Some Mathematical Notes for Reading McKelvey, "Intransitivities in Multidimensional Voting Models"

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General

When working through the proof of Theorem 2, be sure to reconstruct the pictures. They're given (perhaps too small to see) in Figures 2, 3, and 4, but there's no substitute for understanding how to make them yourself.

Recall that for any set of real numbers $C \subset \mathbb{R}$ which is bounded below, $\inf C$ denotes its greatest lower bound. That's the same thing as $\min C$ if it is contained in C. Thus $\inf(0,1) = \inf[0,1] = \min[0,1]$; but $\min(0,1)$ does not exist.

A notational short-cut: As McKelvey only partially explains on p. 473, for sets S occurring in this paper, you have to figure out from context whether $|S|$ represents the number of elements in S (as is conventional) or the number of voters having their ideal points in S. Generally it's the latter if and only if $S \subset \mathbb{R}^m$.

Vectors and inner products

Notice that just about everything in the paper is a vector in $\mathbb{R}^m$. (This paper uses R for $\mathbb{R}$.) Usually these are denoted x, y, θ. Occasionally a scalar pops up; it too is an italic letter, so they can only be distinguished by context. In Theorem 2, the "sequence of alternatives" $\theta_1, \ldots, \theta_N$ is a sequence of vectors and not the components of a vector θ.

When writing the inner product of two vectors, McKelvey transposes one so that it's a row vector times a column vector; redundantly, he also uses the "dot" notation. Thus $x' \cdot y$ represents the inner product of vectors x and y:

$$x' \cdot y = x_1 y_1 + \ldots + x_m y_m.$$

Remember that $x' \cdot y > 0$ means that the two vectors form an acute angle, and $x' \cdot y = 0$ means they are perpendicular. The length or *norm* of a vector is written as

$$||x|| = \sqrt{x \cdot x} = \sqrt{x_1^2 + \ldots + x_m^2},$$

so the distance between two vectors is given by $||x - y||$. Some important rules:

- For any scalar c, $||cx|| = |c|||x||$.
- The *triangle inequality*: $||x + y|| \leq ||x|| + ||y||$.

Hyperplanes

For any nonzero vector $y \in \mathbb{R}^m$ and scalar c, the following set defines a hyperplane in $\mathbb{R}^m$ (if $m = 3$, just a garden-variety plane):

$$H_{y,c} = \{x \mid x' \cdot y = c\}$$

For such a hyperplane, y is called a *normal* of the hyperplane; it is a vector perpendicular to the hyperplane. Hyperplanes with the same y and different c are parallel to each other. The constant c indicates the distance from the origin to the hyperplane in the direction of y; $c < 0$ indicates that the hyperplane lies in the opposite direction of y from the origin.

Strictly speaking, the normal to a hyperplane like that above is usually taken to be the vector y scaled to length 1, that is, $y/||y||$. The same formula applies: if $x \cdot y = c$ for a general vector y, then of course the scalar c can just be re-scaled to fit the definition with a unit-length normal:

$$x \cdot \frac{y}{||y||} = \frac{c}{||y||}.$$

McKelvey doesn't say anything about restricting the y vectors that define his hyperplanes to have length 1. However, the big derivation on p. 477 twice uses the assumption $||y_i|| = 1$, which is without loss of generality. This is done in the third line of expression (3.9), and again in the third line of (3.11), which implicitly uses $y_i' \cdot y_i = ||y_i||^2 = 1^2 = 1$.

As McKelvey notes, any hyperplane partitions all $\mathbb{R}^m$ into three parts: the hyperplane itself and the two *open half-spaces* lying on either side of it. Hence $H^-_{y,c} = \{x \mid x' \cdot y < c\}$ is (assuming $c > 0$) the set of all points lying on the same side of the hyperplane as the origin, and $H^+_{y,c}$ denotes the points lying on the opposite side.

Note that for every direction from the origin, that is for every y with $||y|| = 1$, there will be a (different) median hyperplane, and possibly more than one (if, for example, there is an even number of voters).

Again, notice that $|H^+_{y,c}|$ represents not the cardinality of the open half-space (infinite of course), but the number of individuals having their ideal points in the open half-space.

McKelvey defines a median hyperplane $H_{y,c}$ (p. 474 top) as one for which each of the two open half-spaces contains at most half the voters' ideal points:

$$|H^+_{y,c}| \leq n/2 \text{ and } |H^-_{y,c}| \leq n/2.$$

Equivalently—and more usefully for what follows—the condition could be given in terms of the two *closed* half-spaces containing *at most* half the voters' ideal points:

$$|\bar{H}^+_{y,c}| \geq n/2 \text{ and } |\bar{H}^-_{y,c}| \geq n/2.$$

For in those terms, suppose x is a point in $H^-_{y,c}$. Moving perpendicularly to a point y on the hyperplane $H_{y,c}$ ensures that y is strictly closer than x to the ideal points of at least half the voters, namely, all those in $\bar{H}^+_{y,c}$.

Helly's Theorem

On p. 475 McKelvey uses:

Theorem 1 (Helly's Theorem) *Let $\{S_y\}$ be a possibly infinite collection of compact convex subsets of $\mathbb{R}^m$, such that every subcollection of cardinality at most $m+1$ has nonempty intersection. Then the whole (infinite) collection has nonempty intersection.*

McKelvey actually uses the contrapositive of the theorem: If the collection has empty intersection, then there exists a sub-collection of cardinality $m+1$ having empty intersection.

The collection in question is the set of all the closed half-spaces $\bar{H}_y^+$ defined by the "closest-in" median hyperplanes (those whose constant "c" is $c_y = \inf C_y = \min C_y$). McKelvey asserts that a total median exists if and only if there is some $x*$ in the intersection of the entire collection of $\bar{H}_y^+$. If there's no TM, then, the collection has empty intersection; therefore some sub-collection of $m+1$ has empty intersection. Therefore the system of inequalities defining these closed half-spaces, $x' \cdot y \geq c_i$, has no common solution.

The theorem's condition of compactness of the S_y (that is, each S_y is closed and bounded) is needed only if the collection is infinite. McKelvey's collection is indeed infinite, but the half-spaces are not bounded, and hence not compact. I haven't been able to recover why it is that he can nevertheless apply Helly's Theorem to get his result. Further announcements as events warrant.