Some Mathematical Notes for Reading Schofield, "Instability of Simple Dynamic Games"

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General

Recall that for a utility function $u(\alpha)$ on an alternative space $W \subseteq \mathbb{R}^m$, the gradient $Du(\alpha)$ is a vector pointing in the direction of steepest increase of u. More importantly for this paper, if u is smooth—that is, has first and second derivatives—then, for any direction that makes an acute angle with $Du(\alpha)$, a small enough move from α in that direction will increase $u(\alpha)$. Formally, let v be a vector with $v \cdot Du(\alpha) > 0$. There exists a scalar $\underline{\epsilon}$ such that

$$0 < \epsilon < \underline{\epsilon} \;\Rightarrow\; u(\alpha + \epsilon v) > u(\alpha).$$

That is, a sufficiently small move in direction v will increase utility. This is THE most important thing to know in order to understand this paper.

Three kinds of vectors: W, $T_\alpha W$, and S

The set of alternatives W is "an m-dimensional Euclidean space" (576). Thus each alternative can be thought of as a point (which is what's useful here) or as a vector (informally, the point of an arrow with its "tail" at the origin).

For each point $\alpha \in W$, the space $T_\alpha W$ represents the set of direction vectors $c'(t)$ at α. For this purpose Schofield retains information on the length as well as the direction of those vectors, so for each $\alpha \in W$, $T_\alpha W = \mathbb{R}^m$. Mathematically, then, the vectors in $T_\alpha W$ are identical to whose in W, but you can think of them as having their "tails" at the point α.

Schofield introduces $T_\alpha W$ by talking about a smooth curve, that is, a vector-valued function $c : [0, T] \to W$ that passes through α. In these terms, $c'(t)$ indicates how "fast" the curve is going and in which direction when it passes through $c(t) = \alpha$; that is,

$$c'(t) = \big(c'_1(t), \ldots, c'_m(t)\big)$$

where each $c'_j(t)$ indicates the rate of change of c in the direction of the jth axis of $\mathbb{R}^m$. Later, however, Schofield is also concerned with the gradients *along the curve* of individuals' utility functions u_i defined on W. In expression (1), he denotes by $Du_i(c(t))c'(t)$ the instantaneous slope of i's utility in the direction of the curve c at a point $\alpha = c(t)$. The more conventional notation for this would be to represent it as an inner (dot) product,

$$Du_i(c(t)) \cdot c'(t) = \left(\frac{\partial u_i}{\partial \alpha_1}(c(t)), \ldots, \frac{\partial u_i}{\partial \alpha_m}(c(t))\right) \cdot \big(c'_1(t), \ldots, c'_m(t)\big)$$

which, when multiplied out, gives the right-hand side of equation (1). Each term of that sum represents the change of u_i with respect to t, contributed by the movement of c along a different axis of $\mathbb{R}^m$, using the chain rule.

Finally we also have (p. 577) a set S of "equivalence classes" of nonzero vectors. These equivalence classes are eventually just going to represent the *directions* in which utility gradients might point. The central results in this paper concern sets—the "cones" defined below—of directions of utility gradients at points in W, and what the shapes of those sets imply about cycles of collective preference in W. Of course, at a player's ideal point, that gradient is the zero vector $\mathbf{0}$, so Schofield augments this set of payoff gradient directions as $\bar{S} = S \cup \{\mathbf{0}\}$.

To begin with, think of S as the sphere in $\mathbb{R}^m$ with radius=1, and with its center at some $\alpha \in W$ of interest—in other words, the set of vectors of length=1, again with their "tails" at α. Additional features of S are developed via a lot of work on p. 577. In this initial treatment (only), elements of S are represented by a bracketed vector, such as $[a_1]$. You can ignore all the junk about equivalence classes and just think of $[a_1]$ as the direction of vector $a_1 \in \mathbb{R}^m$, defined by normalizing a_1 to length 1: $[a_1] = a_i/||a_i||$.

Schofield defines the distance $d([a_1],[a_2])$ between two direction-vectors in the sphere S as the length of the chord between those two points. The details of this metric of distance really doesn't have any direct importance in the rest of the paper, but if you're interested in the details:

- The formula in the middle of p. 577 for $d([a_1],[a_2])$ translates as this chord length via the Law of Cosines. The "(a_1, a_2)" in that formula is supposed to represent $a_1 \cdot a_2$, the inner product of the two direction vectors.
- Recall that for any two vectors x, y, we can write $x \cdot y = ||x||\ ||y|| \cos\theta$, where θ is the angle between the two vectors. Suppose that unit-length vectors a_1 and a_2 are two sides of a triangle, and the third side, the chord, has length d. In this setting, the Law of Cosines says the chord length d is given by

  $$d^2 \;=\; ||a_1||^2 + ||a_2||^2 - 2||a_1||\ ||a_2|| \cos\theta \;=\; 2 - 2\cos\theta$$

  the latter since $||a_1|| = ||a_2|| = 1$. Schofield's formula for distance in S follows.
- This distance function on S implies a natural notion of continuous functions on S (which would be based on defining convergence of sequences in terms of that distance), which is what all the business about the topology $\mathscr{S}$ is getting at.

Preference co-cones and the dual of a cone

To begin with, a cone in $\mathbb{R}^m$ is pretty much what the name indicates: its pointy end is at the origin and it fans out to infinity in some direction from there. A "closed" cone includes its boundary. Here we are often working with "finitely generated" cones, which (rather than being nice and round) are polyhedral. To be precise, let $a_1, \ldots, a_k$ be vectors in S that point in sort of the same direction (for all i and j, $a_i \cdot a_j > 0$). Then the set of all their semi-positive linear combinations is a cone,

$$C = \{v \in \mathbb{R}^m : v = \lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_k a_k \text{ where } \lambda_i \geq 0 \text{ and at least one } \lambda_j > 0\}.$$

Schofield's notation for such a finitely generated cone is $C = \sum_{j=1}^{k} [a_j]$ (p. 578).

In general, the *dual* of any cone C in $\mathbb{R}^m$ is the set

$$C^* = \{x \in \mathbb{R}^m : x \cdot y > 0 \text{ for all } y \in C\}.$$

(Elsewhere the dual is sometimes defines as requiring only that the inner product is non-negative and not necessarily positive, but it's important for Schofield's results to use the version above.) That is, all vectors that "point in the same direction as" or "make an acute angle with" *every* vector in the cone. As a special, simple case, if the cone consists of a single vector y, then its dual y^* is the closed half-space that, in the McKelvey paper, we called $H^+_{y,c}$ where $c = 0$.

A co-cone is the dual of a cone. In a couple of cases, Schofield defines a co-cone before he defines the corresponding cone. This is because the co-cone is the more elementary object, but ultimately it is the cone in which we'll ultimately be most interested.

Follow carefully the definitions of cones and co-cones on pp. 578-580: These are the main objects of interest in the paper. They are all located in the set S or sometimes $\bar{S}$.

- Individual i's preference co-cone at α is simply the direction of i's utility gradient there. Schofield writes this as $Du_i(\alpha)$, but sometimes he's really talking about its direction $\frac{Du_i(\alpha)}{||Du_i(\alpha)||} \in S$.
- Individual i's preference cone is the dual of i's preference co-cone:

  $$C_i(\alpha) = Du_i(\alpha)^* = \{a \in S : a \cdot Du_i(\alpha) > 0\}.$$

  This represents the set of directions in S in which some sufficiently small movement from α would increase i's utility, plus the boundary of that set (where $a \cdot Du_i(\alpha) = 0$.
- Let $M \subseteq N$ be a coalition of players. The set of directions in which all $i \in M$ of them could increase utility with a sufficiently small move is $C_M(\alpha) = \cap_{i \in M} C_i(\alpha)$. This turns out to be the dual of $\sum_{i \in M} Du_i(\alpha)$, the cone ($M$'s "preference co-cone") of all semi-positive linear combinations of their utility gradients.
- Suppose the choice of a direction in which to move from any α in W is a *simple game*, which means a game defined by a set ω of winning coalitions: for any $M \in \omega$, if all i in M agree on a direction, then it is chosen regardless of what any other player wants to do. The set of winning coalitions of a simple game must obey certain conditions:
  - $N \in \omega$
  - if $M \subset M'$ and $M \in \omega$ then $M' \in \omega$; and
  - if $M \in \omega$ then $N \setminus M \notin \omega$.

- In such a simple game, any movement from α will be in a direction that is unanimously preferred by the members of some winning coalition. Let C^ω be the set of all such directions, so

$$C^\omega(\alpha) = \bigcup_{M \in \omega} C_M(\alpha) = \bigcup_{M \in \omega} \bigcap_{i \in M} C_i(\alpha).$$

The null dual condition and the instability of simple games

The null dual condition (Definition 2.1, bottom of p. 579), and its consequence, Theorem 2.1 (p. 581), is the central result of the paper. Its essential intuition is that if the dual of $C^\omega(\alpha)$ is empty, then the preference cones of winning coalitions point in many different, even opposite, directions. If the null dual condition holds at α, then since the utility functions are smooth, it will also hold at every point in some neighborhood of α. (A neighborhood is generally defined as a set containing an open set containing α.) And in that case, we can move around from one point in that neighborhood to any other point, and back again, continuously by the unanimous agreement of members of winning coalitions. There are "local preference cycles" in the game.

The second main result, equally important, is embodied in Corollaries 4.4 and 4.5: under broad conditions, the game is "unstable." That is, the null dual condition is satisfied at a subset $\Phi \subseteq W$ that is *dense* in W, that is, it includes nearly every point in W. There are local preference cycles nearly everywhere.

Formally, a Φ is dense in W if every point $\alpha \in W$ is either in Φ or is a boundary point of Φ (the limit of a sequence of points in Φ). For example, the set of rational numbers is dense in the set of real numbers.