4 Statements and Proof: Solutions to the Problems

As is conventional to improve readability, the end of a multi-line proof is indicated by the appearance of a symbol—here, $\square$. (Elsewhere other symbols, or the notation Q.E.D., are sometimes used.)

1. **Proofs using algebra.** Let x, y, a, and b be positive real numbers. Using the basic definition of a logarithm—namely that $\log_b(x)$, or $\log_b x$, is the number y such that $b^y = x$—and the rules of exponentiation, prove that the following statements are true. If no base is indicated for the logarithm, simply "log," then the base doesn't matter, but you can insert it if you need to. Hint: Proving these statements is just like solving the indicated equation or simplifying the indicated expression.

   (a) $b^{\log_b(x)} = x$
   Proof: By definition, $\log_b(x)$ is the number y such that $b^y = x$. So substuting for y in that definition immediately gives the desired equation. $\square$

   (b) $\frac{2a/7b}{11b/5a} = \frac{10a^2}{77b^2}$

   Proof: since $\frac{x}{y/z} = x\frac{z}{y}$,
   $$\frac{2a/7b}{11b/5a} = \frac{2a \cdot 5a}{7b \cdot 11b}.$$
   Since $\frac{w}{x} \cdot \frac{y}{z} = \frac{wy}{xz}$, this is exactly
   $$= \frac{10a^2}{77b^2}.$$
   $\square$

   (c) $y = a^x b^{1-x} \Rightarrow x = \frac{\log(y/b)}{\log(a/b)}$

   Proof: Take the log of both sides of the first equation to get $\log y = \log a^x b^{1-x}$. Using the rule for the logarithm of a product, $\log a^x b^{1-x} = \log a^x + \log b^{1-x}$. Applying (twice) the rule for the logarithm of a number raised to a power, this becomes $x \log a + (1-x)\log b$ or, collecting terms on x, $x(\log a - \log b) + \log b$. Substituting this into the right hand side of the original equation, we now have
   $$\log y = x(\log a - \log b) + \log b$$
   and solving for x yields
   $$x = \frac{\log y - \log b}{\log a - \log b}.$$
   Notice that, according to the rule for the log of a fraction, the numerator is equal to $\log y/b$ and the denominator is $\log a/b$. This gives the desired implication. $\square$

2. **Extra "credit:" the quadratic formula.** Prove that the equation $ax^2+bx+c=0$, where a, b, and c are constants with $a \neq 0$, is satisfied always and only by the following values of x:

$$x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$$

and

$$x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}.$$

Proof: First, divide both sides of the original equation by a and notice that x solves $ax^2+bx+c=0$ if and only if it solves $x^2+(b/a)x+c/a=0$.

The second, and main, step of the proof is to notice that $x^2+(b/a)x+c/a = (x-x_1)(x-x_2)$ where x_1 and x_2 are given by the formulas above. This is shown simply by substituting in those values for x_1 and x_2, and multiplying out and simplifying the result. Third and finally, it is obvious that the equation $(x-x_1)(x-x_2)=0$ is solved always and only by $x=x_1$ or $x=x_2$. Hence from the previous two steps, the original equation $ax^2+bx+c=0$ is satisfied always and only by those values of x. $\square$

Another approach is, after dividing through by a, to subtract c/a from both sides and then "complete the square." That is, take half the coefficient of the x term—that would be $b/2a$—square it, giving $b^2/4a^2$, and add that to both sides of the equation:

$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2} = -\frac{c}{a}+\frac{b^2}{4a^2}.$$

The point of doing this is that it renders the left-hand side a perfect square:

$$\left(x+\frac{b}{2a}\right)^2 = -\frac{c}{a}+\frac{b^2}{4a^2}.$$

Simplifying the right-hand side, this becomes

$$\left(x+\frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2}$$

and taking the square root of both sides,

$$x+\frac{b}{2a} = \frac{\pm\sqrt{b^2-4ac}}{2a}.$$

This is easily re-arranged to give the quadratic formula.

3. **Statements using symbols.** Explain in words what each of the following statements is telling you by using about the symbols and terminology defined above:

   (a) $A \subseteq B \Leftrightarrow [x \in A \Rightarrow x \in B]$. definition of $\subseteq$

(b) $\forall(x \in \mathbb{R})$ [$\exists(y \in \mathbb{R})$ such that $x + y = 0$]. additive inverse exists

(c) $\nexists(x \in \mathbb{R})$ such that $x^2 = -1$. imaginary numbers

(d) if $x > y$ is false, then [NOT $x > y$] is true, and conversely. definition of negation

(e) $\forall(x, y \in \mathbb{R}), x < y \Rightarrow$ [$\exists(z \in \mathbb{R})$ such that $x < z < y$] there are no "adjacent" real numbers

(f) $x \leq y \Leftrightarrow$ NOT $x > y$. definition of $\leq$ in terms of $>$ and vice versa

4. Let A and B be sets contained in a universal set U. Which (one or more) of the following statements is necessarily true if we assume $A \subset B$? Use a Venn diagram to show why.

   (a) $\forall x \in U$, if $x \in A$ then $x \in B$ true

   (b) $\forall x \in U$, if $x \notin A$ then $x \notin B$ false, unless $B \subseteq A$

   (c) $\forall x \in U$, if $x \in B$ then $x \in A$ false, unless $B \subseteq A$

   (d) $\forall x \in U$, if $x \notin B$ then $x \notin A$ true

5. For each of the following statements, explain how you can be absolutely sure the statement is true; or else, give an instance in which it is false. Make use of the formal definitions given above (both in this section and the previous section on Sets) as needed.

   (a) $A \cap B = B \cap A$
   Proof: apply the definition of $\cap$ twice:
   $x \in A \cap B \Leftrightarrow x \in A$ and $x \in B \Leftrightarrow x \in B \cap A$ $\square$

   (b) $A \cup (B \cup C) = (A \cup B) \cup C$
   Proof: applying the definition of $\cup$ repeatedly,
   $x \in A \cup (B \cup C) \Leftrightarrow x \in A$ or $x \in B \cup C$
   $\Leftrightarrow x \in A$ or $(x \in B$ or $x \in C)$
   $\Leftrightarrow x \in A$ or $x \in B$ or $x \in C$
   $\Leftrightarrow (x \in A$ or $x \in B)$ or $x \in C$
   $\Leftrightarrow x \in A \cup B$ or $x \in C$
   $\Leftrightarrow x \in (A \cup B) \cup C$ $\square$

   (c) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
   Proof: apply the definitions of $\cap$ and $\cup$, and remember that R and (P or Q) $\Leftrightarrow$ (R and P) or (R and Q):
   $x \in A \cap (B \cup C) \Leftrightarrow x \in A$ and $[x \in B$ or $x \in C]$
   $\Leftrightarrow [x \in A$ and $x \in B]$ or $[x \in A$ and $x \in C]$
   $\Leftrightarrow x \in A \cap B$ or $x \in A \cap C$
   $\Leftrightarrow x \in (A \cap B) \cup (A \cap C)$ $\square$

   (d) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   Proof: similar to the previous problem

$x \in A \cup (B \cap C) \Leftrightarrow x \in A$ or $[x \in B$ and $x \in C]$
$\Leftrightarrow [x \in A$ or $x \in B]$ and $[x \in A$ or $x \in C]$
$\Leftrightarrow x \in A \cup B$ or $x \in A \cup C$
$\Leftrightarrow x \in (A \cup B) \cap (A \cup C)$ $\square$

(e) $A \cap B \subseteq A$ — $x \in A \cap B \Rightarrow x \in A$

(f) $A \cap B \subset A$ — false if $A \subseteq B$

(g) $A \cup B \subseteq A$ — true only if $B \subseteq A$

(h) $B \setminus A \subseteq A$ — false unless $B \subseteq A$

(i) $B \setminus A \subseteq B$ — true: $x \in B$ and $x \notin A \Rightarrow x \in B$

(j) $B \setminus A \subset B$ — false if $A = \emptyset$, since B not a proper subset of itself

(k) if $A \subseteq B$ then $x \notin B \Rightarrow x \notin A$ — converse of definition

(l) $\forall A, B\ [\ (A \cap B)^c = A^c \cup B^c\ ]$
Proof: $x \in (A \cap B)^c \Leftrightarrow x \notin A \cap B \Leftrightarrow$ not $[x \in A$ and $x \in B]$
$\Leftrightarrow x \notin A$ or $x \notin B$
$\Leftrightarrow x \in A^c$ or $x \in B^c$
$\Leftrightarrow x \in A^c \cup B^c$ $\square$

(m) $\forall A, B\ [\ (A \cup B)^c = A^c \cap B^c\ ]$
Proof: similar to the previous problem:
$x \in (A \cup B)^c \Leftrightarrow x \notin A \cup B \Leftrightarrow$ not $[x \in A$ or $x \in B]$
$\Leftrightarrow x \notin A$ and $x \notin B$
$\Leftrightarrow x \in A^c$ and $x \in B^c$
$\Leftrightarrow x \in A^c \cap B^c$ $\square$