

Exercises on Arrow's theorem and Blau's proof

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Exercises 1-3 refer to this list of conditions on a social welfare function F , where throughout p denotes any preference profile of weak orderings (R_i) :

1. For all profiles p , $R = F(p)$ is a weak order (reflexive, complete, and transitive).
2. universal domain
3. Blau's UPP condition
4. independence of irrelevant alternatives
5. nondictatorship

Exercise 1. Let R° be some given, fixed weak order. Let F be a social welfare function described as follows: for every profile p , $F(p) = R^\circ$. Which of the conditions does F violate?

Exercise 2. Let $R = F(p)$ be the extended Pareto ordering for p , described as follows: for every profile p , for every $x, y \in A$, $xPy \Leftrightarrow [xP_i y \text{ for every } i]$. Which of the conditions does F violate?

Exercise 3. Let $J : A \times A \rightarrow N$ be a function that associates with each pair of alternatives a, b from A a unique individual $j = J(a, b) \in N$. Assume that the function J is onto, that is, for every individual $i \in N$ there is at least one pair of alternatives a, b such that $J(a, b) = i$. For each $p = (R_1, \dots, R_n)$, define the social preference relation $R = F(p)$ as follows: for every $a, b \in A$, $aRb \Leftrightarrow aR_{J(a,b)}b$. That is, different individuals serve as dictator for different pairs of alternatives. Which of the five condition does this F violate? Does it violate any other aspect of Blau's definition of a social welfare function?

Exercise 4. In Blau's proof of the impossibility theorem for $|A| \geq 3$ (page 65), prove the claim that $aDb \Rightarrow yDb$ for all $y \neq b$.

Exercise 5. Prove, as Blau claims in Section 6, that UPP is a stronger assumption than the combination UPR and non-null. That is,

- Prove that UPP implies both UPR and non-null; and
- Construct an example showing that UPR and non-null together do not necessarily imply UPP.