Relations on Sets

- relations on sets
- application: preferences; discrete rational choice
- utility representations; invariance to monotone transformation
- partial orderings
- Note: there are 7 Exercises (some with multiple parts) interspersed below, beginning on p. 4.

Relations: Examples

For any set, a "relation" indicates some connection or comparison between elements that, for each pair of elements in the set, may or may not hold true. Such comparisons may be directional—that is, in talking about relations, the *order* of the two elements does matter, so x may stand in a certain relation to y while y does not stand in that relation to x. Some examples:

1. For any pair of real numbers x, y, we can say whether $x \geq y$.
   - "$\geq$" is a relation on $\mathbb{R}$.
2. Let S be a set, and let $\mathcal{P}(S)$ be the set of all subsets of S. For any two subsets A, B of S, we can say whether or not $A \subseteq B$
   - "$\subseteq$" is a relation on $\mathcal{P}(S)$.
3. Consider a sociogram consisting of a set L of individual members of a legislature, with a connection drawn between every pair of legislators who have ever cosponsored a bill together. Let $C \subseteq L \times L$ be the set of all pairs of legislators, $i, j \in L$, who have ever co-sponsored a bill together. Usually a sociogram is represented as a network of "nodes" connected by "paths," but equivalently we could write iCj if and only if i and j have sponsored a bill together.
   - C is a relation on L.
   - In fact, any network is a relation on the set of nodes.
   - see James H. Fowler, "Connecting the Congress: A Study of Cosponsorship Networks." *Political Analysis*, Vol. 14, No. 4 (Autumn 2006), pp. 456-487.
4. For any two people x, y in the set W of all people who have ever lived in the world, we can say whether x is a biological parent of y. We could write xPy to mean "x is a parent of y."
   - Parenthood, P, is a relation on W.

5. For any two potential presidential candidates A and B and for any voter i, we could write AP_iB to represent the preference, or intention to vote, of voter i between those candidates: i prefers A to B.

    - Voter preference P_i is a relation on the set of all candidates.

Properties of Some Familiar Relations

Relations on Real Numbers

Consider the relation $\leq$ on $\mathbb{R}$. This relation has the following interesting properties: it is

- *transitive* since for every $x, y, z \in \mathbb{R}$ such that $x \leq y$ and $y \leq z$, $x \leq z$—that is,

$$\forall x, y, z \in \mathbb{R}, \big[x \leq y \text{ and } y \leq z\big] \Rightarrow x \leq z.$$

- *complete* since for every distinct $x, y \in \mathbb{R}$, either $x \leq y$ or $y \leq x$
- *reflexive* since for every $x \in \mathbb{R}, x \leq x$.
- *antisymmetric* since for every $x, y \in \mathbb{R}, [x \leq y \text{ and } y \leq x] \Rightarrow x = y$, that is, they are the same number.

Consider the relation $<$ on $\mathbb{R}$. $<$ is also transitive and complete, but not reflexive; the question of antisymmetry never comes up, since the relation $<$ also has the following property:

- *asymmetric*: for every $x, y \in \mathbb{R}$, $x < y \Rightarrow$ NOT $y < x$.

Notice that $<$ is a sort of "sub-relation" of $\leq$, in the sense that $x < y \Rightarrow x \leq y$; in general, we call $<$ the *asymmetric part* of $\leq$.

Relations like $\leq$ and $<$ on $\mathbb{R}$ are sometimes called *orders* or *orderings*, because they rank the elements of the underlying set, in this case $\mathbb{R}$, with no inconsistencies.

Consider as well the relation $=$ on $\mathbb{R}$. This relation is transitive and reflexive but not complete. It also has the following property:

- *symmetric*: for every $x, y \in \mathbb{R}$, $x = y \Leftrightarrow y = x$.

Again, $=$ is a "sub-relation" of $\leq$; in general, we call $=$ the *symmetric part* of $\leq$.

Relations on Pairs of Numbers

The idea of the less-than-or-equal-to relation can be extended to pairs of numbers—that is, to a relation on $\mathbb{R} \times \mathbb{R}$—in several ways. Let a, b, c, d be any real numbers. For example, then

- We could write $(a, b) \leq (c, d)$ whenever $a+b \leq c+d$, to focus on which has the smaller arithmetic total.
- We could write $(a, b) \leq (c, d)$ whenever $\sqrt{a^2+b^2} \leq \sqrt{b^2+d^2}$, telling which pair, plotted on Cartesian coordinates, is closer to the origin.
- We could write $(a, b) \leq (c, d)$ whenever $a \geq c$ and $b \geq d$. This is sometimes referred to as the vector less-than-or-equal-to relation.

The arithmetic sum relation and the distance-from-origin relation are, like the original $\leq$ on $\mathbb{R}$, transitive, reflexive, and complete. However, they are not antisymmetric: there can be two different pairs having identical sums, or standing at identical differences from the origin. (Such relations are often called "weak orders" because the rank the elements of the relevant set, possibly with ties.)

The vector less-than-or-equal-to relation is also transitive and reflexive but not complete; moreover, it is antisymmetric. A relation with those properties is called a "partial order"; it doesn't completely order the set, but at least there are no inconsistencies (in the form of intransitivities).

Formal Definition and Properties of a Binary Relation

Formally, let S be any set; as usual, let $S \times S$ represent the set of all pairs of elements from S:

$$S \times S = \{(x, y) : x \in S \text{ and } y \in S\}$$

In principle, any subset R of $S \times S$ defines a *relation on S*. If $(x, y) \in R$ we also write xRy, which is pronounced either as "xRy" or as "x bears R to y". So for example the relation $\leq$ on $\mathbb{R}$ consists of the following subset of all pairs of real numbers:

$$\{(x, y) \in \mathbb{R}^2 : x \leq y\}$$

In these terms, a *sub-relation* of a relation R is just a subset R' of the ordered pairs making up R: $R' \subseteq R$. Equivalently, R' is a sub-relation of R iff $[xR'y \Rightarrow xRy]$. The relation $<$ on $\mathbb{R}$ is a sub-relation of $\leq$.

In general, any subset of pairs of elements of a set can be treated as if it were a relation on that set. Usually, however, relations of interest obey some of the properties above (or other properties of interest). Here are the definitions of those properties in general terms: the relation R on a set S is

- *transitive* $\Leftrightarrow$ for every $x, y, z \in S$, $[xRy$ and $yRz] \Rightarrow xRz$.
- *complete* $\Leftrightarrow$ for every $x, y \in S$ with $x \neq y$, either xRy or yRx.
- *reflexive* $\Leftrightarrow$ for every $x \in S, xRx$.
- *symmetric* $\Leftrightarrow$ for every $x, y \in S$, $xRy \Rightarrow yRx$.

- *asymmetric* $\Leftrightarrow$ for every $x, y \in S$, $xRy \Rightarrow$ NOT yRx.
- *antisymmetric* $\Leftrightarrow$ for every $x, y \in S, [xRy$ AND $yRx] \Rightarrow x = y$.

Exercise 1. For each of the following relations, tell which of the six properties above are satisfied and which are not, and explain:

(a.) The "strictly greater than" relation $>$ on $\mathbb{R}$

(b.) $\subseteq$ on the set $\mathcal{P}(S)$ consisting of all subsets of S.

(c.) The cosponsorship relation C for the 100th Senate.

(d.) Parenthood on W, the set of all people who have ever lived in the world.

Exercise 2. Prove: if R is transitive and is not reflexive then R is asymmetric. (Hint: suppose R were transitive but failed to be asymmetric.)

Maximal Elements and Greatest Elements

Suppose R be a relation on S, and let T be any subset of S. We say that an element t of T is *R-maximal in T* if and only if there is no other $s \in T$, that is no $s \in T \setminus \{t\}$, such that sRt. The set of R-maximal elements in T is, correspondingly, written as

$$\max(T, R) = \{t \in T : \nexists s \in T \setminus \{t\} \text{ such that } sRt \text{ unless also } tRs\}.$$

If R is an antisymmetric relation, sRt and tRs cannot happen between distinct elements s and t, so the definition is sometimes simplified to $\{t \in T : \nexists s \in T \setminus \{t\} \text{ such that } sRt\}$. If R is transitive and complete, then the definition is sometimes simplified to $\{t \in T : \forall s \in T \setminus \{t\}, tRs\}$.

In words, an element in T is R-maximal if there is no other element that bears R to it. This is not quite the same as saying that t is the greatest, or a greatest, element in T. Rather, t is *R-greatest in T* if and only if tRs for every $s \in T \setminus \{t\}$. For a transitive, complete R, then, an element is R-maximal if and only if it is R-greatest. (Note that there may be more than one R-greatest element unless R is also antisymmetric, including of course the case when it is asymmetric.)

When R is unambiguous, we often write simply $\max(T)$.

This all makes sense for a relation such as $\geq$ on $\mathbb{R}$. In the subset $T = [0, 2]$, 2 is the (only) greatest element and the only maximal element. However, for the relation $\leq$ on $\mathbb{R}$, the definition says that 0 is the greatest, and the only maximal, element of $[0, 2]$! When working with well known relations like these, we freely substitute the terms "minimal elements" and "least element" as long as the meaning is clear. Strictly speaking, however, 2 is a least element under $\leq$. But using the more informal terminology, in the case of $\geq$ and $\leq$ on sets of real numbers, we also write $\max S$ for $\max(S, \geq)$ and $\min S$ for $\max(S, \leq)$.

While a greatest element is always maximal, the reverse need not be true. Let $S = \{1, 2, 3\}$ and consider the relation $\supseteq$ on $\mathcal{P}(S)$. Let T be the set of all *proper* subsets of S—that is, $T = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}\}$. Then there is no greatest element of T under $\supseteq$, but every subset containing all but one element of S is maximal:

$$\max(T, \supseteq) = \{\{1, 2\}, \{2, 3\}, \{1, 3\}\}.$$

Again, the language seems unnatural if we consider instead $\subseteq$ on S, in which case there is a "maximal" element, namely $\varnothing$.

It is important to note that, even under such a regular relation as $\geq$, a set might have no maximal element. This is the case for any open interval, such as $(0, 1)$, or for any set that is not bounded above, such as $\{1, 2, 3, \ldots\}$. (In such cases, then, we write $M(S, \geq) = \varnothing$.)

In a set such as $(0, 1)$, an important related concept is the *least upper bound* of a set. Again let R be a relation on S, and $T \subseteq S$. We define the least upper bound in two steps. First, the set of upper bounds of T is

$$U(T, R) = \{s \in S : sRt \text{ for every } t \in T \setminus \{s\}\}.$$

Second, if $U(T, R)$ has a least element b, then b is the least upper bound of T according to R. The idea of least upper bound is used for relations that are antisymmetric; hence a least upper bound, if it exists, is unique.

Importantly, b need not be an element of T. In the case of $\geq$ on real numbers, the least upper bound of T is also referred to as the *supremum* of T. There are corresponding definitions for the *greatest lower bound* (or *infimum*) of T.

Strictly speaking, the definitions of maximal (minimal) elements, greatest (least) elements, and least upper bound (greatest lower bound) are confined to the case of *partial orders*, that is, relations that are transitive, reflexive, and antisymmetric; the term "partial orders" includes relations such as $\geq$ that are also complete (hence not "partial" at all). However, the concepts make perfectly good sense, and are freely invoked, in some other cases, such as the relation $>$ on $\mathbb{R}$, which is not reflexive and hence not a "partial order".

Exercise 3. Describe the maximal elements in each of the following relations:

a. What is $\max([0, 1], \leq)$?

b. Specify a relation Q on $\{a, b, c\}$ that is complete but not transitive. What is $\max(\{a, b, c\}, Q)$?

Preferences and Choice

An important application of the mathematics of sets and relations is the modeling of choice. Consider the case of a highly informed voter in an early 2016 Republican primary. Let the set A consists of all conceivable candidates for the Republican nomination: {Trump, Bush, Walker, Rubio, ..., Gilmore}. The voter has formed clear opinions about all these candidates and, if polled, is able to say which she thinks would be the better and worse of any pair.

When the voter's state's primary election finally rolls around, however, only some subset of those candidates will be on the ballot and still in the race: a subset B of A. The voter's task will be to choose a candidate from B.

In other applications, A might consist of all conceivable policy alternatives, with the actual set of available choices being constrained by some available budget amount. More generally, suppose A is a universal set of alternatives, all the conceivable items among which a person might need to choose. In any given choice situation, some of those are available, a subset B, and the person must choose from that set.

In rational choice theory, we envision the chooser as having coherent preferences over the set A, and making the choice among any subset B according to that preference.

Weak Orders as a Model of Preference Relations

In choice theory, to have preferences over A is to be able to say, for any pair a, b of alternatives from A, which is the preferable one (or that they are equally good). In this case, the individual's preference can be considered as a relation on the set A. Conventionally, we let P stand for strict preference ("is better than"), R for weak preference ("is at least as good as"), and I for indifference. For reasons of convenience we typically focus on the weak preference relation R, and read aRb as "a is preferred to b" or, to be absolutely clear but slightly ungrammatical, "a is preferred or indifferent to b."

IChoice theory nearly always assumes that these preferences are orderly, in the sense that the preference relation R is a *weak order*:

- R is complete: the individual can compare any two distinct alternatives, although she may be indifferent between them.
- R is reflexive: we conventionally write aRa for any $a \in A$.
- R is transitive: in every case in which the individual weakly prefers some a to some b, and b to some c, she also weakly prefers a to c.

We have already encountered an example of a weak order, namely $\geq$ on a set of numbers. Since $\geq$ has the additional property of antisymmetry (if $a \geq b$ and $b \geq a$ then a and b are identical), $\geq$ is a total order—it orders the numbers completely, with no ties. With preferences, however, we want to allow for the possibility that an individual may be indifferent between two *different* alternatives. A weak order ranks the alternatives from best to worst, but it may include some ties.

Strict preference, P is a sub-relation of R; specifically,

$$aPb \Leftrightarrow aRb \text{ and not } bRa$$

This sub-relation of a weak order is sometimes called the "asymmetric part" of R; notice that, while R need not be asymmetric, P definitely is.

Indifference, I, is another sub-relation, the "symmetric part" of R:

$$aIb \Leftrightarrow aRb \text{ and } bRa.$$

Since R is reflexive (aRa), by this definition we would always say aIa. Since R is complete, we are assured that $aRb \Rightarrow [aPboraIb]$.

Since it is a weak order, R ranks the entire set of alternatives from best to worst, possibly with ties. Often it is referred to as a "preference ranking." Conversely, any consistent ranking with possible ties reflects a weak order: the *US News* ranking of political science graduate programs; the AP Top 25 NCAA basketball teams; or Major League Baseball league leaders in home runs.

If an individual happens to not be indifferent between any two distinct alternatives, her preferences are said to be "strict." In this case P is complete on A. Even though P then ranks all the alternatives from best to worst, it doesn't qualify as a "total order," strictly speaking, since by definition those are reflexive whereas P, being asymmetric, is not reflexive. Such a P is, however, often referred to as a "strict ranking" or "strict ordering."

Exercise 4. Let R be a weak order on A and define P and I from R as above. Prove the following facts about preference relations:

a. For every $a, b \in A$, either aPb, bPa, or aIb.

b. I is a transitive relation.

c. P is a transitive relation.

c. For every $a, b, c \in A$, if aPb and bIc then aPc.

Choosing According to Preferences: Rational Choice

Let $S(A)$ be the set of all non-empty subsets of A, and let us describe all the choices that this individual would make by a *choice function*: a function $C : S(A) \to S(A)$ that obeys

- for every $B \in S(A), C(B) \subseteq B$ and $C(B) \neq \varnothing$.

The prohibition, in the model, of an empty choice is conventional; pretty much any substantive situation one might want to portray as choosing $\varnothing$ could equally well be portrayed as choosing all of B, inasmuch as either choice would amount to "not choosing." Notice that we allow for ties and indecision with this choice function: the individual might not narrow down the choice to a single alternative.

Suppose, now, that choice is to be made "according to" some preference ordering R on A. In that case, surely the chooser should not choose an alternative she regards as inferior to some other available alternative. Suppose that she chooses all the alternatives in B that do not have this flaw: every element in B to which nothing else is preferred. Then we would write

$$C(B) = \{a \in B : \text{ there is no } b \in B \text{ such that } bPa\}.$$

Since by assumption the weak preference relation R is complete, this is equivalent (can you prove it?) to

$$C(B) = \{a \in B : \text{ for all } b \in B, aRb\},$$

which is exactly $\max(B, R)$, the R-maximal elements of B.

Definition: The choice function C is *rational* if there exists some binary relation R on A such that, for every $B \subseteq A$,

$$C(B) = \max(B; R) \stackrel{\text{def}}{=} \{a \in B : \text{ for all } b \in B, aRb\}$$

(Notice that R need not be a weak order.) Given choice function C, we say that such an R *rationalizes* C. For a particular B, given a preference relation R, we also say informally that choice of $\max(B; R)$ is rational.

Exercise 5. Let the set of alternatives be a set of potential candidates {Adams, Buchanan, Clinton, Davis, and Eisenhower}—A, B, C, D, and E, for short. A voter would be willing to support the following candidates from various actual ballots:

- $C(\{\text{A, B, C, D, E}\}) = \{\text{A, B}\}$.
- $C(\{\text{B, C, D}\}) = \{\text{B}\}$.
- $C(\{\text{D, E}\}) = \{\text{E}\}$.
- $C(\{\text{C, D}\}) = \{\text{C}\}$.
- $C(\{\text{A, B}\}) = \{\text{A}\}$.

Is C rational? Explain.

Representation of Preference by a Utility Function

A utility function for the individual is a function that assigns a number to each element in A reflecting the individual's preferences. Formally,

- $u : A \to \mathbb{R}$.
- For all $a, b \in A, u(a) \geq u(b) \Leftrightarrow aRb$.

We say that such a utility function u *represents* the preferences R. The following theorem relates preference relations to utility:

Theorem 1. *A binary relation R can be represented by a utility function if and only if R is a weak order (that is, transitive, reflexive, and complete). Likewise, if $u : A \to \mathbb{R}$ is any function, then the relation R defined as follows is a weak order:*

$$\textit{For every } a, b \in A, aRb \Leftrightarrow u(a) \geq u(b).$$

Sketch of proof: If u represents R, then R inherits transitivity, completeness, and reflexivity from the numerical ordering of utility values by $\geq$.

Conversely, if R is a weak order and A is finite, appropriate utility numbers can be assigned to the alternatives as ordered by R.

Proof for the case of infinite A requires us to appeal to the "between-ness" of real numbers. □

This leaves a little room for choice to be rationalized by R but not representable by a utility function.

Example: Suppose $A = \{a, b, c\}$, and define the choice function by

- $C(\{a, b, c\}) = \{a, b\}$
- $C(\{a, b\}) = \{a, b\}$
- $C(\{b, c\}) = \{b, c\}$
- $C(\{a, c\}) = \{a\}$

Then C is rationalized by the binary relation $R = \{(a, b), (b, a), (b, c), (c, b), (a, c)\}$, which is not transitive since cRb and bRa but not cRa. Hence by Theorem 1, R cannot be represented by a utility function.

However, we do have

Theorem 2. *Suppose preferences over the set of alternatives A are given by a weak order R, and that u represents R. Then for every $B \subseteq A$,*

$$\max(B; R) = \operatorname*{argmax}_{b \in B} u(b) \stackrel{\text{def}}{=} \{a \in B : u(a) \geq u(b) \textit{ for every } b \in B\}$$

Proof. By definition, $\max(B; R) = \{a \in B : \text{ for all } b \in B, aRb\}$.

Since u represents R, $aRb \Leftrightarrow u(a) \geq u(b)$.

Therefore we can rewrite $\max(B; R)$ as $\{a \in B | u(a) \geq u(b) \text{ for every } b \in B\}$,

which, by definition, is $\operatorname{argmax}_{b \in B} u(b)$. □

Notice that by our definition of a utility representation, u, like R, only carries information about the ordering of preference, saying nothing about the "strength" of preference. Formally,

- Suppose that u represents R, and consider another function $v : A \to \mathbb{R}$. If v has the property that for every $a, b \in S, v(a) \geq v(b) \Leftrightarrow u(a) \geq u(b)$, then v also represents R.

Thus there is nothing sacred about any particular utility representation of R. In general, we have the following theorem:

Theorem 3. *Let $f : \mathbb{R} \to \mathbb{R}$ be any strictly increasing function. Suppose u represents R, and for each $a \in A$, define $v(a) = f(u(a))$. Then v also represents R.*

Exercise 6. Prove this assertion formally.

Partially Ordered Sets

(Note: this section was originally written to precede much of the material above; now, it restates some of the definitions already covered above. Everything should be consistent, at least!)

If P, a binary relation on a set S, is reflexive, antisymmetric, and transitive, but not necessarily complete, we call P a *partial order* and we call S (or rather, formally, the pair (S, P)) a partially ordered set (or poset). Important examples of partial orderings:

- Let X be any set of sets. Then set inclusion $\subseteq$ is a partial ordering on X.
- $\mathbb{R}^2$ denotes the set of all ordered pairs of real numbers. For any two ordered pairs $x = (x_1, x_2)$ and $y = (y_1, y_2)$, the *vector less-than-or-equal-to* relation is defined by

  $$x \leq y \Leftrightarrow x_1 \leq y_1 \text{ and } x_2 \leq y_2.$$

  Vector less-than-or-equal-to is a partial ordering of $\mathbb{R}^2$.

Let S be a set partially ordered by P. For any x and y in S, there may be a $z \in S$ such that zPx and zPy; such a z is called an *upper bound* of x and y. Suppose z is an upper bound of x and y, and that for every other upper bound of x and y (if any), z' say, we have $z'Pz$. Then z is a *least upper bound* of x, y. Likewise we define a lower bound of x, y as any w such that xPw and yPw; and w is a greatest lower bound (wPw' for every other lower bound w'.

- (Here the designations "upper" and "lower" make sense if we think of P as being analogous to "vector-$\geq$"; if P happens to be, say, "vector-$\leq$" then the labels seem backward. It doesn't matter; the context usually dictates which is called which.)

Notice that, if x and y have a least upper bound, it is unique. This follows from the assumed antisymmetry of P. If z' and z'' were both least upper bounds, then they are both upper bounds, so we would have to have both $z'Pz''$ and $z''Pz'$; by antisymmetry this is possible only if $z' = z''$.

The *lattice* is a particularly important category of partial orderings. A poset (S, P) is a lattice iff every pair of elements in S has a least upper bound and a greatest lower bound also in S. Examples:

- The set of all subsets of S, ordered by set inclusion $\subseteq$, is a lattice. For any two subsets A and B, the least "upper" bound is their mutual subset $A \cap B$, and their greatest "lower" bound is $A \cup B$.

- Let S^{100} be the set of all pairs of non-negative integers m and n such that $m+n \leq 100$. This set is partially ordered by vector-$\geq$. Every pair of pairs (x_1, x_2) and (y_1, y_2) has a mutual lower bound in the set, $(0, 0)$ for example, and a greatest lower bound (m, n) where $m = \min\{x_1, y_1\}$ and $n = \min\{x_2, y_2\}$. However, not every pair of pairs has a least upper bound, because it may not have an upper bound at all: consider $(100, 0)$ and $(0, 100)$. Any upper bound (m, n) would have to have $m \geq 100$ and $n \geq 100$, and hence would lie outside S^{100}.

The whole point of a partial order is that it can be partial—it does not relate every pair of elements. However, a partial order may satisfy completeness, in which case it is called a "total order." A total order is reflexive, antisymmetric, transitive, and complete. An example of a total order would be the relation $\geq$ on $\mathbb{R}$.

Maximal Elements

If P is a partial order on a set S, and T is any subset of S, we distinguish the *maximal* elements of T according to P, or the "P-maximal elements of T," defined as follows:

$$\max(T; P) = \{t \in T : \text{ there is no } s \neq t \in T \text{ such that } sPt\}$$

Note that the set of maximal elements of T may be empty, or may have one element or several elements. For example, consider the set S of all subsets of positive integers, partially ordered by $\supseteq$, the superset relation: $T \supseteq U$ is equivalent to $U \subseteq T$. Let T be the set of all subsets of $\{1, 2, 3\}$, and T' the set of all proper subsets of $\{1, 2, 3\}$. Then

- $\max(T, \supseteq) = \{\{1, 2, 3\}\}$, but
- $\max(T', \supseteq) = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$, and
- $\max(S, \supseteq) = \varnothing$.

(Notice, there's a difference between "$\max(S, \supseteq) = \varnothing$" and "$\max(S, \subseteq) = \{\varnothing\}$"!)

The same definition applies, of course, to a total order (that is, a complete partial order), so

- $\max([0, 1], \geq) = \{1\}$,
- $\max((0, 1), \geq) = \varnothing$, and
- $\max(\mathbb{R}^+, \geq) = \varnothing$.

Note that the terminology may seem backward. If we consider instead $\mathbb{R}$ ordered by the less-than-or-equal-to relation, then $\max([0, 1], \leq) = \{0\}$. So the "maximal" element under the subset relation is the "smallest" element. We can define the set of *minimal* elements in a manner completely analogous to maximal elements; and can use the terms interchangeably as long as we're consistent.

If R is a weak order on a set S, then its asymmetric part P resembles a partial order on S, except that P is not reflexive. It still makes perfect sense to write $\max(T, P)$, using the same definition as above. In that case, however, the P-maximal elements on any $T \subseteq S$ can be equivalently defined as

$$\max(T; R) = \{t \in T : \text{ for all } s \in T,\ tRs\}$$

because R is complete. $\max(T; R)$ could contain more than one element. Suppose that $T = \{x, y, z\}$, and that R consist of the following relations: xRz, yRz, and xRy, and yRx. Then $\max(T, R) = \{x, y\}$.

Exercise 7. Are any of the relations in Exercise 1 partial orders? Weak orders?

Summary of Relations

Relation on a set S

- Formally, a subset of $S \times S$.
- Partial order: transitive, reflexive, antisymmetric
  - lattice: a partial ordering in which every x, y has a g.l.b and l.u.b. in S
  - total order: a partial order that is complete
- Weak order: transitive, reflexive, complete.
  - asymmetric part of R: $xPy \Leftrightarrow xRy$ and not yRx
  - symmetric part of R: $xIy \Leftrightarrow xRy$ and yRx
  - $R = P \cup I$
- Total order: transitive, reflexive, complete, and asymmetric

Thus a total order is a special case of both a weak order and a partial order.

Hence, a total order is a binary relation on S that rank-orders the elements of S, with no ties. A weak order does so, but possibly with ties. And a partial order does so only partially: it may define several different "chains" within S, each of which is internally rank-ordered without ties.