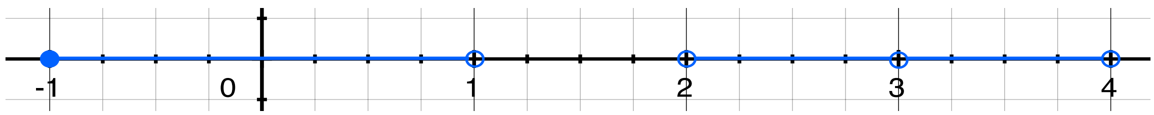
3 Functions and Graphs: Solutions to the Problems

1. Make a drawing of the following sets in $\mathbb{R}$ or $\mathbb{R}^2$ as appropriate:

   (a) $[-1, 1) \cup (2, 3) \cup (3, 4)$



1(a)

   (b) $\{(x, y) \in \mathbb{R}^2 : x = t, y^2 = t\}$ for $t \in \{-2, -1, 0, 1, 2\}$

   - See Figure 1.

2. For the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x + 2$,

   (a) draw the graph of f.

   - See Figure 1.

   (b) What is the slope of the function?

   - 3.

   (c) Does f have an inverse function? If not, why not? If so, what is it?

   - Yes. $x = 3y + 2 \Rightarrow y = \frac{x-2}{3}$.

   (d) Depict the graph of $f(x) = -ax - b$, where a and b are positive constants.

   - See Figure 1.

3. For the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$,

   (a) draw the graph of f.

   - See Figure 1.

   (b) Where is function steepest? Least steep?

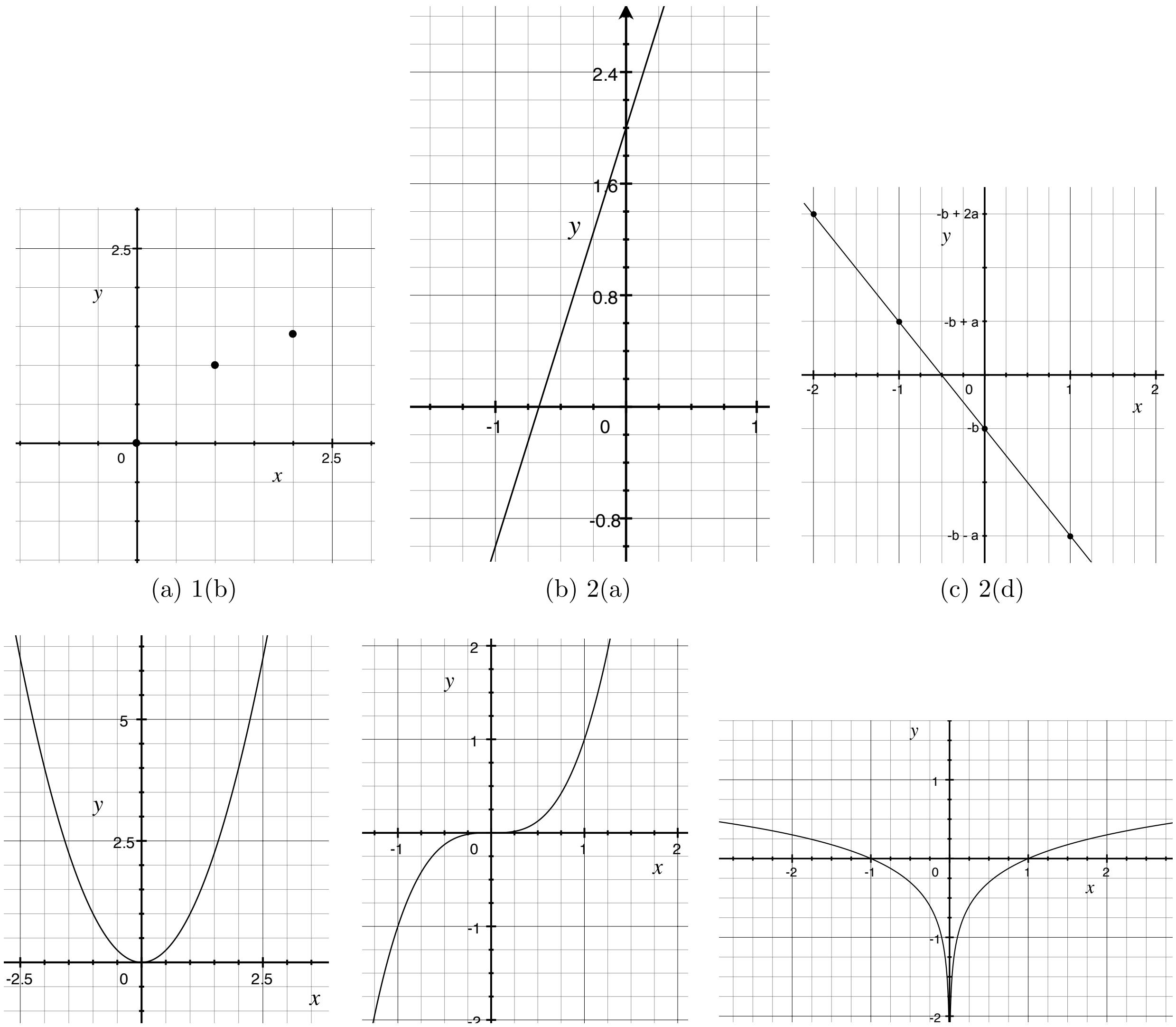
   - In the sense of positive slope: the larger x gets, the steeper is graph of the function; it gets less "steep" the smaller (larger negative) x gets. Or in the sense of absolute slope: the curve gets steeper as $|x|$ gets larger, and is least steep at $x = 0$.

   (c) Does f have an inverse function? If not, why not? If so, what is it?

   - As we have defined it, f has no inverse since for some values of y (namely, all $y > 0$), there are two different values of x for which $f(x) = y$. Hence $f^{-1}(y)$ is not well defined.

   (d) Do the same exercises (a, b, and c) for $f(x) = x^3$.

   - See Figure 1; the graph is steepest at $x = \pm\infty$ and least steep at $x = 0$; f does have an inverse, $f^{-1}(y) = \sqrt[3]{y}$.



(a) 1(b) (b) 2(a) (c) 2(d)

(d) 3(a) (e) 3(d) (f) 3(e)

Figure 1: Problems 1(b), 2(a), 2(d), 3(a), 3(d), and 3(e).

(e) Do the same exercises (a, b, and c) for $f(x) = \log(|x|)$.

- See Figure 1.
- Steepest at $x = 0$ and least steep at $x = \pm\infty$.
- No. $y = \log(|x|) \Rightarrow |x| = 10^y \Rightarrow x = \pm 10^y$, so multiple values of y may correspond to a unique value of x. Thus, not invertible.

4. Graph the following functions for $x \in \mathbb{R}$:

(a) $y = x$

- See Figure 2.

(b) $x + y = a$ for some constant a

- See Figure 2.

(c) $y = 1/x$

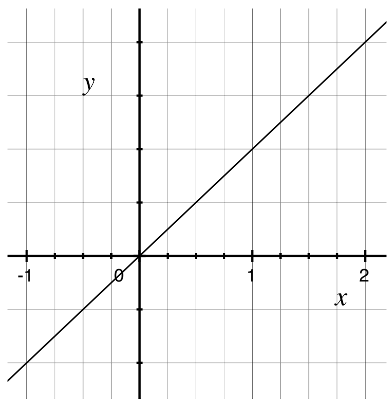
- See Figure 2.

(d) $y = \sqrt{x}$

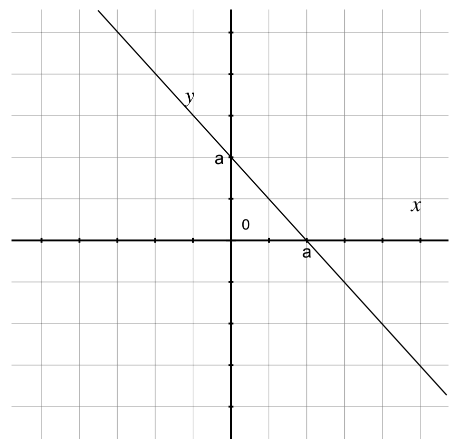
- See Figure 2.

(e) $y = \begin{cases} x+1 & \text{if } -1 < x \leq 0 \\ 1-x & \text{if } \quad 0 < x \leq 1 \\ 3x & \text{otherwise} \end{cases}$

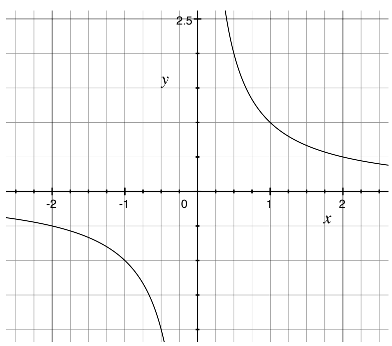
- See Figure 2.



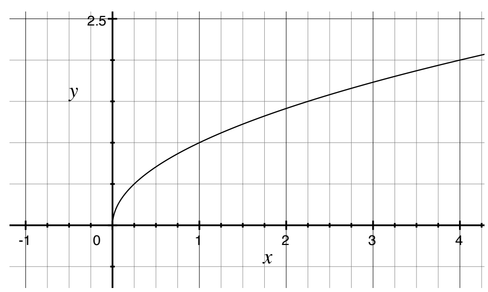
(a) 5(a)



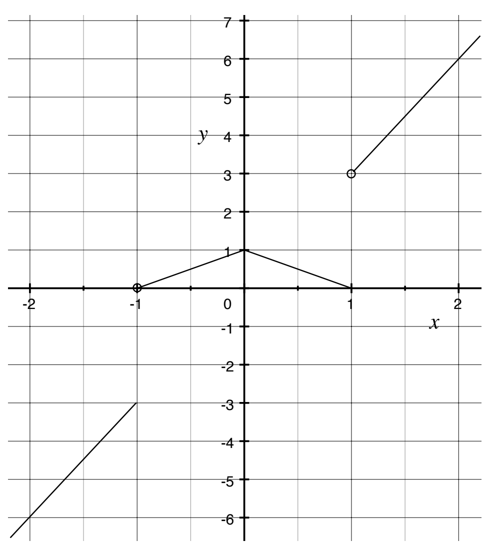
(b) 5(b)



(c) 5(c)



(d) 5(d)



(e) 5(e)

Figure 2: Problems 5(a)-(e).

5. For a legislative system using plurality-rule (that is, “first past the post”) single-member districts with two parties, the Cube Law holds that the ratio between the two parties’ seat shares is approximately equal to the cube of the ratio between their vote shares. That is, if S is the majority party’s seat share and V its vote share,

$$\frac{S}{1-S} = \left(\frac{V}{1-V}\right)^3.$$

(a) Derive a formula for the majority seat share as a function of the majority vote share.

(b) Draw the corresponding graph.

(c) Derive and describe in words the inverse of the function you found in (a).

Solution:

(a) Solving the formula for S in terms of V gives

$$S = \frac{V^3}{1-3V+3V^2}.$$

(b) See the attached page, where the graphing is accomplished using Mathematica.

(c) the inverse function gives the approximate vote share that would be necessary to gain a seat share of S:

$$V = \frac{\sqrt[3]{\frac{S}{1-S}}}{1+\sqrt[3]{\frac{S}{1-S}}}.$$

Cube Law problem

First, solve the given seats-votes relationship to get the seat share S as a function of the vote share V:

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In[5]:= Solve[S / (1 - S) == (V / (1 - V)) ^3, S]
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Out[5]= $\left\{\left\{S \to \frac{V^3}{1 - 3 V + 3 V^2}\right\}\right\}$

Next, for convenience, define this as a function for graphing: S = f(V).

In[6]:= $f[V\_] = \frac{V^3}{1 - 3 V + 3 V^2}$

Out[6]= $\frac{V^3}{1 - 3 V + 3 V^2}$

Finally, plot the seats function for vote shares ranging from 0 .00 to 1.00.

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In[10]:= Plot [f[V], {V, 0, 1}]
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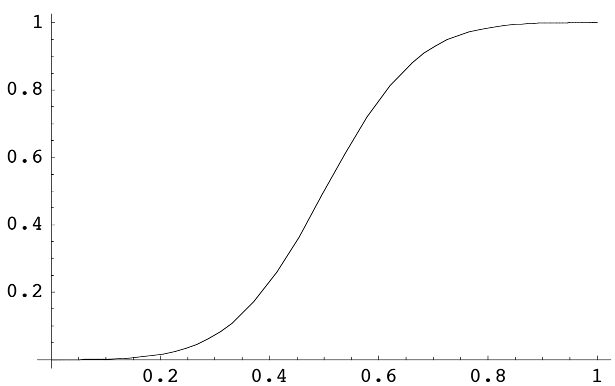

Out[10]= - Graphics -