) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   (e) $A \cap B \subseteq A$
   (f) $A \cap B \subset A$
   (g) $A \cup B \subseteq A$
   (h) $B \setminus A \subseteq A$
   (i) $B \setminus A \subseteq B$
   (j) $B \setminus A \subset B$
   (k) if $A \subseteq B$ then $x \notin B \Rightarrow x \notin A$
   (l) $\forall A, B\ [\ (A \cap B)^c = A^c \cup B^c\ ]$.
   (m) $\forall A, B\ [\ (A \cup B)^c = A^c \cap B^c\ ]$.

Assessment Problems

These are basic versions of the kinds of things we'll be covering during the semester. Let's see where you stand at the outset. **Turn in your answers** (or "don't knows" on Monday, Aug. 29).

5 Sequences and Series

1. For $n = 1, 2, \ldots$, define $x_n = \sqrt{n}$, the positive square root of n. Does the sequence (x_n) converge to a limit? If so, what is the limit? If not, why not?

2. For $n = 1, 2, \ldots$, define $y_n = \frac{(-1)^n}{n}$. Does the sequence (y_n) converge to a limit? If so, what is the limit? If not, why not?

3. What is the value of $1 + 2 + \cdots + 1000$, that is, the sum of the first 1000 positive integers?

4. Find the value of $\sum_{t=1}^{\infty} (.05)^t$.

6 Matrix Arithmetic

1. Rewrite the following system of 3 equations in 3 variables as a single matrix equation:

$$\begin{array}{rcrcrcl} x & + & 2y & + & 2z & = & 1 \\ 2x & - & 2y & + & z & = & 2 \\ x & - & y & + & 3z & = & 3 \end{array}$$

2. Compute the matrix product **AB**, where

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 3 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 0 & 2 \end{bmatrix}$$

3. Find the determinant of each of the following matrices:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 2 & 19 & 37 \\ 0 & 3 & 45 \\ 0 & 0 & 4 \end{bmatrix}$$

7 Differential Calculus

1. What is the slope of the graph of $y = x^3$ at $x = -1$?

2. Using the power rule, find the derivative of $f(x) = 4x^3$.

3. Using the product rule, find the derivative of $f(x) = (6x^3 + x)(x^6 - 3x^4 - 2)$.

4. Using the quotient rule, find the derivative of $f(x) = (3x + a)/(x^2 + b)$.

5. Using the chain rule, find the derivative of $f(x) = (x^4 - 3x^2 + 5x + 1)^{1/4}$.

6. Find the derivative of

$$f(x) = \frac{x^{1/3} - 2}{(x^5 - 2)^3}.$$

8 Integral Calculus

1. What is the indefinite integral of $\dfrac{x^3 + 1}{x}$?

2. In a graph of the function $y = x^3$, what is the area of the shape having the following boundaries:

   - the curve itself;
   - the x-axis;
   - the vertical line at $x = -1$; and
   - the vertical line at $x = 2$?

9 Combinatorics and Probability

1. In how many different orders can you write the 26 letters from A to Z?

2. A conventional deck of cards includes 13 cards in each of four suits (clubs, diamonds, hearts, spades). Of the 13, three (Jack, Queen, King) are the "face cards." One card is drawn at random from a deck. Let H be the event "the card is a heart" and let F be the event "the card is a face card." Find

   (a.) What is the probability that the card is both a heart and a face card?

   (b.) What is the probability that the card is either a heart or a face card?

   (c.) What is the probability that the card is not a heart? The probability that it is not a face card?

   (d.) What is the probability that the card is the ace of spades? Conditional on the card being a spade, what is the probability that it is the ace of spades?

3. The random variable X follows the uniform distribution on the interval $[a, b]$. What is the expected value of X? For a constant c with $a < c < b$, what is the probability that $X < c$?

4. A certain test for drug use will fail to detect the drug in 10% of cases where the drug is present. It will falsely indicate that the drug is present in 5% of cases where there is no drug. Suppose that for a given population, you know that 1% of individuals use the drug regularly, and the remainder never use it. The test is administered to a random individual, and indicates that the drug is present. Using Bayes's Formula, compute the probability that this individual is a user.