

High-order short-time expansions for ATM option prices of exponential Lévy models

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Abstract

The short-time asymptotic behavior of option prices for a variety of models with jumps has received much attention in recent years. In the present work, a novel second-order approximation for ATM option prices is derived for a large class of exponential Lévy models with or without Brownian component. The results hereafter shed new light on the connection between both the volatility of the continuous component and the jump parameters and the behavior of ATM option prices near expiration. In the presence of a Brownian component, the second-order term, in time- t , is of the form $d_2 t^{(3-Y)/2}$, with d_2 only depending on Y , the degree of jump activity, on σ , the volatility of the continuous component, and on an additional parameter controlling the intensity of the “small” jumps (regardless of their signs). This extends the well known result that the leading first-order term is $\sigma t^{1/2}/\sqrt{2\pi}$. In contrast, under a pure-jump model, the dependence on Y and on the separate intensities of negative and positive small jumps are already reflected in the leading term, which is of the form $d_1 t^{1/Y}$. The second-order term is shown to be of the form $\tilde{d}_2 t$ and, therefore, its order of decay turns out to be independent of Y . The asymptotic behavior of the corresponding Black-Scholes implied volatilities is also addressed. Our method of proof is based on an integral representation of the option price involving the tail probability of the log-return process under the share measure and a suitable change of probability measure under which the pure-jump component of the log-return process becomes a Y -stable process. Our approach is sufficiently general to cover a wide class of Lévy processes which satisfy the latter property and whose Lévy density can be closely approximated by a stable density near the origin. Our numerical results show that the first-order term typically exhibits rather poor performance and that the second-order term can significantly improve the approximation’s accuracy, particularly in the absence of a Brownian component.

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1 Introduction

It is generally recognized that the standard Black-Scholes option pricing model is inconsistent with options data, while still being used in practice because of its simplicity and the existence of tractable solutions. Exponential Lévy models generalize the classical Black-Scholes setup by allowing jumps in stock prices while preserving the independence and the stationarity of returns. There are several reasons for introducing jumps in financial modeling.

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First of all, sudden sharp shifts in the price level of financial assets often occur in practice, and these “jumps” are hard to handled within continuous-paths models. Second, the empirical returns of financial assets typically exhibit distributions with heavy tails and high kurtosis, which again are hard to replicate within purely-continuous frameworks. Finally, market prices of vanilla options display skewed implied volatilities (relative to changes in the strikes), in contrast to the classical Black-Scholes model which predicts a flat implied volatility smile. Moreover, the fact that the implied volatility smile and skewness phenomena become much more pronounced for short maturities is a clear indication of the need for jumps in the underlying models used to price options. The reader is referred to Cont and Tankov (2004) for further motivations on the use of jump processes in financial modeling.

One of the first applications of jump processes in financial modeling originates with Mandelbrot (1963), who suggested a pure-jump stable Lévy process to model power-like tails and self-similar behavior in cotton price returns. Merton (1976) and Press (1967) subsequently considered option pricing and hedging problems under an exponential compound Poisson process with Gaussian jumps and an additive independent non-zero Brownian component. A similar exponential compound Poisson jump-diffusion model was more recently studied in Kou (2002), where the jump sizes are distributed according to an asymmetric Laplace law. For infinite activity exponential Lévy models, Barndorff-Nielsen (1997) introduced the normal inverse Gaussian (NIG) model, while the extension to the generalized hyperbolic class was studied by Eberlein et al. (1998). Madan and Seneta (1990) introduced the symmetric variance gamma (VG) model while its asymmetric extension was later analyzed in Madan and Milne (1991) and Madan et al. (1998). Both models are built on Brownian subordination; the main difference being that the log-return process in the NIG model is an infinite variation process with Cauchy-like behavior of small jumps, while in the VG model the log-price is of finite variation with infinite but relatively low activity of small jumps. The class of “truncated stable” processes was first introduced by Koponen (1995). It was advocated for financial modeling by Cont et al. (1997) and Matacz (2000), and was further developed in the context of asset price modeling by Carr et al. (2002), who introduced the terminology CGMY. Nowadays, the CGMY model is considered to be a prototype of the general class of models with jumps and enjoys widespread applicability.

The CGMY model is a particular case of the more general KoBoL class of Boyarchenko and Levendorskii (2002), which in turn is a subclass of the semi-parametric class of tempered stable processes (TSPs) introduced by Rosiński (2007)¹. TSPs form a rich class with appealing features for financial modeling. One of their most interesting features lies in their short-time and long-time behaviors. Indeed, denoting by $Y \in (0, 2)$ the index of a TSP, $L := (L_t)_{t \geq 0}$, it follows that

$$\left(h^{-1/Y} L_{ht} \right)_{t \geq 0} \xrightarrow{\mathfrak{D}} (Z_t)_{t \geq 0}, \quad h \rightarrow 0, \quad (1.1)$$

where $Z := (Z_t)_{t \geq 0}$ is a strictly Y -stable Lévy process and $\xrightarrow{\mathfrak{D}}$ denotes convergence in distribution. In fact, (1.1) holds when $Y \in (0, 1)$ provided L is driftless, while it holds when $Y \in (1, 2)$ regardless of the mean of L . Roughly, (1.1) suggests that the short-time behavior of TSP is akin to that of a stable process, with its heavy-tailed distribution and self-similarity property which are desirable for financial modeling as emphasized, for example, by Mandelbrot (1963). In contrast,

$$\left(h^{-1/2} L_{ht} \right)_{t \geq 0} \xrightarrow{\mathfrak{D}} (W_t)_{t \geq 0}, \quad h \rightarrow \infty, \quad (1.2)$$

where $(W_t)_{t \geq 0}$ is a Brownian Motion, which suggests that in the long-horizon, the process is Brownian-like. In terms of increments, if we were to consider the consecutive “high-frequency” increments of the process L , say $\{L_{\underline{h}i} - L_{\underline{h}(i-1)}\}_{i \geq 1}$ with $\underline{h} \ll 1$, these will exhibit statistical features consistent with those of a self-similar stable time series. But, low-frequency increments, say $\{L_{\bar{h}i} - L_{\bar{h}(i-1)}\}_{i \geq 1}$ with $\bar{h} \gg \underline{h}$, will have Gaussian like distributions.

Stemming in part from their importance for model calibration and testing, small-time asymptotics of option prices have received a lot of attention in recent years (see, e.g., Berestycki et al. (2002), Berestycki (2004), Feng et al. (2010), Feng et al. (2012), Figueroa-López and Forde (2012), Forde and Jacquier (2009), Forde and Jacquier (2011), Forde et al. (2012), Gatheral et al. (2009), Henry-Labordère (2009), Paulot (2009), Roper (2009), Tankov (2010)). Reviewed here are only the studies most closely related to ours, focusing in particular on the at-the-money (ATM) case. Carr and Wu (2003) first analyzed, partially via heuristic arguments, the first-order asymptotic behavior of an Itô semimartingale with jumps. Concretely, Carr and Wu (2003) argued that ATM option prices of pure-jump

¹The terminology Tempered Stable is widely used in the literature, but unfortunately there is no uniform usage. It is worth noting that Rosiński’s class (Rosiński (2007)) is much more general than the “tempered stable” class introduced in several classical references of financial modeling with jumps (e.g. Applebaum (2004), Cont and Tankov (2004), Kyprianou et al. (2005)).

models of bounded variation decrease at the rate $O(t)$, while they are just $O(t^{1/2})$ in the presence of a Brownian component. By analyzing the particular case of a stable pure-jump component, Carr and Wu (2003) also argued that, for other cases, the rate could be $O(t^\beta)$ for some $\beta \in (0, 1)$. Muhle-Karbe and Nutz (2011) formally showed that, in the presence of a continuous-time component, the leading term of ATM option prices is of order \sqrt{t} , for a relatively broad class of Itô models, while for a type of Itô processes with α -stable-like small jumps and $\alpha > 1$, the leading term is $O(t^{1/\alpha})$ (see also (Figueroa-López and Forde, 2012, Proposition 4.2), (Figueroa-López et al., 2012, Theorem 3.7), and (Tankov, 2010, Proposition 5) for related results in exponential Lévy models). However, none of these papers gives higher order asymptotics for the ATM option prices.

The present paper studies the small-time behavior for at-the-money call (or equivalently, put) option prices

$$\mathbb{E} \left[(S_t - S_0)^+ \right] = S_0 \mathbb{E} \left[(e^{X_t} - 1)^+ \right], \quad (1.3)$$

under the exponential Lévy model

$$S_t := S_0 e^{X_t}, \quad t \geq 0, \quad (1.4)$$

where $X := (X_t)_{t \geq 0}$ is the superposition of a tempered-stable-like process $L := (L_t)_{t \geq 0}$ and of an independent Brownian motion $(\sigma W_t)_{t \geq 0}$, i.e.,

$$X_t := \sigma W_t + L_t, \quad t \geq 0, \quad (1.5)$$

where $W := (W_t)_{t \geq 0}$ is a standard Brownian motion independent of L . The term “tempered stable” is understood here in a much more general sense than in several classical sources of financial mathematics (e.g., Applebaum (2004), Cont and Tankov (2004), Kyprianou et al. (2005)) and even more general than in Rosiński (2007). Roughly (see Section 2 below for the explicit conditions), consider a Lévy process whose Lévy measure ν admits a density $s : \mathbb{R} \setminus \{0\} \rightarrow (0, \infty)$ such that, for some $Y \in (1, 2)$, the function $q(x) := s(x)|x|^{Y+1}$ satisfies the following fundamental property:

$$\lim_{x \searrow 0} \frac{1}{x} (C_+ - q(x)) = \beta_+, \quad \lim_{x \nearrow 0} \frac{1}{x} (C_- - q(x)) = -\beta_-, \quad (1.6)$$

for some positive constants C_+ , C_- , β_+ , $\beta_- > 0$. Intuitively, (1.6) indicates that the small jumps of the process behave like those of a Y -stable process. As it turns out (see (Rosenbaum and Tankov, 2011, Proposition 1)), this class still exhibits the appealing short-time and long-time behaviors of Rosiński’s original tempered stable processes as described in (1.1)-(1.2) above. It is worth noting that the previously defined parameter Y coincides with the Blumenthal-Gettoor index of the process L ,

$$BG := \inf \left\{ r \geq 0 : \int_{|x| \leq 1} |x|^r \nu(dx) < \infty \right\},$$

which measures the “degree of jump activity” of the process in that (see, e.g., Sato (1999)):

$$\sum_{u \leq t} |\Delta L_u|^r < \infty \text{ a.s. if and only if } r > Y.$$

As it turns out (see Jacod (2007)), for any $r > Y > 1$,

$$\lim_{t \rightarrow 0} \frac{1}{t} \mathbb{E} (\psi(L_t)) = \int_{\mathbb{R} \setminus \{0\}} \psi(x) \nu(dx),$$

for any bounded continuous function ψ such that $\psi(x) = O(|x|^r)$ as $x \rightarrow 0$.

Under the standing assumption (1.6) (and other conditions), we show that the first-order asymptotic behavior of (1.3) in short-time takes the form

$$\lim_{t \rightarrow 0} t^{-\frac{1}{Y}} \mathbb{E} \left[(S_t - S_0)^+ \right] = S_0 \mathbb{E} (Z^+), \quad (1.7)$$

where Z is a certain centered α -stable random variable under \mathbb{P} . When $\sigma \neq 0$, $Z \sim \mathcal{N}(0, \sigma^2)$ (i.e., $\alpha = 2$) and, as already shown in Tankov (2010) and Roper (2009), (1.7) holds true with $\mathbb{E}(Z^+) = \sigma/\sqrt{2\pi}$. If $\sigma = 0$ and L is symmetric, then $\alpha = Y$ and the characteristic function of Z is explicitly given by

$$\mathbb{E}(e^{iuZ}) = \exp \left\{ -2C\Gamma(-Y) \left| \cos \left(\frac{1}{2}Y\pi \right) \right| |u|^Y \right\},$$

where $C := C_+ = C_-$ can be interpreted as a measure of the ‘‘intensity of small jumps’’ of the process. In that case, (see (Sato, 1999, (25.6))),

$$d_1 := \mathbb{E}(Z^+) = \frac{1}{\pi} \Gamma \left(1 - \frac{1}{Y} \right) \left(2C\Gamma(-Y) \left| \cos \left(\frac{\pi Y}{2} \right) \right| \right)^{\frac{1}{Y}}. \quad (1.8)$$

We refer the reader to Remark 3.5 below for the explicit expression of the leading term in the general case. Interestingly enough, in the presence of a continuous component (i.e., $\sigma \neq 0$ in (1.5)), the first-order asymptotic term only reflects information about the volatility σ , in sharp contrast with the pure-jump case where the leading term depends on the parameter C and the index Y , which in turn respectively control the intensity of small jumps and the degree of jump activity of the process, as already stated above. The intuition behind (1.7) is actually easy to explain. Indeed, since

$$(S_t - S_0)^+ = S_0 (e^{X_t} - 1)^+ = S_0 (e^{X_t^+} - 1) \sim S_0 (X_t^+), \quad (t \rightarrow 0)$$

it is expected that, for $\sigma = 0$,

$$\lim_{t \rightarrow 0} t^{-\frac{1}{Y}} \mathbb{E} \left[(S_t - S_0)^+ \right] = S_0 \lim_{t \rightarrow 0} \mathbb{E} \left(t^{-\frac{1}{Y}} X_t^+ \right) = S_0 \lim_{t \rightarrow 0} \mathbb{E} \left(t^{-\frac{1}{Y}} L_t^+ \right) = S_0 \mathbb{E}(Z_1^+), \quad (1.9)$$

in light of (1.1). In the presence of a Brownian component (i.e., $\sigma \neq 0$), it is known that $t^{-1/2}X_t \xrightarrow{\mathcal{D}} \sigma W_1$ as $t \rightarrow 0$ (see (Sato, 1999, pp. 40)).

The asymptotic result (1.7) is in agreement with the result of (Tankov, 2010, Theorem 5), which showed that for a pure-jump Lévy process $(X_t)_{t \geq 0}$ (i.e., $\sigma = 0$ in (1.5)), whose characteristic function is of the form

$$\mathbb{E}(e^{iuX_t}) = \exp \{ t (iu\gamma - |u|^Y f(u)) \}, \quad (1.10)$$

with $Y \in (1, 2)$ and a function f satisfying $\lim_{u \nearrow \infty} f(u) = \hat{c}_+ \in (0, \infty)$ and $\lim_{u \searrow -\infty} f(u) = \hat{c}_- \in (0, \infty)$,

$$\lim_{t \rightarrow 0} t^{-\frac{1}{Y}} \mathbb{E} \left[(S_t - S_0)^+ \right] = \frac{S_0}{2\pi} \left(\hat{c}_+^{1/Y} + \hat{c}_-^{1/Y} \right). \quad (1.11)$$

An expression similar to (1.7) was also obtained in (Muhle-Karbe and Nutz, 2011, Theorem 4.4) for a more general class of pure-jump martingales. Nevertheless, as previously mentioned, none of these papers obtained second or higher order terms for ATM option prices.

The main result of the present paper establishes, for the exponential Lévy model (1.4)-(1.5), a second-order correction term to (1.7). To the best of our knowledge, this is the first second-order result in the literature of exponential Lévy models. To wit, the second-order asymptotic behavior of the ATM call option price (1.3) in short-time is of the form

$$\frac{1}{S_0} \mathbb{E} \left[(S_t - S_0)^+ \right] = d_1 t^{\frac{1}{Y}} + d_2 t + o(t), \quad t \rightarrow 0, \quad (1.12)$$

in the pure-jump case (i.e., $\sigma = 0$ in (1.5)), while in the presence of an independent Brownian component (i.e., $\sigma \neq 0$ in (1.5)),

$$\frac{1}{S_0} \mathbb{E} \left[(S_t - S_0)^+ \right] = d_1 t^{\frac{1}{2}} + d_2 t^{\frac{3-Y}{2}} + o \left(t^{\frac{3-Y}{2}} \right), \quad t \rightarrow 0, \quad (1.13)$$

for different constants d_1 and d_2 determined explicitly. In the presence of an independent Brownian component, the second-order term d_2 only depends on the degree of jump activity Y and on the parameter $C := (C_+ + C_-)/2$,

which measures the net intensity of the small jumps of X (regardless of their sign). In particular, it is impossible to discern the difference between C_+ and C_- at this second-order approximation. In contrast, in the pure-jump case (i.e., $\sigma = 0$ in (1.5)), the degree of jump activity of the Lévy process is already present in the first-order term. As a byproduct of our asymptotic results for ATM option prices, we also give the asymptotic behavior of the corresponding ATM Black-Scholes implied volatility, denoted by $\hat{\sigma}(t)$. Concretely, in the presence of an independent Brownian component (i.e., $\sigma \neq 0$ in (1.5)),

$$\hat{\sigma}(t) = \sigma + \frac{(C_+ + C_-)2^{-\frac{Y}{2}}}{Y(Y-1)} \Gamma\left(1 - \frac{Y}{2}\right) \sigma^{1-Y} t^{1-\frac{Y}{2}} + o\left(t^{1-\frac{Y}{2}}\right), \quad t \rightarrow 0, \quad (1.14)$$

while, in the absence of the continuous component,

$$\hat{\sigma}(t) = \sigma_1 t^{\frac{1}{Y}-\frac{1}{2}} + \sigma_2 t^{\frac{1}{2}} + o(t^{\frac{1}{2}}), \quad t \rightarrow 0, \quad (1.15)$$

for some constants σ_1, σ_2 that we will determine explicitly.

The results (1.12)-(1.15) are novel and, moreover, not as easily guessed as the first-order term. Indeed, for instance, the intuition behind (1.9) does not seem to be directly transferable to (1.12)-(1.13) since, from the second-order Taylor expansion of the exponential,

$$\frac{1}{S_0} (S_t - S_0)^+ = e^{X_t^+} - 1 \approx X_t^+ + \frac{1}{2} X_t^2 \stackrel{\mathfrak{D}}{\approx} t^{\frac{1}{Y}} Z_1^+ + \frac{1}{2} t^{\frac{2}{Y}} Z_1^2,$$

where for the third approximation we again used that $t^{-1/Y} X_t \stackrel{\mathfrak{D}}{\rightarrow} Z_1$ as $t \rightarrow 0$. However, $\mathbb{E} Z_1^2 = \infty$ and the argument fails after taking expectations. Our derivation is fully probabilistic and builds on two facts. First, make use of the following model-free representation due to Carr and Madan (2009):

$$\frac{1}{S_0} \mathbb{E} \left[(S_t - S_0)^+ \right] = \mathbb{P}^* (X_t > E) = \int_0^\infty e^{-x} \mathbb{P}^* (X_t > x) dx, \quad (1.16)$$

where \mathbb{P}^* is the martingale probability measure obtained when one takes the stock as the numéraire (i.e., $\mathbb{P}^*(A) := \mathbb{E}(S_t \mathbf{1}_A)$ for every $A \in \mathcal{F}_t$, $t \geq 0$) and E is an independent mean-one exponential random variable under \mathbb{P}^* . The measure \mathbb{P}^* is also called the share measure (see Carr and Madan (2009)). Second, further change the probability measure \mathbb{P}^* to another probability measure, say $\tilde{\mathbb{P}}$, under which the pure jump-component of X becomes a stable Lévy process, independent of the continuous component of X . This change, in turn, enables to exploit some key features of stable processes such as self-similarity and the asymptotic behavior of their marginal densities.

Let us finish this introductory section with a brief digression on the relevance as well as further extensions and applications of our results:

- Lévy models are often criticized for their partial ability to fit volatility surfaces across maturities and to account for some stylized features of stock prices such as volatility clustering. A natural question is then whether the results hereafter can be extended to more complex models. The short answer is “yes”. Even though a complete exposition of this point is beyond the scope of the present manuscript, the reason can be broadly justified here. This has its origins in the work of Muhle-Karbe and Nutz (2011), where first-order asymptotics of ATM option prices for a relatively general martingale process with jumps and stochastic volatility are obtained by first showing the analog for a suitable class of Lévy models and then proving that the option prices under the general model can be closely approximated by those of a suitably chosen Lévy model in the class. It is therefore expected that our second-order expansions will be valid for a much more general class of Itô models satisfying certain regularity conditions. Thus, for instance, it is expected that (1.14) will hold with σ replaced with the spot volatility at time 0, say σ_0 , which itself can be random depending on an additional risky factor such as in the Heston model. We refer the reader to Figueroa-López and Ólafsson (2014) for more details about the latter type of results.
- From a qualitative point of view, the short-time expansions stated here allow to connect information of the model’s parameters to key features of the option prices and the implied volatility smile and, at a more basic level, to identify (and rank) the parameters (or numerical features of functional parameters) that most

influence the behavior of the option prices and implied volatilities in short-time. Hence, for instance, (1.14) asserts that, under a continuous component and an infinite stable-like jump activity component, the most important feature that determines the short-time behavior of the implied volatility, below the spot volatility σ , is the intensity of small-jumps as measured by $C := (C_+ + C_-)/2$. Moreover, this influence is “felt at the rate” $t^{1-Y/2}$, which is much stronger than how the jump features start to be “felt” in the case of a finite jump activity, where the rate is of order $t^{1/2}$ (see Remark 4.4 below).

- From a quantitative point of view, as already mentioned, short-time asymptotics of option prices are relevant in model testing and calibration. The first type of application is already present in the seminal work of Carr and Wu (2003). For the second type of application, high-order expansions, as the ones obtained here, can facilitate numerical calibration by suggesting, for instance, proper functional forms for extrapolation purposes or for setting starting points of numerical calibration methods. We refer to Remark 4.5 below for more information about this latter type of application.

The present article is organized as follows. Section 2 introduces the class of Lévy processes studied thereafter and some probability measures transformations needed throughout the paper. Section 3 establishes the second-order asymptotics of the call option price under the pure-jump model (i.e., $\sigma = 0$ in (1.5)). Section 4 establishes the second-order asymptotics of the call option price under an additional independent Brownian component (i.e., $\sigma \neq 0$ in (1.5)). Section 5 illustrates our second-order asymptotics for the important particular class of CGMY models, recovering our preliminary results first presented in Figueroa-López et al. (2011). Section 6 assesses the performance of the asymptotic expansions through a detailed numerical analysis for the CGMY model. The proofs of our main results are deferred to the appendices.

2 A Tempered-Stable-Like Model With Brownian Component

Let $L := (L_t)_{t \geq 0}$ be a pure-jump Lévy process with triplet $(0, b, \nu)$ and let $W := (W_t)_{t \geq 0}$ be a Wiener process, independent of L , defined on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. Assume zero interest rate and that \mathbb{P} is a martingale measure for the exponential Lévy model

$$S_t = S_0 e^{X_t}, \quad \text{with } X_t := \sigma W_t + L_t, \quad t \geq 0,$$

where $(S_t)_{t \geq 0}$ represents the price process of a non-dividend paying risky asset. Equivalently, the Lévy triplet (σ^2, b, ν) of $X := (X_t)_{t \geq 0}$ is such that

$$(i) \int_1^\infty e^x \nu(dx) < \infty, \quad \text{and} \quad (ii) \mathbb{E}(e^{X_1}) = \exp\left(b + \frac{\sigma^2}{2} + \int_{\mathbb{R}_0} (e^x - 1 - x \mathbf{1}_{\{|x| \leq 1\}}) \nu(dx)\right) = 1, \quad (2.1)$$

where hereafter $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$ and the Lévy triplets are given relative to the truncation function $\mathbf{1}_{\{|x| \leq 1\}}$ (see (Sato, 1999, Section 8)). Without loss of generality, also assume that $(X_t)_{t \geq 0}$ is the canonical process $\bar{X}_t(\omega) := \omega(t)$ defined on the canonical space $\Omega = \mathbb{D}([0, \infty), \mathbb{R})$ (the space of càdlàg functions $\omega : [0, \infty) \rightarrow \mathbb{R}$) equipped with the σ -field $\mathcal{F} := \sigma(X_s : s \geq 0)$ and the right-continuous filtration $\mathcal{F}_t := \cap_{s > t} \sigma(X_u : u \leq s)$.

As explained in the introduction, we consider a tempered-stable-like Lévy processes L where the Lévy measure ν admits a density $s : \mathbb{R}_0 \rightarrow [0, \infty)$ of the form

$$s(x) = |x|^{-Y-1} q(x),$$

for $Y \in (1, 2)$ and a bounded measurable function $q : \mathbb{R}_0 \rightarrow [0, \infty)$ such that

$$\lim_{x \searrow 0} \frac{1}{x} (C_+ - q(x)) = \beta_+, \quad \lim_{x \nearrow 0} \frac{1}{x} (C_- - q(x)) = -\beta_-, \quad (2.2)$$

for some positive constants C_+ , C_- , β_+ and β_- . Throughout, the following are the standing assumptions:

Assumption 2.1. The function $q : \mathbb{R}_0 \rightarrow [0, \infty)$ is such that

$$(i) \quad q(x) \leq C_-, \quad \text{for all } x < 0; \quad (ii) \quad q(x) \leq C_+ e^{-x}, \quad \text{for all } x > 0; \quad (2.3)$$

$$(iii) \quad \limsup_{|x| \rightarrow \infty} \frac{|\ln q(x)|}{|x|} < \infty; \quad (iv) \quad \inf_{|x| < \delta} q(x) > 0, \quad \text{for any } \delta > 0. \quad (2.4)$$

A prototypical tempered-stable-like process, as understood here, is the CGMY process of widespread use in mathematical finance. For the CGMY process,

$$\nu(dx) = \left(\frac{C e^{-Mx}}{x^{1+Y}} \mathbf{1}_{\{x>0\}} + \frac{C e^{Gx}}{|x|^{1+Y}} \mathbf{1}_{\{x<0\}} \right) dx, \quad (2.5)$$

so that $C_+ = C_- = C$, $\beta_+ = C_+ M$, and $\beta_- = C_- G$. From this analogy, let us choose to re-express the model in terms of the parameters $M := \beta_+/C_+$ and $G := \beta_-/C_-$, and the function

$$\bar{q}(x) := \frac{q(x)}{C_+} \mathbf{1}_{\{x>0\}} + \frac{q(x)}{C_-} \mathbf{1}_{\{x<0\}}. \quad (2.6)$$

In that case, (2.2) can equivalently be written as

$$\lim_{x \searrow 0} \frac{1}{x} (1 - \bar{q}(x)) = M, \quad \lim_{x \nearrow 0} \frac{1}{x} (1 - \bar{q}(x)) = -G. \quad (2.7)$$

The following relations are direct consequences of (2.7) and are only stated here for future reference:

$$(i) \quad \lim_{\delta \searrow 0} \bar{q}(\delta y)^{\frac{1}{\delta}} = e^{-yM} \mathbf{1}_{\{y>0\}} + e^{-|y|G} \mathbf{1}_{\{y<0\}}, \quad (ii) \quad \lim_{\delta \searrow 0} \frac{1}{\delta} \ln \bar{q}(\delta y) = -yM \mathbf{1}_{\{y>0\}} - |y|G \mathbf{1}_{\{y<0\}}, \quad (2.8)$$

for any $y \in \mathbb{R} \setminus \{0\}$. Let us also note that (2.3-ii) implies the martingale condition (2.1-i) and also that $M > 1$.

Remark 2.2. The class of processes considered here is similar to the unifying class of Regular Lévy Processes of Exponential type (RLPE) as introduced in Boyarchenko and Levendorskii (2002). It also covers the class of proper tempered stable processes as defined in Rosiński (2007) as well as several parametric models typically used in mathematical finance, including the CGMY processes and the more general class of normal tempered stable processes (cf. (Cont and Tankov, 2004, Section 4.4.3)). Let us also remark that the range of Y considered here (namely, $Y \in (1, 2)$) may arguably be the most relevant for financial applications in that several recent econometric studies of high-frequency financial data suggest that the Blumenthal-Gettoor index is larger than 1 (cf. Ait-Sahalia and Jacod (2009), Belomestny (2010), and references therein). Nevertheless, admittedly the Blumenthal-Gettoor index is in general relatively hard to estimate and different, less recent, studies have indicated values of $Y < 1$ for some financial data set (see, e.g., Carr et al. (2002)).

Following a density transformation construction as given in (Sato, 1999, Definition 33.4 and Example 33.14) and using the martingale condition $\mathbb{E}(e^{X_t}) = 1$, define a probability measure \mathbb{P}^* on (Ω, \mathcal{F}) via

$$\left. \frac{d\mathbb{P}^*}{d\mathbb{P}} \right|_{\mathcal{F}_t} = e^{X_t}, \quad t \geq 0, \quad (2.9)$$

i.e., $\mathbb{P}^*(B) = \mathbb{E}(e^{X_t} \mathbf{1}_B)$, for any $B \in \mathcal{F}_t$ and $t \geq 0$. The measure \mathbb{P}^* , sometimes called the share measure, can be interpreted as the martingale measure when using the stock price as the numéraire. Under this probability measure, $(X_t)_{t \geq 0}$ has the representation $(\sigma W_t^* + L_t^*)_{t \geq 0}$ where, under \mathbb{P}^* , $W^* := (W_t^*)_{t \geq 0}$ is a Wiener process and $L^* := (L_t^*)_{t \geq 0}$ is a Lévy process with triplet $(0, b^*, \nu^*)$ given by

$$\nu^*(dx) := e^x \nu(dx) = e^x s(x) dx, \quad b^* := b + \int_{|x| \leq 1} x (e^x - 1) s(x) dx + \sigma^2,$$

which is moreover independent of W^* . Hereafter, set

$$q^*(x) := q(x) e^x, \quad s^*(x) := |x|^{-Y-1} q^*(x), \quad \bar{q}^*(x) := \bar{q}(x) e^x.$$

The next relations are direct consequences of (2.8):

$$(i) \lim_{\delta \searrow 0} \bar{q}^*(\delta y)^{\frac{1}{\delta}} = e^{-yM^*} \mathbf{1}_{\{y>0\}} + e^{-|y|G^*} \mathbf{1}_{\{y<0\}}, \quad (ii) \lim_{\delta \searrow 0} \frac{1}{\delta} \ln \bar{q}^*(\delta y) = -yM^* \mathbf{1}_{\{y>0\}} - |y|G^* \mathbf{1}_{\{y<0\}}, \quad (2.10)$$

where $M^* := M - 1$ and $G^* := G + 1$.

An important tool thereafter is to change probability measures from \mathbb{P}^* to another probability measure, denoted by $\tilde{\mathbb{P}}$, under which $(L_t^*)_{t \geq 0}$ is a stable Lévy process and $(W_t^*)_{t \geq 0}$ is a Wiener process independent of L^* . Concretely, let

$$\tilde{q}(x) := C_+ \mathbf{1}_{\{x>0\}} + C_- \mathbf{1}_{\{x<0\}}, \quad \tilde{\nu}(dx) := |x|^{-Y-1} \tilde{q}(x) dx, \quad \tilde{b} := b^* + \int_{|x| \leq 1} x(\tilde{\nu} - \nu^*)(dx). \quad (2.11)$$

Note that $\tilde{\nu}$ is the Lévy measure of a Y -stable Lévy process. Moreover, $\tilde{\nu}$ is equivalent to ν^* since clearly $\nu^*(dx) = e^x q(x) \tilde{q}(x)^{-1} \tilde{\nu}(dx)$ and q is strictly positive in view of (2.4-iv). For future reference, it is convenient to write $\tilde{\nu}$ as

$$\tilde{\nu}(dx) = e^{\varphi(x)} \nu^*(dx), \quad \text{with} \quad \varphi(x) := -\ln \bar{q}^*(x) = -\ln \bar{q}(x) - x.$$

By virtue of (Sato, 1999, Theorem 33.1), there exists a probability measure $\tilde{\mathbb{P}}$ locally equivalent² to \mathbb{P}^* such that $(X_t)_{t \geq 0}$ is a Lévy process with Lévy triplet $(\sigma^2, \tilde{b}, \tilde{\nu})$ under $\tilde{\mathbb{P}}$, provided that the following condition is satisfied:

$$\int_{\mathbb{R}_0} \left(e^{\varphi(x)/2} - 1 \right)^2 \nu^*(dx) = \int_{\mathbb{R}_0} \left(1 - e^{-\varphi(x)/2} \right)^2 \tilde{\nu}(dx) < \infty.$$

To see that the previous condition holds under our assumptions, note that the integral therein can be expressed as

$$C_+ \int_0^\infty \left(1 - e^{\frac{1}{2} \ln \bar{q}^*(x)} \right)^2 |x|^{-Y-1} dx + C_- \int_{-\infty}^0 \left(1 - e^{\frac{1}{2} \ln \bar{q}^*(x)} \right)^2 |x|^{-Y-1} dx, \quad (2.12)$$

and that, by (2.10-ii), the integrand in the first integral is such that

$$\left(e^{\frac{1}{2} \ln \bar{q}^*(x)} - 1 \right)^2 \sim \frac{1}{4} (\ln \bar{q}^*(x))^2 \sim \frac{(M^*)^2}{4} x^2, \quad \text{as } x \searrow 0.$$

This shows that the first integral is finite on any interval $(0, \varepsilon)$. Outside any neighborhood of the origin, this integral is finite in view of (2.1-i). The second integral in (2.12) can be similarly handled using (2.10) and the fact that

$$\varphi(x) = -\ln \bar{q}^*(x) \geq 0, \quad \text{for any } x < 0, \quad (2.13)$$

as seen from the assumption (2.3-i).

Now that the existence of the probability measure $\tilde{\mathbb{P}}$ is established, let us state some of its properties and introduce some related terminology. Throughout, $\tilde{\mathbb{E}}$ denotes the expectation under $\tilde{\mathbb{P}}$. Letting $\tilde{\gamma} := \tilde{\mathbb{E}}(X_1) = \tilde{\mathbb{E}}(L_1^*)$, recall that the centered process $(Z_t)_{t \geq 0}$, defined by

$$Z_t := L_t^* - t\tilde{\gamma}, \quad t \geq 0, \quad (2.14)$$

is a strictly Y -stable under $\tilde{\mathbb{P}}$ and, thus, also self-similar, i.e.,

$$\left(t^{-1/Y} Z_{ut} \right)_{u \geq 0} \stackrel{\mathcal{D}}{=} (Z_u)_{u \geq 0}, \quad \text{for any } t > 0. \quad (2.15)$$

Let p_Z denote the marginal density function of Z_1 under $\tilde{\mathbb{P}}$. It is well known (see, e.g., (Sato, 1999, (14.37)) and references therein) that

$$p_Z(v) \sim C_\pm |v|^{-Y-1}, \quad \text{as } v \rightarrow \pm\infty, \quad \text{respectively.} \quad (2.16)$$

²Equivalently, there exists a process $(U_t)_{t \geq 0}$ such that $\tilde{\mathbb{P}}(B) = \mathbb{E}^*(e^{U_t} \mathbf{1}_B)$, for any $B \in \mathcal{F}_t$ and $t \geq 0$.

In particular, for any $v > 0$, as $t \rightarrow 0$,

$$\tilde{\mathbb{P}}(Z_t \geq v) = \tilde{\mathbb{P}}\left(Z_1 \geq t^{-1/Y}v\right) \sim t\tilde{\nu}([v, \infty)) = t\frac{C_+}{Y}v^{-Y}, \quad (2.17)$$

$$\tilde{\mathbb{P}}(Z_t \leq -v) = \tilde{\mathbb{P}}\left(Z_1 \leq -t^{-1/Y}v\right) \sim t\tilde{\nu}((-\infty, -v]) = t\frac{C_-}{Y}v^{-Y}. \quad (2.18)$$

The following tail estimate can also be deduced from (2.16) (see Appendix C for its proof):

$$\tilde{\mathbb{P}}\left(|Z_1| \geq t^{-1/Y}v\right) = \tilde{\mathbb{P}}(|Z_t| \geq v) \leq \kappa t|v|^{-Y}, \quad (2.19)$$

for all $v > 0$ and $0 < t \leq 1$ and some absolute constant $0 < \kappa < \infty$.

Also of use, below, is the following representation of the log density process

$$U_t := \ln \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^*} \Big|_{\mathcal{F}_t} = \lim_{\epsilon \rightarrow 0} \left(\sum_{\substack{s \leq t: \\ |\Delta X_s| > \epsilon}} \varphi(\Delta X_s) - t \int_{|x| > \epsilon} (e^{\varphi(x)} - 1) \nu^*(dx) \right), \quad t \geq 0, \quad (2.20)$$

(cf. (Sato, 1999, Theorem 33.2)). The process $(U_t)_{t \geq 0}$ can be expressed in terms of the jump-measure $N(dt, dx) := \#\{(s, \Delta X_s) \in dt \times dx\}$ of the process $(X_t)_{t \geq 0}$ and its compensated measure $\bar{N}(dt, dx) := N(dt, dx) - \tilde{\nu}(dx)dt$ (under $\tilde{\mathbb{P}}$). Indeed, for any $t \geq 0$,

$$\begin{aligned} U_t &= \lim_{\epsilon \rightarrow 0} \left(\int_0^t \int_{|x| > \epsilon} \varphi(x) N(ds, dx) - t \int_{|x| > \epsilon} (e^{\varphi(x)} - 1) e^{-\varphi(x)} \tilde{\nu}(dx) \right) \\ &= \lim_{\epsilon \rightarrow 0} \left(\int_0^t \int_{|x| > \epsilon} \varphi(x) \bar{N}(ds, dx) + t \int_{|x| > \epsilon} (e^{-\varphi(x)} - 1 + \varphi(x)) \tilde{\nu}(dx) \right). \end{aligned}$$

Thus, from the definition of the Poisson integral under a compensated Poisson measure as in (Kallenberg, 1997, Theorem 33.2),

$$U_t = \tilde{U}_t + \eta t, \quad t \geq 0, \quad (2.21)$$

with

$$\tilde{U}_t := \int_0^t \int_{\mathbb{R}_0} \varphi(x) \bar{N}(ds, dx), \quad \eta := \int_{\mathbb{R}_0} (e^{-\varphi(x)} - 1 + \varphi(x)) \tilde{\nu}(dx). \quad (2.22)$$

In particular, from the definition of $(U_t)_{t \geq 0}$ in (2.20), the process e^{-U_t} is a $\tilde{\mathbb{P}}$ -martingale and, so,

$$\tilde{\mathbb{E}}\left(e^{-\tilde{U}_t}\right) = e^{\eta t}, \quad t \geq 0. \quad (2.23)$$

Remark 2.3. The constant η above is well defined and, therefore, so is \tilde{U}_t . Indeed, from (2.4-iv) and (2.10-ii),

$$\int_{|x| \leq 1} \left| e^{-\varphi(x)} - 1 + \varphi(x) \right| |x|^{-Y-1} dx \leq C_1 \int_{|x| \leq 1} \varphi^2(x) |x|^{-Y-1} dx \leq C_2 \int_{|x| \leq 1} |x|^{1-Y} dx < \infty,$$

for some constants $0 < C_1, C_2 < \infty$. Similarly, from (2.13) and both conditions in (2.4),

$$\int_{x < -1} \left| e^{-\varphi(x)} - 1 + \varphi(x) \right| |x|^{-Y-1} dx \leq C_1 \int_{x < -1} |\varphi(x)| |x|^{-Y-1} dx \leq C_2 \int_{x < -1} |x|^{-Y} dx < \infty.$$

Finally, from (2.1-i) and (2.4), for some constant $0 < C_1 < \infty$,

$$\int_{x > 1} \left| e^{-\varphi(x)} - 1 + \varphi(x) \right| |x|^{-Y-1} dx \leq C_1 \int_{x > 1} e^{\ln q(x) + x} x^{-Y-1} dx + \int_{x > 1} |-1 + \varphi(x)| x^{-Y-1} dx < \infty.$$

The following decomposition of the process X in terms of the compensated measure \tilde{N} is also of use throughout:

$$X_t = \sigma W_t^* + Z_t + t\tilde{\gamma} = \sigma W_t^* + \int_0^t \int_{\mathbb{R}_0} x \tilde{N}(ds, dx) + t\tilde{\gamma}, \quad t \geq 0. \quad (2.24)$$

The representation (2.24) can be deduced from the Lévy-Itô decomposition of the process $(X_t)_{t \geq 0}$ (cf. (Kallenberg, 1997, Theorem 13.4 and Corollary 13.7) or (Sato, 1999, Theorem 19.2)) with the stated Lévy triplet $(0, \tilde{b}, \tilde{\nu})$ of X under $\tilde{\mathbb{P}}$ and where, by construction, \tilde{b} must be such that $\tilde{\mathbb{E}}(X_1) = \tilde{\gamma}$.

3 The Pure-Jump Model

This section finds the second-order asymptotic behavior for the at-the-money call option prices (1.3) in the pure-jump model (i.e., $\sigma = 0$ in (1.5) and, thus, $X_t = L_t$, for $t \geq 0$). The proofs of all the results of this section are deferred to Appendix A. Before stating our results, the call option price (1.3) needs to be rewritten in a suitable form.

Lemma 3.1. *With the probability measure $\tilde{\mathbb{P}}$ defined in (2.20) and the parameter $\tilde{\gamma} := \tilde{\mathbb{E}}(X_1) = \tilde{\mathbb{E}}(L_1^*)$,*

$$t^{-\frac{1}{Y}} \frac{1}{S_0} \mathbb{E} \left[(S_t - S_0)^+ \right] = e^{-(\tilde{\gamma} + \eta)t} \int_{-\tilde{\gamma}t^{1-\frac{1}{Y}}}^{\infty} e^{-t^{\frac{1}{Y}}v} \tilde{\mathbb{E}} \left(e^{-\tilde{U}_t} \mathbf{1}_{\{t^{-\frac{1}{Y}}Z_t \geq v\}} \right) dv. \quad (3.1)$$

The next two lemmas are crucial to obtain the main result of the section:

Lemma 3.2. *In the setting and under the assumptions of Section 2, for any $\xi \geq 0$,*

$$(i) \lim_{t \rightarrow 0} \tilde{\mathbb{E}} \left(e^{-\xi t^{-\frac{1}{Y}}(\tilde{U}_t + Z_t)} \right) = e^{\tilde{\eta}\xi^Y}, \quad (ii) \lim_{t \rightarrow 0} \tilde{\mathbb{E}} \left(e^{-\xi t^{-\frac{1}{Y}}\tilde{U}_t} \right) = e^{\tilde{\eta}^* \xi^Y}, \quad (3.2)$$

where $\tilde{\eta} := \Gamma(-Y)(C_+ M^Y + C_- G^Y)$ and $\tilde{\eta}^* = \Gamma(-Y)(C_+(M^*)^Y + C_-(G^*)^Y)$. In particular, both $t^{-1/Y}(\tilde{U}_t + Z_t)$ and $t^{-1/Y}\tilde{U}_t$ converge to a Y -stable distribution as $t \rightarrow 0$.

Lemma 3.3. *In the setting and under the assumptions of Section 2, the following two assertions hold true:*

1. For any $v > 0$,

$$\lim_{t \rightarrow 0} t^{-1} \tilde{\mathbb{P}} \left(Z_t^+ + \tilde{U}_t \geq v \right) = \int_{\mathbb{R}_0} \mathbf{1}_{\{x - \ln \bar{q}(x) \geq v\}} \tilde{\nu}(dx). \quad (3.3)$$

2. There exist constants $0 < \tilde{\kappa} < \infty$ and $t_0 > 0$ such that

$$(i) t^{-1} \tilde{\mathbb{P}} \left(\tilde{U}_t \geq v \right) \leq \tilde{\kappa} v^{-Y}, \quad (ii) t^{-1} \tilde{\mathbb{P}} \left(Z_t^+ + \tilde{U}_t \geq v \right) \leq \tilde{\kappa} v^{-Y}, \quad (3.4)$$

for any $0 < t \leq t_0$ and $v > 0$.

The following theorem gives the second-order asymptotic behavior of ATM call option prices under an exponential tempered-stable-like Lévy model. It is useful to recall from Section 2 that, under $\tilde{\mathbb{P}}$, $\{Z_t\}_{t \geq 0}$ is a strictly stable Lévy process with Lévy measure $\tilde{\nu}(dx) := |x|^{-Y-1} (C_+ \mathbf{1}_{\{x > 0\}} + C_- \mathbf{1}_{\{x < 0\}}) dx$.

Theorem 3.4. *For the exponential Lévy model (1.4) without Brownian component (i.e., $\sigma = 0$ in (1.5)) and under the conditions of Section 2,*

$$\lim_{t \rightarrow 0} t^{\frac{1}{Y}-1} \left(t^{-\frac{1}{Y}} \frac{1}{S_0} \mathbb{E} \left[(S_t - S_0)^+ \right] - \tilde{\mathbb{E}}(Z_1^+) \right) = \tilde{\vartheta} + \tilde{\gamma} \tilde{\mathbb{P}}(Z_1 \geq 0), \quad (3.5)$$

where, in terms of the function \bar{q} introduced in (2.6),

$$\tilde{\vartheta} := C_+ \int_0^\infty (e^x \bar{q}(x) - \bar{q}(x) - x) x^{-Y-1} dx, \quad (3.6)$$

$$\tilde{\gamma} := \tilde{\mathbb{E}}(L_1^*) = b + \frac{C_+ - C_-}{Y-1} + C_+ \int_0^1 x^{-Y} (1 - \bar{q}(x)) dx - C_- \int_{-1}^0 |x|^{-Y} (1 - \bar{q}(x)) dx. \quad (3.7)$$

Remark 3.5. The leading term $\tilde{\mathbb{E}}(Z_1^+)$ can be explicitly computed via the absolute first moment of Z_1 since $\tilde{\mathbb{E}}|Z_1| = 2\tilde{\mathbb{E}}(Z_1^+) - \tilde{\mathbb{E}}(Z_1) = 2\tilde{\mathbb{E}}(Z_1^+)$. It turns out that

$$\begin{aligned} \tilde{\mathbb{E}}(Z_1^+) &= \frac{1}{2}\tilde{\mathbb{E}}|Z_1| = \frac{1}{\pi}(C_+ + C_-)^{\frac{1}{Y}} \Gamma(-Y)^{\frac{1}{Y}} \left| \cos\left(\frac{Y\pi}{2}\right) \right|^{\frac{1}{Y}} \Gamma\left(1 - \frac{1}{Y}\right) \left(1 + \left(\frac{C_+ - C_-}{C_+ + C_-}\right)^2 \tan^2\left(\frac{Y\pi}{2}\right)\right)^{\frac{1}{2Y}} \\ &\quad \times \cos\left(\frac{1}{Y} \arctan\left(\frac{C_+ - C_-}{C_+ + C_-} \tan\left(\frac{Y\pi}{2}\right)\right)\right), \end{aligned} \quad (3.8)$$

(see the proof of (Samorodnitsky and Taqqu, 1994, (1.2.13))). In the symmetric case (i.e. $C_- = C_+ = C$), (3.8) simplifies to (1.8). The following closed form formula for the probability $\tilde{\mathbb{P}}(Z_1 \geq 0)$ is also known:

$$\tilde{\mathbb{P}}(Z_1 \geq 0) = \frac{1}{2} + \frac{1}{\pi Y} \arctan\left(\frac{C_+ - C_-}{C_+ + C_-} \tan\left(\frac{Y\pi}{2}\right)\right); \quad (3.9)$$

see (Bertoin, 1996, Section VIII.1) and (Zolotarev, 1986, Section 2.2). Furthermore, after plugging the value b , as determined by the condition (2.1-ii), in (3.7), we get the following expression for the limit in (3.5):

$$\begin{aligned} \tilde{\vartheta} + \tilde{\gamma}\tilde{\mathbb{P}}(Z_1 \geq 0) &= C_+\tilde{\mathbb{P}}(Z_1 < 0) \int_0^\infty (e^x \bar{q}(x) - \bar{q}(x) - x) x^{-Y-1} dx \\ &\quad - C_-\tilde{\mathbb{P}}(Z_1 \geq 0) \int_{-\infty}^0 (e^x \bar{q}(x) - \bar{q}(x) - x) |x|^{-Y-1} dx. \end{aligned}$$

Remark 3.6. From (3.5), it follows that the short-time second-order asymptotic behavior of the ATM call option price (1.3) has the form:

$$\frac{1}{S_0} \mathbb{E} \left[(S_t - S_0)^+ \right] = d_1 t^{\frac{1}{Y}} + d_2 t + o(t), \quad t \rightarrow 0, \quad (3.10)$$

with $d_1 := \tilde{\mathbb{E}}(Z_1^+)$ given in (3.8) and $d_2 := \tilde{\vartheta} + \tilde{\gamma}\tilde{\mathbb{P}}(Z_1 \geq 0)$. In particular, the first-order term only synthesizes information on the degree of jump activity as measured by Y and the intensity of small jumps as measured by C_+ and C_- . However, the second-order term also incorporates some information about the tempering function \bar{q} . Let us also point out that even though the parameters M and G introduced in (2.7) do not explicitly show up in the second-order expansion, their existence guarantees that the last two terms in the right-hand side of (3.7) are well-defined. In particular, the result is expected to be true if $M = 0$ or $G = 0$, but not if either M or G is ∞ .

Let us proceed to study the asymptotic behavior of the corresponding Black-Scholes implied volatility. Throughout, let $\hat{\sigma}(t)$ denote the ATM Black-Scholes implied volatility at maturity t with zero interest rates and zero dividend yield. The following result gives an easy derivation of the second-order asymptotic expansion of $\hat{\sigma}(t)$ as $t \rightarrow 0$. For a more cumbersome, yet methodical, procedure to derive an asymptotic expansion for $\hat{\sigma}(t)$ of arbitrary order, we refer the reader to the recent manuscript Gao and Lee (2013).

Corollary 3.7. *For the exponential Lévy model (1.4) without Brownian component, the implied volatility $\hat{\sigma}$ has the following small-time behavior:*

$$\hat{\sigma}(t) = \sigma_1 t^{\frac{1}{Y} - \frac{1}{2}} + \sigma_2 t^{\frac{1}{2}} + o(t^{\frac{1}{2}}), \quad t \rightarrow 0, \quad (3.11)$$

where

$$\sigma_1 := \sqrt{2\pi} \tilde{\mathbb{E}}(Z_1^+), \quad \sigma_2 := \sqrt{2\pi} \left(\tilde{\vartheta} + \tilde{\gamma}\tilde{\mathbb{P}}(Z_1 \geq 0) \right), \quad (3.12)$$

and where $\tilde{\vartheta}$ and $\tilde{\gamma}$ are respectively given in (3.6) and (3.7).

4 The Pure-Jump Model With A Nonzero Brownian Component

In this part, the case of a nonzero independent Brownian component is considered. Concretely, throughout, $(X_t)_{t \geq 0}$ is a Lévy process with triplet (σ^2, b, ν) as introduced in Section 2 and $\sigma \neq 0$. The first-order asymptotic behavior

for the ATM European call options in this mixed model is obtained in Tankov (2010) using Fourier methods. The second-order correction term for the at-the-money European call option price is given next. As previously done, we change the probability measure \mathbb{P}^* to $\tilde{\mathbb{P}}$ so that $X_t = t\tilde{\gamma} + \sigma W_t^* + Z_t$, $t \geq 0$, with $(Z_t)_{t \geq 0}$ a strictly Y -stable Lévy process under $\tilde{\mathbb{P}}$ (see (2.14)). Recall also that, under both \mathbb{P}^* and $\tilde{\mathbb{P}}$, W^* is still a standard Brownian motion, independent of $(Z_t)_{t \geq 0}$. We will also make use of the decompositions (2.21) and (2.24). The proofs of the results below are presented in Appendix B.

Theorem 4.1. *For the exponential Lévy model (1.4) with a nonzero independent Brownian component and under the conditions of Section 2, the ATM European call option price is such that*

$$\lim_{t \rightarrow 0} t^{\frac{Y}{2}-1} \left(t^{-\frac{1}{2}} \frac{1}{S_0} \mathbb{E} \left[(S_t - S_0)^+ \right] - \sigma \mathbb{E}^* \left((W_1^*)^+ \right) \right) = \frac{C_- + C_+}{2Y(Y-1)} \sigma^{1-Y} \mathbb{E}^* \left(|W_1^*|^{1-Y} \right). \quad (4.1)$$

Remark 4.2. The $(1-Y)$ -centered moment of a standard normal distribution is given by (see, e.g., (Sato, 1999, (25.6))):

$$\mathbb{E}^* \left(|W_1^*|^{1-Y} \right) = \frac{2^{\frac{1-Y}{2}}}{\sqrt{\pi}} \Gamma \left(1 - \frac{Y}{2} \right).$$

Thus, the second-order asymptotic behavior of the ATM call option price (1.3) in short-time takes the form

$$\frac{1}{S_0} \mathbb{E} \left[(S_t - S_0)^+ \right] = d_1 t^{\frac{1}{2}} + d_2 t^{\frac{3-Y}{2}} + o \left(t^{\frac{3-Y}{2}} \right), \quad t \rightarrow 0, \quad (4.2)$$

with

$$d_1 := \frac{\sigma}{\sqrt{2\pi}}, \quad d_2 := \frac{2^{\frac{1-Y}{2}}}{\sqrt{\pi}} \Gamma \left(1 - \frac{Y}{2} \right) \frac{(C_- + C_+) \sigma^{1-Y}}{2Y(Y-1)}. \quad (4.3)$$

The first-order term only synthesizes the information on the volatility parameter σ of the continuous component. In fact, the first-order term of the ATM call option price under the mixed tempered-stable-like model is the same as the one under the Black-Scholes model. The second-order term further incorporates the information on the degree of jump activity Y and the net intensity of small jumps as measured by the parameter $\bar{C} := C_- + C_+$. However, these two-terms do not reflect the individual intensities of the small negative or positive jumps as measured by the values of C_- and C_+ , respectively. This fact suggests that it could be useful to consider a third-order approximation. For more information on the latter in the CGMY model, we refer the reader to Figueroa-López et al. (2013).

The forthcoming proposition provides the small-time asymptotic behavior for the ATM Black-Scholes implied volatility under the tempered-stable-like model with a nonzero independent Brownian component. Unlike the pure-jump case, the first-order asymptotics can only be derived from Theorem 4.1. The second-order term for the implied volatility requires the third-order asymptotics for the ATM call option price.

Corollary 4.3. *For the exponential Lévy model (1.4) with a nonzero independent Brownian component, the implied volatility $\hat{\sigma}$ has the following small-time behavior:*

$$\hat{\sigma}(t) = \sigma + \frac{(C_+ + C_-) 2^{-\frac{Y}{2}}}{Y(Y-1)} \Gamma \left(1 - \frac{Y}{2} \right) \sigma^{1-Y} t^{1-\frac{Y}{2}} + o \left(t^{1-\frac{Y}{2}} \right), \quad t \rightarrow 0. \quad (4.4)$$

Remark 4.4. It is of interest to compare (4.4) with the corresponding asymptotics under the presence of a jump component L of finite activity (namely, a compound Poisson process). Indeed, Medvedev and Scaillet (2007) argued, under a relatively strong technical condition (see Proposition 5 therein for details), that the implied volatility is asymptotically $\sigma + I_1 \sqrt{t} + O(t)$, as $t \rightarrow 0$, for a certain constant I_1 . As seen from (4.4), $\hat{\sigma}(t)$ converges at a slower rate under a tempered stable-like infinite jump activity component. In fact, the higher the degree of jump activity, the slower the rate of convergence is and, thus, the stronger is the effect of the intensity of small jumps in the asymptotic behavior of the implied volatility in short-time.

Remark 4.5. As mentioned in the introduction, short-time asymptotics of option prices are relevant in numerical calibration. Here, we illustrate an approach for this purpose inspired by a method proposed in Medvedev and

Scaillet (2007). As outlined in the introduction, it is expected that (4.4) is valid under a stochastic volatility model with σ replaced with the spot volatility at time 0, say σ_0 , which itself can be random depending on an additional risky factor (see Figueroa-López and Ólafsson (2014) for further information about this type of results). Now, two major interrelated issues arise. The resulting asymptotic formula would involve the unobserved spot volatility, which is changing in time. Thus, apparently the calibration will have to be carried out day by day, but unfortunately, a limited number of short-term options are available in the market at any given day. To overcome these issues, argue as follows: First, it is reasonable that observed prices of options “sufficiently close-to-the-money” can be considered as being at-the-money (see Figueroa-López and Ólafsson (2014) for a more formal justification of this step). Hence, if κ denotes the log-moneyness $\ln(K/S_0)$ of the option and $\hat{\sigma}(t; \kappa)$ denotes the corresponding Black-Scholes implied volatility under the model, it is expected that

$$\hat{\sigma}(t; \kappa) \approx \sigma_0 + \frac{(C_+ + C_-)2^{-Y}}{Y(Y-1)} \Gamma\left(1 - \frac{Y}{2}\right) \sigma_0^{1-Y} t^{1-\frac{Y}{2}}, \quad (4.5)$$

when $\kappa \approx 0$. Next, consider the observed implied volatility corresponding to the closest-to-the-money option, hereafter denoted by $\hat{\sigma}^*$, and solve the equation

$$\hat{\sigma}^* = \sigma_0 + \frac{(C_+ + C_-)2^{-Y}}{Y(Y-1)} \Gamma\left(1 - \frac{Y}{2}\right) \sigma_0^{1-Y} t^{1-\frac{Y}{2}}, \quad (4.6)$$

for the spot volatility σ_0 . Denote the smallest of such solutions by $\sigma_0(t, \hat{\sigma}^*; C_+, C_-, Y)$. Then, substitute it into (4.5) to get the equation:

$$\hat{\sigma}(t; \kappa) \approx \sigma_0(t, \hat{\sigma}^*; C_+, C_-, Y) + \frac{(C_+ + C_-)2^{-Y}}{Y(Y-1)} \Gamma\left(1 - \frac{Y}{2}\right) \sigma_0(t, \hat{\sigma}^*; C_+, C_-, Y)^{1-Y} t^{1-\frac{Y}{2}}, \quad (4.7)$$

which is free of the unobserved spot volatility σ_0 . Finally, calibrate (4.7) to other sufficiently close-to-the-money implied volatilities by minimizing a weighted sum of squared errors. As in Medvedev and Scaillet (2007), this approach has the advantage of enabling to calibrate the parameters across calendar dates simultaneously and not day by day. Note that, alternatively, one can plug $\sigma_0(t, \hat{\sigma}^*; C_+, C_-, Y)$ into other type of asymptotics such as the out-of-the-money asymptotics obtained in Figueroa-López et al. (2012).

5 Asymptotics For ATM Option Prices Under A CGMY Model

In this section, the second-order asymptotic expansions of the two previous sections is specialized to the CGMY model. The result presented here were first reported in Figueroa-López et al. (2011). Recall that under the CGMY model, the Lévy measure of the pure-jump component $(L_t)_{t \geq 0}$ is given by

$$\nu(dx) = |x|^{-Y-1} q(x) dx = |x|^{-Y-1} (C e^{-Mx} \mathbf{1}_{\{x>0\}} + C e^{Gx} \mathbf{1}_{\{x<0\}}) dx, \quad (5.1)$$

with corresponding parameters $C, G, M > 0$ and $Y \in (1, 2)$. Then, the characteristic function of the log-return process $X_t = \sigma W_t + L_t$ takes the form

$$\varphi_t(u) = \mathbb{E}(e^{iuX_t}) = \exp\left(t \left[icu - \frac{\sigma^2 u^2}{2} + C\Gamma(-Y) ((M - iu)^Y + (G + iu)^Y - M^Y - G^Y) \right]\right), \quad (5.2)$$

for a constant $c \in \mathbb{R}$. The martingale condition (2.1) implies that $M > 1$ and

$$c = -C\Gamma(-Y) ((M - 1)^Y + (G + 1)^Y - M^Y - G^Y) - \frac{\sigma^2}{2}, \quad (5.3)$$

(see, e.g., (Tankov, 2010, Proposition 4.2)). In particular, the center $\gamma := \mathbb{E}(X_1) = \mathbb{E}(L_1)$ of X and the parameter b of X (relative to the truncation function $x \mathbf{1}_{\{|x| \leq 1\}}$) are given by

$$\gamma = c - CY\Gamma(-Y) (M^{Y-1} - G^{Y-1}), \quad b = c - \int_{|x|>1} x\nu(dx) - CY\Gamma(-Y) (M^{Y-1} - G^{Y-1}).$$

Under the share measure \mathbb{P}^* introduced in Section 2, $(X_t)_{t \geq 0}$ has Lévy triplet $(b^*, (\sigma^*)^2, \nu^*)$ given by

$$\sigma^* := \sigma, \quad b^* := c + \sigma^2 - \int_{|x| > 1} x \nu^*(dx) - \text{CYT}(-Y) \left((M^*)^{Y-1} - (G^*)^{Y-1} \right), \quad \nu^*(dx) := e^x \nu(dx), \quad (5.4)$$

with $M^* = M - 1$ and $G^* = G + 1$. Under the probability measure $\tilde{\mathbb{P}}$, the centered process $(Z_t)_{t \geq 0}$ is symmetric, and its center $\tilde{\gamma} := \tilde{\mathbb{E}}(X_1) = \tilde{\mathbb{E}}(L_1^*)$ is given by (see Figueroa-López et al. (2011) for the detailed computation)

$$\begin{aligned} \tilde{\gamma} &= \tilde{b} + \int_{\{|x| > 1\}} x \tilde{\nu}(dx) = b^* + \int_{|x| \leq 1} x (\tilde{\nu} - \nu^*)(dx) + \int_{\{|x| > 1\}} x \tilde{\nu}(dx) \\ &= -\text{CT}(-Y) \left((M-1)^Y + (G+1)^Y - M^Y - G^Y \right) + \frac{\sigma^2}{2}. \end{aligned} \quad (5.5)$$

The value of η defined in (2.22) is also needed. Under the CGMY model, it is now given by

$$\eta = C \int_{0^+}^{\infty} \left(e^{-M^* x} - 1 + M^* x \right) x^{-Y-1} dx + C \int_{-\infty}^{0^-} \left(e^{G^* x} - 1 - G^* x \right) |x|^{-Y-1} dx = \text{CT}(-Y) \left((M^*)^Y + (G^*)^Y \right), \quad (5.6)$$

where, to obtain the last equality above, we used the analytic continuation of the representation (14.19) given in Sato (1999).

Let us now explicitly write the second-order expansions. First, compute the term ϑ of (3.6). To this end, it is convenient to use the representation given in (A.21) below noting that in the CGMY case, $\bar{q}(x) := e^{-Mx} \mathbf{1}_{\{x > 0\}} + e^{Gx} \mathbf{1}_{\{x < 0\}}$. Hence,

$$\begin{aligned} \vartheta &= C \int_0^{\infty} (e^{-v} - 1) \int_{\mathbb{R}_0} \mathbf{1}_{\{x - \ln \bar{q}(x) \geq v\}} |x|^{-Y-1} dx dv \\ &= C \int_0^{\infty} (e^{-v} - 1) \int_0^{\infty} \mathbf{1}_{\{Mx \geq v\}} x^{-Y-1} dx dv + C \int_0^{\infty} (e^{-v} - 1) \int_{-\infty}^0 \mathbf{1}_{\{-x - Gx \geq v\}} (-x)^{-Y-1} dx dv \\ &= \frac{C}{Y} (M^Y + (G+1)^Y) \int_0^{\infty} (e^{-v} - 1) v^{-Y} dv \\ &= -\text{CT}(-Y) \left((M^* + 1)^Y + (G^*)^Y \right), \end{aligned} \quad (5.7)$$

where for the last equality $\int_0^{\infty} (e^{-v} - 1) v^{-Y} dv = \Gamma(1 - Y) = -Y\Gamma(-Y)$ is used (see (Sato, 1999, (14.18))). Next, from (5.5)-(5.7) and recalling that $\tilde{\vartheta} = \vartheta + \eta$ (see (A.22)), it follows that in the pure-jump CGMY model,

$$d_2 := \vartheta + \eta + \frac{\tilde{\gamma}}{2} = \frac{\text{CT}(-Y)}{2} \left((M-1)^Y - M^Y - (G+1)^Y + G^Y \right),$$

while in the general CGMY model with non-zero Brownian component, $C_+ = C_- = C$ and, thus,

$$d_2 := \frac{C\sigma^{1-Y} 2^{\frac{1-Y}{2}}}{Y(Y-1)\sqrt{\pi}} \Gamma\left(1 - \frac{Y}{2}\right).$$

6 Numerical Examples

In this part, the performance of the previous approximations is assessed through a detailed numerical analysis for the CGMY model.

6.1 The Numerical Methods

Let us first select a suitable numerical scheme to compute the ATM option prices by considering two methods: Inverse Fourier Transform (IFT) and Monte Carlo (MC). Before introducing the IFT, let us set some notations. The characteristic function corresponding to the Black-Scholes model with volatility Σ is given by

$$\varphi_t^{BS,\Sigma}(v) = \exp\left(-\frac{\Sigma^2 t}{2}(v^2 + iv)\right).$$

The corresponding call option price at the log-moneyness $k = \log(S_0/K)$, under the Black-Scholes model with volatility Σ is denoted by $C_{BS}^\Sigma(k)$; that is,

$$C_{BS}^\Sigma(k) = S_0 e^{-rt} \mathbb{E}\left[\left(e^{(r-\Sigma^2/2)t+\Sigma W_t} - e^k\right)^+\right].$$

Recall also that the characteristic function, under the mixed CGMY model with a Brownian component, is denoted by φ_t (see (5.2)) and denote the corresponding call option price at log-moneyness k by $C(k)$. The IFT method is based on the following inversion formula (see (Cont and Tankov, 2004, Section 11.1.3)):

$$z_T(k) := C(k) - C_{BS}^\Sigma(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ivk} \zeta_T(v) dv, \quad (6.1)$$

where

$$\zeta_T(v) := e^{ivr} \frac{\varphi_T(v-i) - \varphi_T^{BS,\Sigma}(v-i)}{iv(1+iv)}. \quad (6.2)$$

Fix $r = 0$ and, since our interest is only in ATM option prices, set $k = 0$. The integral in (6.1) is numerically computed using the Simpson's rule:

$$z_T(0) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \zeta_T(v) dv = \Delta \sum_{m=0}^{P-1} w_m^* \zeta_T(v_m^*),$$

with $\Delta = Q/(P-1)$, $v_m^* = -Q/2 + m\Delta$, and $w_0 = 1/2$, $w_{2\ell-1} = 4/3$, and $w_{2\ell} = 2/3$, for $\ell = 1, \dots, P/2$.

Let us introduce a Monte Carlo method based on the risk-neutral option price representation under the probability measure $\tilde{\mathbb{P}}$. Under this probability measure and using the notation (2.22) as well as the relations (2.21) and (2.24), we have:

$$\mathbb{E}\left[(e^{X_T} - 1)^+\right] = \mathbb{E}^*\left[e^{-X_T} (e^{X_T} - 1)^+\right] = \tilde{\mathbb{E}}\left[e^{-U_T} (1 - e^{-X_T})^+\right] = \tilde{\mathbb{E}}\left[e^{-M^* \bar{U}_T^+ + G^* \bar{U}_T^- - \eta T} \left(1 - e^{-\bar{U}_T^+ - \bar{U}_T^- - T\tilde{\gamma} - \sigma W_T}\right)^+\right],$$

which can be easily computed by Monte Carlo method using that, under $\tilde{\mathbb{P}}$, the variables \bar{U}_T^+ and $-\bar{U}_T^-$ are independent Y -stable random variables with scale, skewness, and location parameters $TC|\cos(\pi Y/2)|\Gamma(-Y)$, 1 and 0, respectively. Standard simulation methods are available to generate stable random variables.

Next, take the following set of parameters

$$C = 0.5, \quad G = 2, \quad M = 3.6, \quad Y = 1.5.$$

Figure 1 compares the first- and second-order approximations as given in Remarks 3.6 and 4.2 to the prices based on the Inverse Fourier Transform (IFT-based price) and the Monte Carlo method (MC-based price) under both the pure-jump case and the mixed CGMY case with $\sigma = 0.4$. For the MC-based price, 100,000 simulations are used, while for the IFT-based method, $P = 2^{14}$ and $Q = 800$. As it can be seen, it is not easy to integrate numerically the characteristic function (6.2) since T is quite small and, therefore, the characteristic functions φ_T and $\varphi_T^{BS,\Sigma}$ are quite flat. The Monte Carlo method turns out to be much more accurate and faster. It is also interesting to note that the second-order approximation is in general much more accurate in the pure-jump model than in the mixed model with nonzero continuous component. This observation is consistent with the last comment of Remark 4.2.

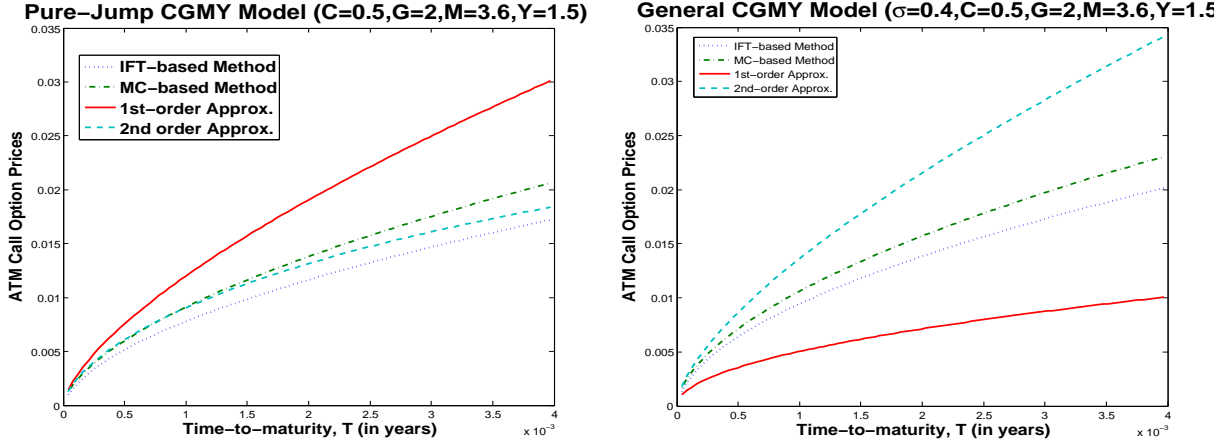


Figure 1: Comparisons of ATM call option prices for two methods (Inverse Fourier Transform and Monte-Carlo method) with the first- and second-order approximations. MC-based price is based on 100,000 simulations while the IFT-based method is based on the parameter values $P = 2^{14}$ and $Q = 800$. The parameter σ in the mixed CGMY model is set to be 0.1.

6.2 Results For Different Parameter Settings

The performances of the approximations for different settings of parameters are investigated below:

1. Figure 2 compares the first- and second-order approximations with the MC prices for different values of C , fixing the values of all the other parameters. In the pure-jump case, the second-order approximation is significantly better for moderately small values of C , but for larger values of C , this is not the case unless T is extremely small. For a nonzero continuous component, the first-order approximation is extremely poor as it only takes into account the parameter σ .
2. Figure 3 compares the first- and second-order approximations with the MC prices for different values of Y , fixing the values of all the other parameters. In both cases, the second-order approximation is significantly better for values of Y around 1.5, which is consistent with the observation that $|d_2| \rightarrow \infty$ as $Y \rightarrow 1$ or $Y \rightarrow 2$. For a nonzero continuous component, the first-order approximation is again extremely poor when compared to the 2nd order approximation.
3. The left panel of Figure 4 analyzes the effect of the relative intensities of the negative jumps compared to the positive jumps in the pure-jump CGMY case. That is, the value M is fixed to be 4 and consider different values for G . As expected, since the first-order approximation does not take into account this information, the second-order approximation performs significantly better.
4. In the right panel of Figure 4, we analyze the effect of the volatility of the continuous component in the mixed CGMY case. The second-order approximation is, in general, much better than the first-order approximation and, interestingly enough, the quality of the second-order approximations improves as the values of σ increases. In fact, it seems that the second-order approximation and the MC prices collapse to a steady curve as σ increases.

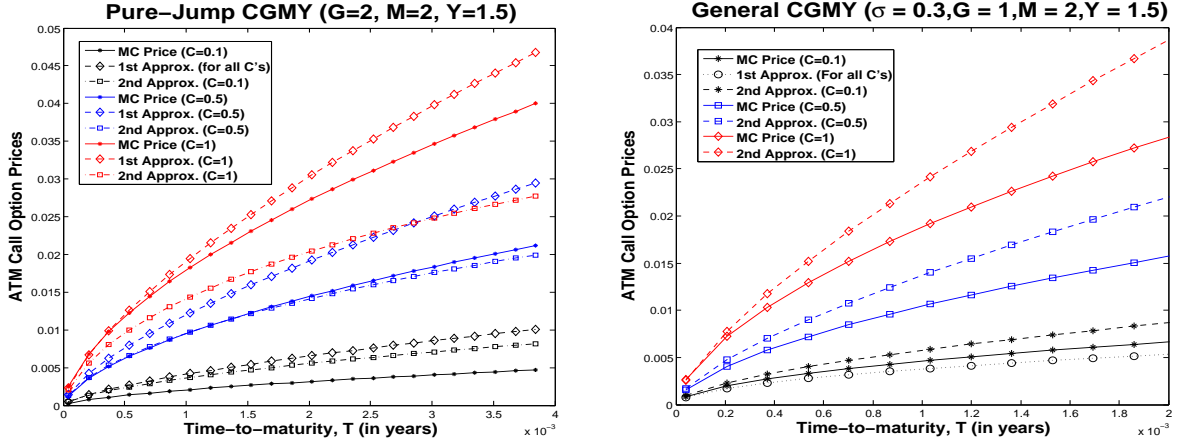


Figure 2: Comparisons of ATM call option prices with short-time approximations for different values of the jump intensity parameter C .

A Proofs of Section 3: The Pure-Jump Model

For simplicity, unless otherwise stated, throughout this section, fix $S_0 = 1$.

Proof of Lemma 3.1.

From (1.16),

$$\begin{aligned}
 t^{-\frac{1}{Y}} \mathbb{E} \left[(S_t - S_0)^+ \right] &= t^{-\frac{1}{Y}} \mathbb{P}^* (X_t \geq E) = t^{-\frac{1}{Y}} \int_0^\infty e^{-x} \mathbb{P}^* (X_t \geq x) dx \\
 &= \int_0^\infty e^{-t^{\frac{1}{Y}} u} \mathbb{P}^* \left(t^{-\frac{1}{Y}} X_t \geq u \right) du.
 \end{aligned} \tag{A.1}$$

Next, using the change of probability measure (2.20),

$$\mathbb{P}^* \left(t^{-\frac{1}{Y}} X_t \geq u \right) = \mathbb{E}^* \left(\mathbf{1}_{\{t^{-\frac{1}{Y}} X_t \geq u\}} \right) = \tilde{\mathbb{E}} \left(e^{-U_t} \mathbf{1}_{\{t^{-\frac{1}{Y}} X_t \geq u\}} \right),$$

and, moreover, (2.21) and (2.24),

$$\mathbb{P}^* \left(t^{-\frac{1}{Y}} X_t \geq u \right) = e^{-\eta t} \tilde{\mathbb{E}} \left(e^{-\tilde{U}_t} \mathbf{1}_{\{t^{-\frac{1}{Y}} Z_t \geq u - \tilde{\gamma} t^{1-\frac{1}{Y}}\}} \right).$$

Then,

$$t^{-\frac{1}{Y}} \mathbb{E} \left[(S_t - S_0)^+ \right] = \int_0^\infty e^{-t^{\frac{1}{Y}} u - \eta t} \tilde{\mathbb{E}} \left(e^{-\tilde{U}_t} \mathbf{1}_{\{t^{-\frac{1}{Y}} Z_t \geq u - \tilde{\gamma} t^{1-\frac{1}{Y}}\}} \right) du, \tag{A.2}$$

and, changing variables ($v = u - \tilde{\gamma} t^{1-1/Y}$), the result follows. \square

Proof of Lemma 3.2.

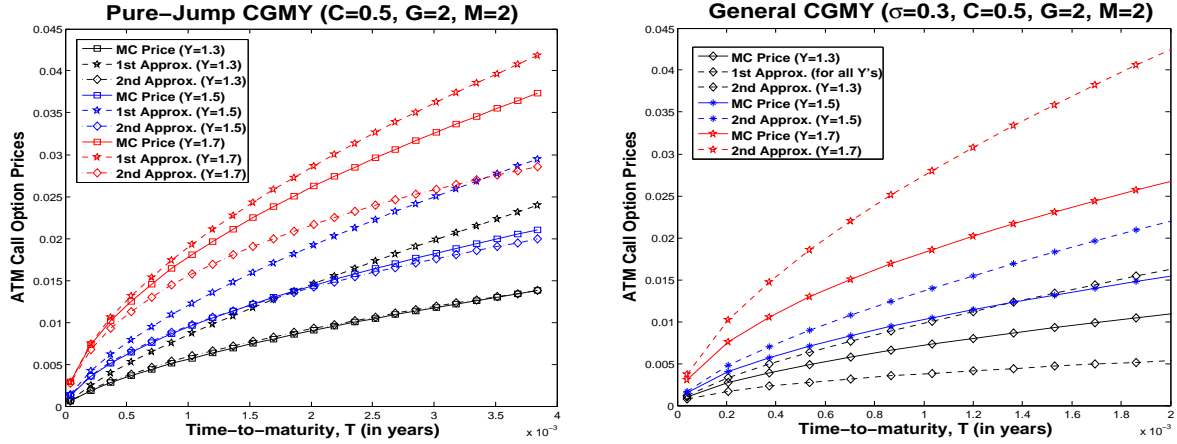


Figure 3: Comparisons of ATM call option prices with short-time approximations for different values of the tail-heaviness parameter Y .

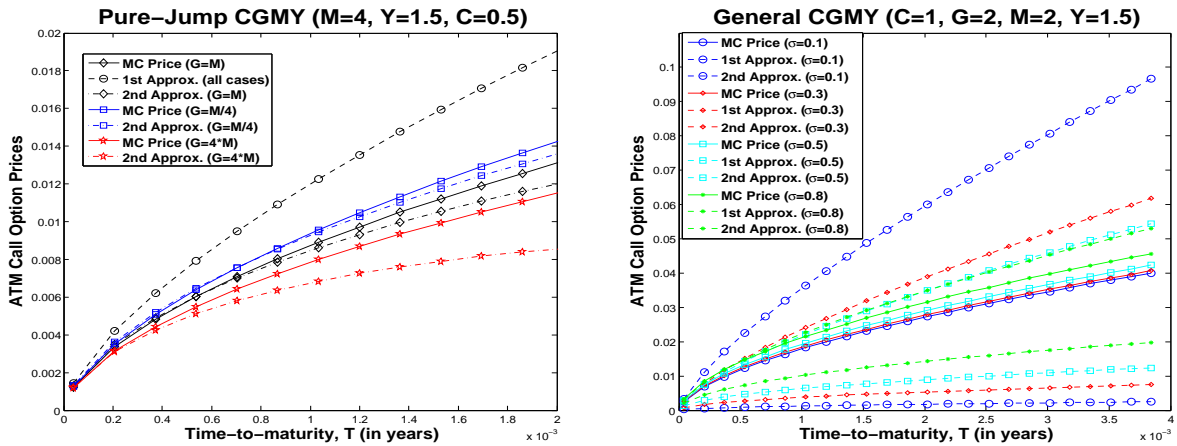


Figure 4: Comparisons of ATM call option prices with short-time approximations for different values of G or M and different values of the volatility parameter σ .

Fix $Q_t := \tilde{U}_t + Z_t$. From the usual moment generating formula for Poisson integrals, the change of variables $y = t^{-1/Y}x$, and recalling that $\varphi(x) + x = -\ln \bar{q}(x)$,

$$\begin{aligned} \tilde{\mathbb{E}} \left(e^{-\xi t^{-\frac{1}{Y}} Q_t} \right) &= \tilde{\mathbb{E}} \left(\exp \left\{ -\xi t^{-\frac{1}{Y}} \int_0^t \int (\varphi(x) + x) \bar{N}(ds, dx) \right\} \right) \\ &= \exp \left\{ t \int \left(e^{\xi t^{-\frac{1}{Y}} \ln \bar{q}(x)} - 1 - \xi t^{-\frac{1}{Y}} \ln \bar{q}(x) \right) \tilde{\nu}(dx) \right\} \\ &= \exp \left\{ C_+ \int_0^\infty \left(e^{\xi t^{-\frac{1}{Y}} \ln \bar{q}(t^{\frac{1}{Y}} y)} - 1 - \xi t^{-\frac{1}{Y}} \ln \bar{q}(t^{\frac{1}{Y}} y) \right) y^{-Y-1} dy \right. \\ &\quad \left. + C_- \int_{-\infty}^0 \left(e^{\xi t^{-\frac{1}{Y}} \ln \bar{q}(t^{\frac{1}{Y}} y)} - 1 - \xi t^{-\frac{1}{Y}} \ln \bar{q}(t^{\frac{1}{Y}} y) \right) |y|^{-Y-1} dy \right\}. \end{aligned} \quad (\text{A.3})$$

Let $\Xi(u) := e^u - 1 - u$ and $B := \sup_w |\ln \bar{q}(w)|/|w|$, which is necessarily finite in light of (2.4) and (2.8-ii). Now, using that $\ln \bar{q}(u) \leq 0$, for any $u \in \mathbb{R}_0$ (from both conditions in (2.3)), and that $|\Xi(u)| \leq K(u^2 \wedge |u|)$, for $u \leq 0$ and some constant $0 < K < \infty$,

$$\begin{aligned} \Xi \left(\xi t^{-\frac{1}{Y}} \ln \bar{q}(t^{\frac{1}{Y}} y) \right) |y|^{-Y-1} &\leq K \left[\left(\xi t^{-\frac{1}{Y}} \ln \bar{q}(t^{\frac{1}{Y}} y) \right)^2 \wedge \left| \xi t^{-\frac{1}{Y}} \ln \bar{q}(t^{\frac{1}{Y}} y) \right| \right] |y|^{-Y-1} \\ &\leq K \left((\xi y B)^2 \wedge |\xi y B| \right) |y|^{-Y-1} \\ &\leq K \left((\xi B)^2 \vee |\xi B| \right) (y^2 \wedge |y|) |y|^{-Y-1}, \end{aligned}$$

which is integrable. Therefore, one can pass the limit inside the integrals in (A.3) and, using (2.8),

$$\begin{aligned} \lim_{t \rightarrow 0} \tilde{\mathbb{E}} \left(e^{-\xi t^{-\frac{1}{Y}} Q_t} \right) &= \exp \left\{ C_+ \int_0^\infty (e^{-y \xi M} - 1 + y \xi M) y^{-Y-1} dy + C_- \int_{-\infty}^0 (e^{-|y| \xi G} - 1 + |y| \xi G) |y|^{-Y-1} dy \right\} \\ &= \exp \left\{ \left(C_+ \int_0^\infty (e^{-u M} - 1 + u M) u^{-Y-1} du + C_- \int_{-\infty}^0 (e^{-|u| G} - 1 + |u| G) |u|^{-Y-1} du \right) \xi^Y \right\}. \end{aligned}$$

Finally, the analytic continuation of the representation (14.19) given in Sato (1999) shows that the last expression is of the form $\exp(\bar{\eta} \xi^Y)$, with $\bar{\eta}$ given as in the statement of the lemma.

For (3.2-ii), proceed as above to get

$$\tilde{\mathbb{E}} \left(e^{-\xi t^{-\frac{1}{Y}} \tilde{U}_t} \right) = \exp \left\{ C_+ \int_0^\infty \Xi \left(\xi t^{-\frac{1}{Y}} \ln \bar{q}^*(t^{\frac{1}{Y}} y) \right) y^{-Y-1} dy + C_- \int_{-\infty}^0 \Xi \left(\xi t^{-\frac{1}{Y}} \ln \bar{q}^*(t^{\frac{1}{Y}} y) \right) |y|^{-Y-1} dy \right\}. \quad (\text{A.4})$$

Now, for all $y \in \mathbb{R}_0$,

$$\xi t^{-\frac{1}{Y}} \ln \bar{q}^*(t^{\frac{1}{Y}} y) = \xi t^{-\frac{1}{Y}} \ln \bar{q}(t^{\frac{1}{Y}} y) + \xi y = \xi y \left(\frac{1}{y t^{\frac{1}{Y}}} \ln \bar{q}(t^{\frac{1}{Y}} y) + 1 \right) \leq 0, \quad (\text{A.5})$$

from both conditions in (2.3). Using (A.5), proceed as above to justify passing the limit into the integrals in (A.4). Next, using (2.10), conclude that

$$\begin{aligned} \lim_{t \rightarrow 0} \tilde{\mathbb{E}} \left(e^{-\xi t^{-\frac{1}{Y}} \tilde{U}_t} \right) &= \exp \left\{ C_+ \int_0^\infty (e^{-y \xi M^*} - 1 + y \xi M^*) y^{-Y-1} dy + C_- \int_{-\infty}^0 (e^{-|y| \xi G^*} - 1 + |y| \xi G^*) |y|^{-Y-1} dy \right\} \\ &= \exp \left\{ \left(C_+ \int_0^\infty (e^{-u M^*} - 1 + u M^*) u^{-Y-1} du + C_- \int_{-\infty}^0 (e^{-|u| G^*} - 1 + |u| G^*) |u|^{-Y-1} du \right) \xi^Y \right\}. \end{aligned}$$

Finally, (3.2-ii) follows once more from the analytic continuation of (Sato, 1999, (14.19)). \square

Proof of Lemma 3.3.

(1) The proof of the first assertion of this lemma makes use of the following “small/large jumps decomposition” of Z :

$$\bar{Z}_t^{(\varepsilon)} := \int_0^t \int_{|x| \geq \varepsilon} x N(ds, dx), \quad Z_t^{(\varepsilon)} := Z_t - \bar{Z}_t^{(\varepsilon)}, \quad t \geq 0, \quad (\text{A.6})$$

for a suitably chosen $\varepsilon > 0$. Under $\tilde{\mathbb{P}}$, $(\bar{Z}_t^{(\varepsilon)})_{t \geq 0}$ is a drift-less Lévy process with finite Lévy measure $\mathbf{1}_{\{|x| \geq \varepsilon\}} \tilde{\nu}(dx)$ and, thus, is a compound Poisson process. Denote respectively by $(N_t^{(\varepsilon)})_{t \geq 0}$ and $(\xi_i^{(\varepsilon)})_{i \geq 1}$ the counting process and the sizes of the jumps of $(\bar{Z}_t^{(\varepsilon)})_{t \geq 0}$, so that $\bar{Z}_t^{(\varepsilon)} = \sum_{i=1}^{N_t^{(\varepsilon)}} \xi_i^{(\varepsilon)}$, for any $t \geq 0$. In particular, $(N_t^{(\varepsilon)})_{t \geq 0}$ is a Poisson process with intensity $\lambda_\varepsilon := \tilde{\mathbb{E}} N_1^{(\varepsilon)} = \tilde{\nu}(|x| \geq \varepsilon)$ and $(\xi_i^{(\varepsilon)})_{i \geq 1}$ are i.i.d, random variables with distribution $\mathbf{1}_{\{|x| \geq \varepsilon\}} \tilde{\nu}(dx) / \lambda_\varepsilon$. Next, define the corresponding processes for \tilde{U} :

$$\tilde{U}_t^{(\varepsilon)} := \int_0^t \int_{|x| \geq \varepsilon} \varphi(x) N(ds, dx) = - \sum_{i=1}^{N_t^{(\varepsilon)}} \ln \bar{q}^* \left(\xi_i^{(\varepsilon)} \right), \quad \tilde{U}_t^{(\varepsilon)} := \tilde{U}_t - \tilde{U}_t^{(\varepsilon)}, \quad t \geq 0. \quad (\text{A.7})$$

For any $t \geq 0$, recalling that $N_t^{(\varepsilon)}$ is Poisson distributed with mean $\lambda_\varepsilon t$ and using (A.6) and (A.7), conditioning on $N_t^{(\varepsilon)}$ gives

$$\begin{aligned} t^{-1} \tilde{\mathbb{P}} \left(Z_t^+ + \tilde{U}_t \geq v \right) &= t^{-1} \tilde{\mathbb{P}} \left(\left(Z_t^{(\varepsilon)} + \sum_{i=1}^{N_t^{(\varepsilon)}} \xi_i^{(\varepsilon)} \right)^+ + \tilde{U}_t^{(\varepsilon)} - \sum_{i=1}^{N_t^{(\varepsilon)}} \ln \bar{q}^* \left(\xi_i^{(\varepsilon)} \right) \geq v \right) \\ &= t^{-1} \tilde{\mathbb{P}} \left(\left(Z_t^{(\varepsilon)} \right)^+ + \tilde{U}_t^{(\varepsilon)} \geq v \right) e^{-\lambda_\varepsilon t} + e^{-\lambda_\varepsilon t} \lambda_\varepsilon \tilde{\mathbb{P}} \left(\left(Z_t^{(\varepsilon)} + \xi_1^{(\varepsilon)} \right)^+ + \tilde{U}_t^{(\varepsilon)} - \ln \bar{q}^* \left(\xi_1^{(\varepsilon)} \right) \geq v \right) + O(t). \end{aligned}$$

The first term above can be made $O(t)$ by taking $\varepsilon \in (0, \varepsilon_0)$, for some small enough $\varepsilon_0 > 0$ (see, e.g., (Sato, 1999, Section 26) and (Rüschendorf and Woerner, 2002, Lemma 3.2)). Indeed, first note that the supports of the Lévy measures of $(Z_t^{(\varepsilon)})_{t \geq 0}$ and $(\tilde{U}_t^{(\varepsilon)})_{t \geq 0}$ are respectively $\{x : |x| \leq \varepsilon\}$ and $\{\varphi(x) : |x| \leq \varepsilon\} = \{-\ln \bar{q}(x) - x : |x| \leq \varepsilon\}$. Next, since $\lim_{x \rightarrow 0} \bar{q}(x) = 1$, one can choose ε small enough so that the supports are contained in a ball of arbitrarily small radius δ , which in turn implies that $\tilde{\mathbb{P}} \left(Z_t^{(\varepsilon)} \geq v/2 \right) = O(t^2)$ and $\tilde{\mathbb{P}} \left(\tilde{U}_t^{(\varepsilon)} \geq v/2 \right) = O(t^2)$, by taking $\delta > 0$ small enough. For the second term, first from (2.7), there exists $\varepsilon_0 > 0$ such that, for all $0 < \varepsilon < \varepsilon_0$,

$$\begin{aligned} a_0(v) &:= \lambda_\varepsilon \tilde{\mathbb{P}} \left(\left(\xi_1^{(\varepsilon)} \right)^+ - \ln \bar{q}^* \left(\xi_1^{(\varepsilon)} \right) \geq v \right) = \int_{\mathbb{R}_0} \mathbf{1}_{\{x^+ - \ln \bar{q}^*(x) \geq v\}} \tilde{\nu}(dx) = \int_{\mathbb{R}_0} \mathbf{1}_{\{x - \ln \bar{q}(x) \geq v\}} \tilde{\nu}(dx) \\ &= C_+ \int_0^\infty \mathbf{1}_{\{-\ln \bar{q}(x) \geq v\}} x^{-Y-1} dx + C_- \int_{-\infty}^0 \mathbf{1}_{\{-x - \ln \bar{q}(x) \geq v\}} |x|^{-Y-1} dx, \end{aligned}$$

where for the second equality recall that $\xi_i^{(\varepsilon)}$ has distribution $\mathbf{1}_{\{|x| \geq \varepsilon\}} \tilde{\nu}(dx) / \lambda_\varepsilon$. Also, since

$$F_{v,\varepsilon}(z, u) := \tilde{\mathbb{P}} \left(\left(z + \xi_1^{(\varepsilon)} \right)^+ + u - \ln \bar{q}^* \left(\xi_1^{(\varepsilon)} \right) \geq v \right),$$

is continuous at $(0, 0)$, for any fixed $0 < \varepsilon < \varepsilon_0$, the function

$$A_t(v) := t^{-1} \tilde{\mathbb{P}} \left(Z_t^+ + \tilde{U}_t \geq v \right) - a_0(v),$$

is such that

$$\begin{aligned} \lim_{t \rightarrow 0} A_t(v) &= \lambda_\varepsilon \lim_{t \rightarrow 0} \left[\tilde{\mathbb{P}} \left(\left(Z_t^{(\varepsilon)} + \xi_1^{(\varepsilon)} \right)^+ + \tilde{U}_t^{(\varepsilon)} - \ln \bar{q}^* \left(\xi_1^{(\varepsilon)} \right) \geq v \right) - \lambda_\varepsilon \tilde{\mathbb{P}} \left(\left(\xi_1^{(\varepsilon)} \right)^+ - \ln \bar{q}^* \left(\xi_1^{(\varepsilon)} \right) \geq v \right) \right] \\ &= \lambda_\varepsilon \lim_{t \rightarrow 0} \tilde{\mathbb{E}} \left[F_{v,\varepsilon} \left(Z_t^{(\varepsilon)}, \tilde{U}_t^{(\varepsilon)} \right) - F_{v,\varepsilon}(0, 0) \right] \\ &= \lambda_\varepsilon \tilde{\mathbb{E}} \left[\lim_{t \rightarrow 0} F_{v,\varepsilon} \left(Z_t^{(\varepsilon)}, \tilde{U}_t^{(\varepsilon)} \right) - F_{v,\varepsilon}(0, 0) \right] = 0, \end{aligned}$$

where dominated convergence is used to obtain the last equality.

(2) Throughout this part, $\tilde{\kappa} > 0$ denotes a generic finite constant that may vary from line to line. First, note that (2.19) implies that

$$t^{-1}\tilde{\mathbb{P}}\left(Z_t^+ \geq v\right) \leq \tilde{\kappa}v^{-Y}, \quad (\text{A.8})$$

for any $0 < t \leq 1$ and $v > 0$. So, it suffices to show the analog inequality for $(\tilde{U}_t)_{t \geq 0}$. To this end, use the following “small/large jumps decomposition” of \tilde{U} :

$$\tilde{U}_t^{(\varepsilon)} := \int_0^t \int_{|\varphi(x)| \geq \varepsilon} \varphi(x) N(ds, dx), \quad \tilde{U}_t^{(\varepsilon)} := \tilde{U}_t - \tilde{U}_t^{(\varepsilon)}, \quad t \geq 0, \quad (\text{A.9})$$

with $\varepsilon := v/4$. Note that, in view of both (2.4) and (2.10-ii), there exists a constant $r < \infty$ such that

$$|\varphi(x)| = |\ln \bar{q}^*(x)| \leq r|x|, \quad (\text{A.10})$$

for all $x \neq 0$. In particular, $\tilde{U}^{(\varepsilon)}$ in (A.9) is a compound Poisson process with intensity of jumps $\lambda_\varepsilon := \tilde{\nu}(|\varphi(x)| \geq \varepsilon)$ and, thus, denoting the counting process of the jumps of $\tilde{U}^{(\varepsilon)}$ by $(N_t^{(\varepsilon)})_{t \geq 0}$,

$$t^{-1}\tilde{\mathbb{P}}\left(\tilde{U}_t^{(\varepsilon)} \geq \frac{v}{2}\right) \leq t^{-1}\tilde{\mathbb{P}}\left(N_t^{(\varepsilon)} \neq 0\right) = t^{-1}(1 - e^{-\lambda_\varepsilon t}) \leq \lambda_\varepsilon \leq \bar{C} \int_{|\varphi(x)| \geq \varepsilon} |x|^{-Y-1} dx,$$

where $\bar{C} := C_+ + C_-$. Next, using (A.10) and $\varepsilon = v/4$,

$$t^{-1}\tilde{\mathbb{P}}\left(\tilde{U}_t^{(\varepsilon)} \geq \frac{v}{2}\right) \leq \bar{C} \int_{|x| \geq \frac{v}{4r}} |x|^{-Y-1} dx \leq \tilde{\kappa}v^{-Y}, \quad (\text{A.11})$$

for some constant $\tilde{\kappa}$. Also,

$$\begin{aligned} \tilde{\mathbb{E}}\left(\tilde{U}_t^{(\varepsilon)}\right) &= -t \int_{|\varphi(x)| \geq \varepsilon} \varphi(x) \tilde{\nu}(dx) \\ &= t \int_{|\varphi(x)| \geq \varepsilon} \ln \bar{q}(x) \tilde{\nu}(dx) + tC_- \int_{|\varphi(x)| \geq \varepsilon, x < 0} x|x|^{-Y-1} dx + tC_+ \int_{|\varphi(x)| \geq \varepsilon, x > 0} x|x|^{-Y-1} dx, \end{aligned}$$

and, since $\bar{q}(x) \leq 1$ for all $x \in \mathbb{R} \setminus \{0\}$ (from (2.3)),

$$\tilde{\mathbb{E}}\left(\tilde{U}_t^{(\varepsilon)}\right) \leq tC_+ \int_{x \geq \frac{\varepsilon}{r}} x^{-Y} dx = C_+(Y-1)^{-1}(4r)^{Y-1}tv^{1-Y}.$$

Thus, whenever t and v are such that $C_+(Y-1)^{-1}(4r)^{Y-1}tv^{1-Y} \leq v/4$, we have

$$\tilde{\mathbb{P}}\left(\tilde{U}_t^{(\varepsilon)} \geq \frac{v}{2}\right) = \tilde{\mathbb{P}}\left(\tilde{U}_t^{(\varepsilon)} - \tilde{\mathbb{E}}\left(\tilde{U}_t^{(\varepsilon)}\right) + \tilde{\mathbb{E}}\left(\tilde{U}_t^{(\varepsilon)}\right) \geq \frac{v}{2}\right) \leq \tilde{\mathbb{P}}\left(\tilde{U}_t^{(\varepsilon)} - \tilde{\mathbb{E}}\left(\tilde{U}_t^{(\varepsilon)}\right) \geq \frac{v}{4}\right).$$

Next, by a concentration inequality for centered random variables (see (Houdré, 2002, Corollary 1)) together with the identity $e^{-x \log(1+x)} \mathbf{1}_{\{x>0\}} \leq x^{-1} \mathbf{1}_{\{x>0\}}$,

$$\tilde{\mathbb{P}}\left(\tilde{U}_t^{(\varepsilon)} \geq \frac{v}{2}\right) \leq \tilde{\mathbb{P}}\left(\tilde{U}_t^{(\varepsilon)} - \tilde{\mathbb{E}}\left(\tilde{U}_t^{(\varepsilon)}\right) \geq \frac{v}{4}\right) \leq e^{\frac{v}{4\varepsilon} - \left(\frac{v}{4\varepsilon} + \frac{tV_\varepsilon^2}{\varepsilon^2}\right) \log\left(1 + \frac{\varepsilon v}{4tV_\varepsilon^2}\right)} \leq \left(\frac{4eV_\varepsilon^2}{\varepsilon v}\right)^{\frac{v}{4\varepsilon}} t^{\frac{v}{4\varepsilon}} \leq \frac{16eV_\varepsilon^2}{v^2} t,$$

where $V_\varepsilon^2 := \text{Var}\left(\tilde{U}_1^{(\varepsilon)}\right)$ and, in the last inequality, $\varepsilon = v/4$. Now, using again (A.10) and setting $\bar{C} := C_+ + C_-$,

$$\begin{aligned} V_{v/4}^2 &= \int_{|\varphi(x)| < \frac{v}{4}} (\varphi(x))^2 \tilde{\nu}(dx) \\ &\leq \bar{C} \int_{|\varphi(x)| < \frac{v}{4}, |x| < \frac{v}{4}} (\varphi(x))^2 |x|^{-Y-1} dx + \bar{C} \int_{|\varphi(x)| < \frac{v}{4}, |x| \geq \frac{v}{4}} (\varphi(x))^2 |x|^{-Y-1} dx \\ &\leq \bar{C} r^2 \int_{|x| < \frac{v}{4}} |x|^{1-Y} dx + \frac{v^2}{16} \bar{C} \int_{|x| \geq \frac{v}{4}} |x|^{-Y-1} dx \\ &\leq \tilde{\kappa}v^{2-Y}, \end{aligned}$$

for some $0 < \tilde{\kappa} < \infty$. Therefore, whenever $C_+(Y-1)^{-1}(4r)^{Y-1}tv^{1-Y} \leq v/4$ (or equivalently, $C_+(Y-1)^{-1}r^{Y-1}4^Y tv^{-Y} \leq 1$),

$$t^{-1}\tilde{\mathbb{P}}\left(\tilde{U}_t^{(\varepsilon)} \geq \frac{v}{2}\right) \leq \frac{eV_{v/4}^2}{v^2} \leq \tilde{\kappa}v^{-Y},$$

for some $0 < \tilde{\kappa} < \infty$. Moreover, for any $t > 0$ and $v > 0$,

$$\begin{aligned} t^{-1}\tilde{\mathbb{P}}\left(\tilde{U}_t^{(\varepsilon)} \geq \frac{v}{2}\right) &= t^{-1}\tilde{\mathbb{P}}\left(\tilde{U}_t^{(\varepsilon)} \geq \frac{v}{2}\right) \mathbf{1}_{\{C_+(Y-1)^{-1}r^{Y-1}4^Y tv^{-Y} \leq 1\}} + t^{-1}\tilde{\mathbb{P}}\left(\tilde{U}_t^{(\varepsilon)} \geq \frac{v}{2}\right) \mathbf{1}_{\{C_+(Y-1)^{-1}r^{Y-1}4^Y tv^{-Y} > 1\}} \\ &\leq \tilde{\kappa}v^{-Y} + \frac{C_+}{t}(Y-1)^{-1}r^{Y-1}4^Y tv^{-Y} \leq \tilde{\kappa}'v^{-Y}, \end{aligned} \quad (\text{A.12})$$

for some constant $0 < \tilde{\kappa}' < \infty$. Combining (A.11) and (A.12), finally leads to

$$t^{-1}\tilde{\mathbb{P}}\left(\tilde{U}_t \geq v\right) \leq t^{-1}\tilde{\mathbb{P}}\left(\tilde{U}_t^{(\varepsilon)} \geq \frac{v}{2}\right) + t^{-1}\tilde{\mathbb{P}}\left(\tilde{U}_t^{(\varepsilon)} \geq \frac{v}{2}\right) \leq \tilde{\kappa}v^{-Y}, \quad (\text{A.13})$$

for all $v > 0$ and $t > 0$ and some constant $0 < \tilde{\kappa} < \infty$. □

Proof of Theorem 3.4.

For simplicity, the case $\tilde{\gamma} = 0$ is treated first, so that, in light of Lemma 3.1,

$$t^{-\frac{1}{Y}}\mathbb{E}\left[(S_t - S_0)^+\right] = e^{-\eta t} \int_0^\infty e^{-t^{\frac{1}{Y}}v} \tilde{\mathbb{E}}\left(e^{-\tilde{U}_t} \mathbf{1}_{\{t^{-\frac{1}{Y}}Z_t \geq v\}}\right) dv. \quad (\text{A.14})$$

The general case is resolved in Lemma A.1 below. Let

$$D(t) := t^{-\frac{1}{Y}}\mathbb{E}\left[(S_t - S_0)^+\right] - \tilde{\mathbb{E}}(Z_1^+),$$

which can be written as

$$\begin{aligned} D(t) &:= \left[\int_0^\infty e^{-t^{\frac{1}{Y}}v} \tilde{\mathbb{E}}\left(e^{-\tilde{U}_t} \mathbf{1}_{\{t^{-\frac{1}{Y}}Z_t \geq v\}}\right) dv - \tilde{\mathbb{E}}(Z_1^+) \right] + (e^{-\eta t} - 1) \tilde{\mathbb{E}}(Z_1^+) \\ &\quad + (e^{-\eta t} - 1) \left[\int_0^\infty e^{-t^{\frac{1}{Y}}v} \tilde{\mathbb{E}}\left(e^{-\tilde{U}_t} \mathbf{1}_{\{t^{-\frac{1}{Y}}Z_t \geq v\}}\right) dv - \tilde{\mathbb{E}}(Z_1^+) \right] \\ &=: D_1(t) + D_2(t) + D_3(t). \end{aligned} \quad (\text{A.15})$$

Let us show that

$$t^{\frac{1}{Y}-1}D_1(t) \rightarrow \vartheta + \eta =: \tilde{\vartheta}, \quad \text{as } t \rightarrow 0, \quad (\text{A.16})$$

for a certain constant ϑ , while it is clear that $D_3(t) = o(D_1(t))$ and that $t^{1/Y-1}D_2(t) = o(1)$, as $t \rightarrow 0$. First, note that

$$\begin{aligned} D_1(t) &= \tilde{\mathbb{E}}\left(e^{-\tilde{U}_t} \int_0^{t^{-\frac{1}{Y}}Z_t^+} e^{-t^{\frac{1}{Y}}v} dv\right) - \tilde{\mathbb{E}}(Z_1^+) \\ &= t^{-\frac{1}{Y}} \left[\tilde{\mathbb{E}}\left(e^{-\tilde{U}_t}\right) - \tilde{\mathbb{E}}\left(e^{-(\tilde{U}_t + Z_t^+)}\right) \right] - \tilde{\mathbb{E}}(Z_1^+) \\ &= t^{-\frac{1}{Y}} \left[e^{\eta t} - \tilde{\mathbb{E}}\left(e^{-(\tilde{U}_t + Z_t^+)}\right) \right] - \tilde{\mathbb{E}}(Z_1^+), \end{aligned} \quad (\text{A.17})$$

where to obtain the last equality use (2.23). Next, by the self-similarity of $(Z_t)_{t \geq 0}$ under $\tilde{\mathbb{P}}$ (see (2.15)), $\tilde{\mathbb{E}}(Z_1^+) =$

$t^{-1/Y} \tilde{\mathbb{E}}(Z_t^+)$ and, since $\tilde{\mathbb{E}}(\tilde{U}_t) = 0$,

$$\begin{aligned}
t^{\frac{1}{Y}-1} D_1(t) &= t^{\frac{1}{Y}-1} \left[\frac{e^{\eta t} - 1}{t^{\frac{1}{Y}}} + \frac{1 - \tilde{\mathbb{E}}(e^{-(Z_t^+ + \tilde{U}_t)}) - \tilde{\mathbb{E}}(Z_t^+ + \tilde{U}_t)}{t^{\frac{1}{Y}}} \right] \\
&= \frac{e^{\eta t} - 1}{t} + t^{-1} \tilde{\mathbb{E}} \left(\int_0^{Z_t^+ + \tilde{U}_t} (e^{-v} - 1) dv \mathbf{1}_{\{Z_t^+ + \tilde{U}_t \geq 0\}} \right) - t^{-1} \tilde{\mathbb{E}} \left(\int_{Z_t^+ + \tilde{U}_t}^0 (e^{-v} - 1) dv \mathbf{1}_{\{Z_t^+ + \tilde{U}_t \leq 0\}} \right) \\
&= \frac{e^{\eta t} - 1}{t} + t^{-1} \int_0^\infty (e^{-v} - 1) \tilde{\mathbb{P}}(Z_t^+ + \tilde{U}_t \geq v) dv - t^{-1} \int_0^\infty (e^v - 1) \tilde{\mathbb{P}}(Z_t^+ + \tilde{U}_t \leq -v) dv \\
&=: D_{11}(t) + D_{12}(t) + D_{13}(t). \tag{A.18}
\end{aligned}$$

Clearly,

$$D_{11}(t) \rightarrow \eta, \quad \text{as } t \rightarrow 0. \tag{A.19}$$

For any $t \geq 0$, let $Q_t := Z_t + \tilde{U}_t$ and note that $Q_t \leq Z_t^+ + \tilde{U}_t$. Then, for $D_{13}(t)$, using respectively $e^y - 1 \leq ye^y$, $y > 0$, Markov's inequality, and since $Y < 2$,

$$\begin{aligned}
0 \leq D_{13}(t) &\leq t^{\frac{1}{Y}-1} \int_0^\infty \left(e^{t^{\frac{1}{Y}} u} - 1 \right) \tilde{\mathbb{P}}(t^{-\frac{1}{Y}} Q_t \leq -u) du \\
&\leq t^{\frac{2}{Y}-1} \int_0^\infty e^{t^{\frac{1}{Y}} u} u \tilde{\mathbb{P}}(t^{-\frac{1}{Y}} Q_t \leq -u) du \\
&\leq t^{\frac{2}{Y}-1} \int_0^\infty e^{(t^{\frac{1}{Y}} - 1)u} u du \cdot \tilde{\mathbb{E}} \left(e^{-t^{-\frac{1}{Y}} Q_t} \right) \rightarrow 0, \quad t \rightarrow 0, \tag{A.20}
\end{aligned}$$

where in the last step we applied (3.2-i).

For D_{12} , from (3.4-ii), by dominated convergence, one passes the limit inside the integrals so that

$$\lim_{t \rightarrow 0} D_{12}(t) = \int_0^\infty (e^{-v} - 1) \lim_{t \rightarrow 0} \left[t^{-1} \tilde{\mathbb{P}}(Z_t^+ + \tilde{U}_t \geq v) \right] dv.$$

Then, from (3.3),

$$\lim_{t \rightarrow 0} D_{12}(t) = \int_0^\infty (e^{-v} - 1) \int_{\mathbb{R}_0} \mathbf{1}_{\{x - \ln \bar{q}(x) \geq v\}} \tilde{\nu}(dx) dv =: \vartheta. \tag{A.21}$$

Combining (A.18), (A.19), (A.20), and (A.21), it follows that

$$\lim_{t \rightarrow 0} t^{\frac{1}{Y}-1} \left(t^{-\frac{1}{Y}} \mathbb{E} \left[(S_t - S_0)^+ \right] - \tilde{\mathbb{E}}(Z_1^+) \right) = \lim_{t \rightarrow 0} t^{\frac{1}{Y}-1} D_1(t) = \vartheta + \eta. \tag{A.22}$$

Finally, to get the expression in (3.6), recall that $\tilde{\nu}(dx) = |x|^{-Y-1} (C_+ \mathbf{1}_{\{x>0\}} + C_- \mathbf{1}_{\{x<0\}}) dx$ and, thus, applying Fubini's theorem to the right-hand side of (A.21) gives

$$\begin{aligned}
\vartheta &= C_+ \int_0^\infty (e^{-v} - 1) \left(\int_0^\infty \mathbf{1}_{\{-\ln \bar{q}(x) \geq v\}} x^{-Y-1} dx \right) dv + C_- \int_0^\infty (e^{-v} - 1) \left(\int_{-\infty}^0 \mathbf{1}_{\{-x - \ln \bar{q}(x) \geq v\}} |x|^{-Y-1} dx \right) dv \\
&= C_+ \int_0^\infty \left(\int_0^{-\ln \bar{q}(x)} (e^{-v} - 1) dv \right) x^{-Y-1} dx + C_- \int_{-\infty}^0 \left(\int_0^{-x - \ln \bar{q}(x)} (e^{-v} - 1) dv \right) |x|^{-Y-1} dx \\
&= C_+ \int_0^\infty \left(1 - e^{\ln \bar{q}(x)} + \ln \bar{q}(x) \right) x^{-Y-1} dx + C_- \int_{-\infty}^0 \left(1 - e^{x + \ln \bar{q}(x)} + x + \ln \bar{q}(x) \right) |x|^{-Y-1} dx.
\end{aligned}$$

One can similarly show that the constant η defined in (2.22) can be written as:

$$\eta = C_+ \int_0^\infty \left(e^{x + \ln \bar{q}(x)} - 1 - \ln \bar{q}(x) - x \right) x^{-Y-1} dx + C_- \int_{-\infty}^0 \left(e^{x + \ln \bar{q}(x)} - 1 - \ln \bar{q}(x) - x \right) |x|^{-Y-1} dx.$$

Combining the expressions for ϑ and η yields (3.6). The expression for $\tilde{\gamma}$ in (3.7) follows from

$$\tilde{\gamma} = \tilde{\mathbb{E}}(L_1^*) = \tilde{b} + \int_{\{|x|>1\}} x \tilde{\nu}(dx) = b^* + \int_{|x|\leq 1} x (\tilde{\nu} - \nu^*)(dx) + \int_{\{|x|>1\}} x \tilde{\nu}(dx),$$

and standard simplifications. This concludes the proof. \square

Lemma A.1. *If $\tilde{\gamma} \neq 0$ in (3.1), then*

$$\lim_{t \rightarrow 0} t^{\frac{1}{Y}-1} \left(t^{-\frac{1}{Y}} \frac{1}{S_0} \mathbb{E} \left[(S_t - S_0)^+ \right] - S_0 \tilde{\mathbb{E}}(Z_1^+) \right) = \tilde{\vartheta} + \tilde{\gamma} \tilde{\mathbb{P}}(Z_1 \geq 0). \quad (\text{A.23})$$

Proof. Without loss of generality, fix $S_0 = 1$, and also assume that $\tilde{\gamma} > 0$ (the case $\tilde{\gamma} < 0$ being similar). Using (3.1),

$$\begin{aligned} t^{\frac{1}{Y}-1} \left(t^{-\frac{1}{Y}} \mathbb{E} \left[(S_t - S_0)^+ \right] - \tilde{\mathbb{E}}(Z_1^+) \right) &= t^{\frac{1}{Y}-1} \left[e^{-(\tilde{\gamma}+\eta)t} \int_0^\infty e^{-t^{\frac{1}{Y}} v} \tilde{\mathbb{E}} \left(e^{-\tilde{U}_t} \mathbf{1}_{\{t^{-\frac{1}{Y}} Z_t \geq v\}} \right) dv - \tilde{\mathbb{E}}(Z_1^+) \right] \\ &\quad + t^{\frac{1}{Y}-1} e^{-(\tilde{\gamma}+\eta)t} \int_{-\tilde{\gamma}t^{1-\frac{1}{Y}}}^0 e^{-t^{\frac{1}{Y}} v} \tilde{\mathbb{E}} \left(e^{-\tilde{U}_t} \mathbf{1}_{\{t^{-\frac{1}{Y}} Z_t \geq v\}} \right) dv \\ &=: \tilde{D}_{11}(t) + \tilde{D}_{12}(t). \end{aligned}$$

As in the proof of (A.16), it can be shown that

$$\lim_{t \rightarrow 0} \tilde{D}_{11}(t) = \tilde{\vartheta}. \quad (\text{A.24})$$

For $\tilde{D}_{12}(t)$, changing variables to $u = t^{1/Y-1}v$ and probability measure to \mathbb{P}^* , we have

$$\tilde{D}_{12}(t) = e^{-\tilde{\gamma}t} \int_{-\tilde{\gamma}}^0 e^{-tu} \tilde{\mathbb{E}} \left(e^{-(\tilde{U}_t + \eta t)} \mathbf{1}_{\{Z_t \geq tu\}} \right) du = e^{-\tilde{\gamma}t} \int_{-\tilde{\gamma}}^0 e^{-tu} \mathbb{P}^* \left(t^{-\frac{1}{Y}} Z_t \geq t^{1-\frac{1}{Y}} u \right) du.$$

Next, recall from Section 2 that, under \mathbb{P}^* , $(L_t^*)_{t \geq 0}$ is a Lévy process with Lévy measure ν^* given by

$$\nu^*(dx) := e^x \nu(dx) = e^x s(x) dx = q^*(x) |x|^{-Y-1} dx.$$

In particular, $\lim_{x \searrow 0} q^*(x) = C_+$ and $\lim_{x \nearrow 0} q^*(x) = C_-$ and, since $1 < Y < 2$, the assumptions of (Rosenbaum and Tankov, 2011, Proposition 1) are satisfied. Therefore, both $t^{-1/Y} L_t^*$ and $t^{-1/Y} Z_t$ converge in distribution to a Y -stable random variable \tilde{Z} under \mathbb{P}^* with center (or mean) 0 and Lévy measure $|x|^{-Y-1} (C_+ \mathbf{1}_{\{x>0\}} + C_- \mathbf{1}_{\{x<0\}}) dx$. Hence, the distribution of \tilde{Z} (under \mathbb{P}^*) is the same as the distribution of Z_1 under $\tilde{\mathbb{P}}$. Thus, Slutsky's lemma implies that $t^{-1/Y} Z_t - t^{1-1/Y} u \xrightarrow{\mathcal{D}} \tilde{Z}$ and, thus,

$$\lim_{t \rightarrow 0} \mathbb{P}^* \left(t^{-\frac{1}{Y}} Z_t - t^{1-\frac{1}{Y}} u \geq 0 \right) = \mathbb{P}^* \left(\tilde{Z} \geq 0 \right) = \tilde{\mathbb{P}}(Z_1 \geq 0).$$

Finally, by the dominated convergence theorem,

$$\lim_{t \rightarrow 0} \tilde{D}_{12}(t) = \tilde{\gamma} \tilde{\mathbb{P}}(Z_1 \geq 0). \quad (\text{A.25})$$

Combining (A.25) with (A.24) leads to (A.23). \square

Proof of Corollary 3.7.

The small-time asymptotic behavior of the ATM call option price $C_{BS}(t, \sigma)$ at maturity t under the Black-Scholes model with volatility σ and zero interest rates is given by (e.g., see (Forde et al., 2012, Corollary 3.4) and recall also that $S_0 = 1$)

$$C_{BS}(t, \sigma) = \frac{\sigma}{\sqrt{2\pi}} t^{\frac{1}{2}} - \frac{\sigma^3}{24\sqrt{2\pi}} t^{\frac{3}{2}} + O\left(t^{\frac{5}{2}}\right), \quad t \rightarrow 0. \quad (\text{A.26})$$

To derive the small-time asymptotics for the implied volatility, a result analogous to (A.26) is needed when σ is replaced by $\hat{\sigma}(t)$. The following representation taken from (Roper and Rutkowski, 2007, Lemma 3.1) is useful,

$$C_{BS}(t, \sigma) = F(\sigma\sqrt{t}) \quad \text{with} \quad F(\theta) := \int_0^\theta \Phi' \left(\frac{v}{2} \right) dv = \frac{1}{\sqrt{2\pi}} \int_0^\theta \exp \left(-\frac{v^2}{8} \right) dv,$$

together with the Taylor expansion for F at $\theta = 0$ (see (Roper and Rutkowski, 2007, Lemma 5.1)), i.e.,

$$F(\theta) = \frac{1}{\sqrt{2\pi}}\theta - \frac{1}{24\sqrt{2\pi}}\theta^3 + O(\theta^5), \quad \theta \rightarrow 0.$$

Then, since $\hat{\sigma}(t) \rightarrow 0$ as $t \rightarrow 0$ (see, e.g., (Tankov, 2010, Proposition 5)),

$$C_{BS}(t, \hat{\sigma}(t)) = \frac{\hat{\sigma}(t)}{\sqrt{2\pi}}t^{\frac{1}{2}} - \frac{\hat{\sigma}(t)^3}{24\sqrt{2\pi}}t^{\frac{3}{2}} + O \left(\left(\hat{\sigma}(t)t^{\frac{1}{2}} \right)^5 \right), \quad \text{as } t \rightarrow 0. \quad (\text{A.27})$$

Returning to the proof of Proposition 3.7, equating (3.10) and (A.27) gives

$$\frac{d_1 t^{\frac{1}{Y}}}{\frac{\hat{\sigma}(t)}{\sqrt{2\pi}}t^{\frac{1}{2}}} = \frac{1 + O(t^{1-1/Y})}{1 + O \left(\left(\hat{\sigma}(t)t^{\frac{1}{2}} \right)^2 \right)},$$

showing that

$$\hat{\sigma}(t) \sim \sqrt{2\pi} d_1 t^{\frac{1}{Y} - \frac{1}{2}}, \quad t \rightarrow 0. \quad (\text{A.28})$$

Next, using (A.28) and setting $\tilde{\sigma}(t) := \hat{\sigma}(t) - \sqrt{2\pi} d_1 t^{\frac{1}{Y} - \frac{1}{2}}$, rewrite (A.27) as follows:

$$C_{BS} = d_1 t^{\frac{1}{Y}} + \frac{\tilde{\sigma}(t)}{\sqrt{2\pi}}t^{\frac{1}{2}} + O \left(t^{\frac{3}{Y}} \right).$$

Equating (3.10) and the previous expression leads to

$$d_2 t + o(t) = \frac{\tilde{\sigma}(t)}{\sqrt{2\pi}}t^{\frac{1}{2}} + O \left(t^{\frac{3}{Y}} \right), \quad (\text{A.29})$$

which, together with the fact that the second-term on the right of (A.29) is $o(t)$, implies that

$$d_2 t \sim \frac{\tilde{\sigma}(t)}{\sqrt{2\pi}}\sqrt{t}, \quad t \rightarrow 0.$$

Therefore, $\tilde{\sigma}(t) \rightarrow 0$ as $t \rightarrow 0$ and, moreover, recalling the definition of d_2 ,

$$\tilde{\sigma}(t) \sim \sqrt{2\pi} \left(\tilde{\vartheta} + \tilde{\gamma} \tilde{\mathbb{P}}(Z_1 \geq 0) \right) \sqrt{t}, \quad t \rightarrow 0. \quad (\text{A.30})$$

Combining (A.28) and (A.30) finishes the proof. \square

B Proofs of Section 4: The Pure-Jump Model With A Nonzero Brownian Component

Proof of Theorem 4.1.

For simplicity, fix $S_0 = 1$. Recalling that $X_t = \sigma W_t^* + L_t^*$ under \mathbb{P}^* , and using (1.16), the self-similarity of W^* , and the change of variable $u = t^{-1/2}x$,

$$R_t := t^{-\frac{1}{2}} \mathbb{E} \left[(S_t - S_0)^+ \right] - \sigma \mathbb{E}^* (W_1^*)^+ = \int_0^\infty e^{-\sqrt{t}u} \mathbb{P}^* \left(\sigma W_1^* \geq u - t^{-\frac{1}{2}} L_t^* \right) du - \int_0^\infty \mathbb{P}^* (\sigma W_1^* \geq u) du.$$

Next, changing the probability measure to $\tilde{\mathbb{P}}$, using that $L_t^* = Z_t + \tilde{\gamma}t$, $U_t = \tilde{U}_t + \eta t$, and the change of variable $y = u - t^{1/2}\tilde{\gamma}$ in the first integral above, lead to

$$\begin{aligned} R_t &= \int_{-\sqrt{t}\tilde{\gamma}}^{\infty} e^{-\sqrt{t}y - \tilde{\gamma}t} \tilde{\mathbb{E}} \left(e^{-\tilde{U}_t - \eta t} \mathbf{1}_{\{\sigma W_1^* \geq y - t^{-\frac{1}{2}}Z_t\}} \right) dy - \int_0^{\infty} \tilde{\mathbb{E}} \left(e^{-\tilde{U}_t - \eta t} \mathbf{1}_{\{\sigma W_1^* \geq u\}} \right) du \\ &= e^{-(\eta + \tilde{\gamma})t} \int_0^{\infty} e^{-\sqrt{t}y} \left[\tilde{\mathbb{E}} \left(e^{-\tilde{U}_t} \mathbf{1}_{\{\sigma W_1^* \geq y - t^{-\frac{1}{2}}Z_t\}} \right) - \tilde{\mathbb{E}} \left(e^{-\tilde{U}_t} \mathbf{1}_{\{\sigma W_1^* \geq y\}} \right) \right] dy \\ &\quad + e^{-(\eta + \tilde{\gamma})t} \int_{-\sqrt{t}\tilde{\gamma}}^0 e^{-\sqrt{t}y} \tilde{\mathbb{E}} \left(e^{-\tilde{U}_t} \mathbf{1}_{\{\sigma W_1^* \geq y - t^{-\frac{1}{2}}Z_t\}} \right) dy + \int_0^{\infty} \left(e^{-\tilde{\gamma}t - \sqrt{t}y} - 1 \right) \mathbb{P}^* (\sigma W_1^* \geq y) dy. \end{aligned} \quad (\text{B.1})$$

Above, the last term is clearly $O(t^{1/2})$ as $t \rightarrow 0$, while the middle term can be shown to be asymptotically equivalent to a term that is also $O(t^{1/2})$ by arguments analogous to those of (A.25). Thus, only the first term in (B.1), which we hereafter denote by A_t , needs to be studied. Setting $\tilde{\eta} := \eta + \tilde{\gamma}$, this term can further be expressed as:

$$A_t = e^{-\tilde{\eta}t} \tilde{\mathbb{E}} \left[e^{-\tilde{U}_t} \int_0^{\infty} e^{-\sqrt{t}y} \left(\mathbf{1}_{\{\sigma W_1^* \geq y - t^{-\frac{1}{2}}Z_t\}} - \mathbf{1}_{\{\sigma W_1^* \geq y\}} \right) dy \right],$$

To study the asymptotic behavior of A_t , decompose it into the following three parts:

$$\begin{aligned} A_t &= e^{-\tilde{\eta}t} \tilde{\mathbb{E}} \left(e^{-\tilde{U}_t} \mathbf{1}_{\{W_1^* \geq 0, \sigma W_1^* + t^{-\frac{1}{2}}Z_t \geq 0\}} \int_{\sigma W_1^*}^{\sigma W_1^* + t^{-\frac{1}{2}}Z_t} e^{-\sqrt{t}y} dy \right) - e^{-\tilde{\eta}t} \tilde{\mathbb{E}} \left(e^{-\tilde{U}_t} \mathbf{1}_{\{0 \leq \sigma W_1^* \leq t^{-\frac{1}{2}}Z_t\}} \int_0^{\sigma W_1^*} e^{-\sqrt{t}y} dy \right) \\ &\quad + e^{-\tilde{\eta}t} \tilde{\mathbb{E}} \left(e^{-\tilde{U}_t} \mathbf{1}_{\{0 \leq -\sigma W_1^* \leq t^{-\frac{1}{2}}Z_t\}} \int_0^{\sigma W_1^* + t^{-\frac{1}{2}}Z_t} e^{-\sqrt{t}y} dy \right) \\ &=: I_1(t) - I_2(t) + I_3(t). \end{aligned} \quad (\text{B.2})$$

Each of these terms is analyzed in the following three steps:

Step 1. Since $(Z_t)_{t \geq 0}$ and $(W_t^*)_{t \geq 0}$ are independent,

$$\begin{aligned} I_1(t) &= e^{-\tilde{\eta}t} \tilde{\mathbb{E}} \left(\mathbf{1}_{\{W_1^* \geq 0, \sigma W_1^* + t^{-\frac{1}{2}}Z_t \geq 0\}} \frac{e^{-\tilde{U}_t} - e^{-(\tilde{U}_t + Z_t)}}{\sqrt{t}} e^{-\sqrt{t}\sigma W_1^*} \right) \\ &= e^{-\tilde{\eta}t} \int_0^{\infty} \tilde{\mathbb{E}} \left(\mathbf{1}_{\{Z_t \geq -t^{\frac{1}{2}}y\}} \frac{e^{-\tilde{U}_t} (1 - e^{-Z_t})}{\sqrt{t}} \right) e^{-\sqrt{t}y} \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy \\ &=: e^{-\tilde{\eta}t} \int_0^{\infty} J_1(t, y) e^{-\sqrt{t}y} \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy. \end{aligned} \quad (\text{B.3})$$

Using the self-similarity of $(Z_t)_{t \geq 0}$ and since $\tilde{\mathbb{E}}Z_t = 0$, $J_1(t, y)$ is then decomposed as:

$$\begin{aligned} J_1(t, y) &= \tilde{\mathbb{E}} \left[\mathbf{1}_{\{Z_t \geq -t^{\frac{1}{2}}y\}} \left(\frac{e^{-\tilde{U}_t} - e^{-(\tilde{U}_t + Z_t)}}{\sqrt{t}} - t^{-\frac{1}{2}}Z_t \right) \right] + t^{\frac{1}{Y} - \frac{1}{2}} \tilde{\mathbb{E}} \left[(-Z_1) \mathbf{1}_{\{-Z_1 \geq t^{\frac{1}{2} - \frac{1}{Y}}y\}} \right] \\ &=: J_{11}(t, y) + J_{12}(t, y). \end{aligned} \quad (\text{B.4})$$

Let us first consider $J_{12}(t, y)$. From (2.16)-(2.18), there exists a constant $\lambda > 0$ such that

$$t^{\frac{Y}{2} - 1} J_{12}(t, y) \leq \lambda y^{1-Y}, \quad (\text{B.5})$$

for any $0 < t \leq 1$ and $y > 0$ (see Appendix C for proof of this claim). Moreover, for any fixed $y > 0$,

$$t^{\frac{Y}{2} - 1} J_{12}(t, y) = t^{\frac{Y}{2} + \frac{1}{Y} - \frac{3}{2}} \int_{t^{\frac{1}{2} - \frac{1}{Y}}y}^{\infty} u p_Z(-u) du = t^{\frac{Y}{2} - \frac{1}{Y} - \frac{1}{2}} \int_y^{\infty} w p_Z \left(-t^{\frac{1}{2} - \frac{1}{Y}}w \right) dw.$$

Using (2.16), there exists $0 < t_0 < 1$ such that

$$t^{\frac{Y}{2} - \frac{1}{Y} - \frac{1}{2}} w p_Z \left(-t^{\frac{1}{2} - \frac{1}{Y}} w \right) \leq 2(C_+ \vee C_-) w^{-Y},$$

for any $0 < t < t_0$ and $w \geq y$. Therefore, by the dominated convergence theorem, and in light of (2.16),

$$\begin{aligned} \lim_{t \rightarrow 0} t^{\frac{Y}{2} - 1} e^{-\tilde{\eta}t} \int_0^\infty J_{12}(t, y) e^{-\sqrt{t}y} \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy &= \int_0^\infty \left(\lim_{t \rightarrow 0} t^{\frac{Y}{2} - 1} J_{12}(t, y) \right) \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy \\ &= \int_0^\infty \left[\int_y^\infty w \left(\lim_{t \rightarrow 0} t^{\frac{Y}{2} - \frac{1}{Y} - \frac{1}{2}} p_Z \left(-t^{\frac{1}{2} - \frac{1}{Y}} w \right) \right) dw \right] \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy \\ &= C_- \int_0^\infty \left(\int_y^\infty w^{-Y} dw \right) \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy \\ &= \frac{C_-}{Y-1} \int_0^\infty y^{1-Y} \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy. \end{aligned} \quad (\text{B.6})$$

For $J_{11}(t, y)$,

$$\begin{aligned} J_{11}(t, y) &= t^{-\frac{1}{2}} \tilde{\mathbb{E}} \left(\mathbf{1}_{\{Z_t \geq 0\}} \int_{\tilde{U}_t}^{\tilde{U}_t + Z_t} (e^{-x} - 1) dx \right) - t^{-\frac{1}{2}} \tilde{\mathbb{E}} \left(\mathbf{1}_{\{-t^{\frac{1}{2}}y \leq Z_t \leq 0\}} \int_{\tilde{U}_t + Z_t}^{\tilde{U}_t} (e^{-x} - 1) dx \right) \\ &= t^{-\frac{1}{2}} \int_{\mathbb{R}} (e^{-x} - 1) T_1(t, x, y) dx - t^{-\frac{1}{2}} \int_{\mathbb{R}} (e^{-x} - 1) T_2(t, x, y) dx, \end{aligned} \quad (\text{B.7})$$

where, for $t > 0$ and $y > 0$, we set

$$T_1(t, x, y) := \tilde{\mathbb{P}} \left(Z_t \geq 0, \tilde{U}_t \leq x \leq \tilde{U}_t + Z_t \right), \quad T_2(t, x, y) := \tilde{\mathbb{P}} \left(-t^{\frac{1}{2}}y \leq Z_t \leq 0, \tilde{U}_t + Z_t \leq x \leq \tilde{U}_t \right).$$

By (3.4-ii), there exists $0 < \tilde{\kappa} < \infty$ such that, for any $x > 0$ and $0 < t \leq 1$,

$$T_1(t, x, y) \leq \tilde{\mathbb{P}} \left(x \leq \tilde{U}_t + Z_t \right) \leq \tilde{\mathbb{P}} \left(x \leq \tilde{U}_t + Z_t^+ \right) \leq \tilde{\kappa} t x^{-Y}. \quad (\text{B.8})$$

Hence,

$$\begin{aligned} 0 &\leq e^{-\tilde{\eta}t} t^{\frac{Y}{2} - 1} \int_0^\infty \left(\int_0^\infty \frac{(1 - e^{-x})}{\sqrt{t}} T_1(t, x, y) dx \right) \frac{e^{-\sqrt{t}y} e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy \\ &\leq e^{-\tilde{\eta}t} \tilde{\kappa} t^{\frac{Y-1}{2}} \int_0^\infty \left(\int_0^\infty (1 - e^{-x}) x^{-Y} dx \right) \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy \rightarrow 0, \quad \text{as } t \rightarrow 0, \end{aligned} \quad (\text{B.9})$$

since $Y > 1$. Similarly, using (3.4-i), there exists a constant $0 < \tilde{\kappa} < \infty$ such that

$$T_2(t, x, y) \leq \tilde{\mathbb{P}} \left(\tilde{U}_t \geq x \right) \leq \tilde{\kappa} t x^{-Y}, \quad (\text{B.10})$$

for any $x > 0$ and $0 < t \leq 1$, and thus, as in (B.9),

$$\lim_{t \rightarrow 0} t^{\frac{Y}{2} - 1} e^{-\tilde{\eta}t} \int_0^\infty \left(\int_0^\infty \frac{(1 - e^{-x})}{\sqrt{t}} T_2(t, x, y) dx \right) \frac{e^{-\sqrt{t}y} e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy = 0. \quad (\text{B.11})$$

For $x < 0$, using (3.2-ii) and Markov's inequality, there exist $0 < t_0 < 1$ and $0 < \tilde{\kappa} < \infty$ such that

$$T_1(t, x, y) \leq \tilde{\mathbb{P}} \left(t^{-\frac{1}{Y}} \tilde{U}_t \leq t^{-\frac{1}{Y}} x \right) \leq \tilde{\mathbb{E}} \left(e^{-t^{-\frac{1}{Y}} \tilde{U}_t} \right) e^{t^{-\frac{1}{Y}} x} \leq \tilde{\kappa} e^{t^{-\frac{1}{Y}} x}, \quad (\text{B.12})$$

for any $0 < t \leq t_0$. Therefore,

$$0 \leq t^{\frac{Y-3}{2}} \int_{-\infty}^0 (e^{-x} - 1) T_1(t, x, y) dx \leq \tilde{\kappa} t^{\frac{Y-3}{2}} \int_{-\infty}^0 \left(e^{(t^{-\frac{1}{Y}} - 1)x} - e^{t^{-\frac{1}{Y}}x} \right) dx = \tilde{\kappa} t^{\frac{Y-3}{2}} \frac{t^{\frac{2}{Y}}}{1 - t^{\frac{1}{Y}}}.$$

Hence, by the dominated convergence theorem, as $t \rightarrow 0$,

$$0 \leq e^{-\tilde{\eta}t} t^{\frac{Y}{2}-1} \int_0^\infty \left(\int_{-\infty}^0 \frac{e^{-x} - 1}{\sqrt{t}} T_1(t, x, y) dx \right) \frac{e^{-\sqrt{t}y} e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy \leq \tilde{\kappa} t^{\frac{Y-3}{2}} \frac{t^{\frac{2}{Y}}}{1 - t^{\frac{1}{Y}}} \int_0^\infty \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy \rightarrow 0, \quad (\text{B.13})$$

since $2/Y > 1 > (3 - Y)/2$, for $1 < Y < 2$. Similarly, using (3.2-i), for $x < 0$,

$$T_2(t, x, y) \leq \tilde{\mathbb{P}}(\tilde{U}_t + Z_t \leq x) \leq \tilde{\mathbb{E}}\left(e^{-t^{-\frac{1}{Y}}(\tilde{U}_t + Z_t)}\right) e^{t^{-\frac{1}{Y}}x} \leq \tilde{\kappa} e^{t^{-\frac{1}{Y}}x}, \quad (\text{B.14})$$

for any $0 < t \leq t_0$ and some constant $0 < \tilde{\kappa} < \infty$. Therefore, as in (B.13),

$$\lim_{t \rightarrow 0} e^{-\tilde{\eta}t} t^{\frac{Y}{2}-1} \int_0^\infty \left(\int_{-\infty}^0 \frac{(1 - e^{-x})}{\sqrt{t}} T_2(t, x, y) dx \right) \frac{e^{-\sqrt{t}y} e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy = 0. \quad (\text{B.15})$$

Combining (B.6), (B.9), (B.11), (B.13) and (B.15) finally gives

$$\lim_{t \rightarrow 0} t^{\frac{Y}{2}-1} I_1(t) = \frac{C_-}{Y-1} \int_0^\infty y^{1-Y} \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy. \quad (\text{B.16})$$

Step 2. The asymptotic behavior of $I_2(t)$ is now studied. Using the independence of $(Z_t)_{t \geq 0}$ and $(W_t^*)_{t \geq 0}$,

$$\begin{aligned} I_2(t) &= e^{-\tilde{\eta}t} \tilde{\mathbb{E}} \left(e^{-\tilde{U}_t} \mathbf{1}_{\{0 \leq \sigma W_1^* \leq -t^{-\frac{1}{2}} Z_t\}} \frac{1 - e^{-\sqrt{t}\sigma W_1^*}}{\sqrt{t}} \right) \\ &= e^{-\tilde{\eta}t} \int_0^\infty \tilde{\mathbb{E}} \left(e^{-\tilde{U}_t} \mathbf{1}_{\{Z_t \leq -t^{\frac{1}{2}} y\}} \right) \frac{1 - e^{-\sqrt{t}y}}{\sqrt{t}} \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy \\ &= e^{-\tilde{\eta}t} \int_0^\infty \tilde{\mathbb{E}} \left[(e^{-\tilde{U}_t} - 1) \mathbf{1}_{\{Z_t \leq -t^{\frac{1}{2}} y\}} \right] \frac{1 - e^{-\sqrt{t}y}}{\sqrt{t}} \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy + e^{-\tilde{\eta}t} \int_0^\infty \tilde{\mathbb{P}}(Z_t \leq -t^{\frac{1}{2}} y) \frac{1 - e^{-\sqrt{t}y}}{\sqrt{t}} \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy. \end{aligned} \quad (\text{B.17})$$

By (2.19) and the self-similarity of $(Z_t)_{t \geq 0}$, for $y > 0$,

$$\tilde{\mathbb{P}}(Z_t \leq -t^{\frac{1}{2}} y) \frac{1 - e^{-\sqrt{t}y}}{\sqrt{t}} = \tilde{\mathbb{P}}(Z_1 \leq -t^{\frac{1}{2}-\frac{1}{Y}} y) \frac{1 - e^{-\sqrt{t}y}}{\sqrt{t}} \leq \kappa t^{1-\frac{Y}{2}} y^{1-Y} \leq \kappa y^{1-Y},$$

which, when multiplied by $\exp(-y^2/2\sigma^2)$, becomes integrable on $[0, \infty)$. Hence, by (2.18) and the dominated convergence theorem,

$$\lim_{t \rightarrow 0} t^{\frac{Y}{2}-1} e^{-\tilde{\eta}t} \int_0^\infty \tilde{\mathbb{P}}(Z_t \leq -t^{\frac{1}{2}} y) \frac{1 - e^{-\sqrt{t}y}}{\sqrt{t}} \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy = \frac{C_-}{Y} \int_0^\infty y^{1-Y} \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy. \quad (\text{B.18})$$

To find the asymptotic behavior of the first integral in (B.17), decompose it as:

$$\begin{aligned} \tilde{\mathbb{E}} \left[(e^{-\tilde{U}_t} - 1) \mathbf{1}_{\{Z_t \leq -t^{\frac{1}{2}} y\}} \right] &= \tilde{\mathbb{E}} \left(\mathbf{1}_{\{Z_t \leq -t^{\frac{1}{2}} y, \tilde{U}_t < 0\}} \int_{\tilde{U}_t}^0 e^{-u} du \right) - \tilde{\mathbb{E}} \left(\mathbf{1}_{\{Z_t \leq -t^{\frac{1}{2}} y, \tilde{U}_t \geq 0\}} \int_0^{\tilde{U}_t} e^{-u} du \right) \\ &=: J_{21}(t, y) + J_{22}(t, y). \end{aligned} \quad (\text{B.19})$$

For $J_{21}(t, y)$, using Markov's inequality and (3.2-ii), there exist $0 < \tilde{\kappa} < \infty$ and $0 < t_0 < 1$ such that

$$J_{21}(t, y) = \int_{-\infty}^0 e^{-x} \tilde{\mathbb{P}}\left(Z_t \leq -t^{\frac{1}{2}}y, t^{-\frac{1}{Y}}\tilde{U}_t \leq t^{-\frac{1}{Y}}x\right) dx \leq \tilde{\mathbb{E}}\left(e^{-t^{-1/Y}\tilde{U}_t}\right) \int_{-\infty}^0 e^{-x} \exp\left(t^{-\frac{1}{Y}}x\right) dx \leq \frac{\tilde{\kappa}t^{\frac{1}{Y}}}{1-t^{\frac{1}{Y}}},$$

for any $0 < t < t_0$ and $y \geq 0$. Since $1 - Y/2 < 1/2 < 1/Y$, for $1 < Y < 2$, by the dominated convergence theorem,

$$0 \leq t^{\frac{Y}{2}-1}e^{-\tilde{\eta}t} \int_0^\infty J_{21}(t, y) \frac{1-e^{-\sqrt{t}y}}{\sqrt{t}} \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy \leq \tilde{\kappa}t^{\frac{Y}{2}-1} \frac{t^{\frac{1}{Y}}}{1-t^{\frac{1}{Y}}} \int_0^\infty y \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy \rightarrow 0, \quad \text{as } t \rightarrow 0. \quad (\text{B.20})$$

Next, further decompose the second term $J_{22}(t, y)$ in (B.19) as:

$$\begin{aligned} J_{22}(t, y) &= \tilde{\mathbb{E}}\left[\left(e^{-\tilde{U}_t} - 1 + \tilde{U}_t\right) \mathbf{1}_{\{Z_t \leq -t^{\frac{1}{2}}y, \tilde{U}_t \geq 0\}}\right] - \tilde{\mathbb{E}}\left(\tilde{U}_t \mathbf{1}_{\{Z_t \leq -t^{\frac{1}{2}}y, \tilde{U}_t \geq 0\}}\right) \\ &=: J_{22}^{(1)}(t, y) - J_{22}^{(2)}(t, y). \end{aligned} \quad (\text{B.21})$$

Using (3.4-i) and the fact that $1 - Y/2 < 1/2 < 1/Y$, for $1 < Y < 2$,

$$\begin{aligned} 0 \leq J_{22}^{(2)}(t, y) &\leq t^{\frac{1}{Y}} \tilde{\mathbb{E}}\left[\left(t^{-\frac{1}{Y}}\tilde{U}_t\right)^+\right] = t^{\frac{1}{Y}} \int_0^\infty \mathbb{P}\left(t^{-\frac{1}{Y}}\tilde{U}_t \geq u\right) du \leq t^{\frac{1}{Y}} \left(1 + \int_1^\infty \mathbb{P}\left(t^{-\frac{1}{Y}}\tilde{U}_t \geq u\right) du\right) \\ &\leq t^{\frac{1}{Y}} \left(1 + \int_1^\infty \tilde{\kappa}t(t^{\frac{1}{Y}}u)^{-Y} du\right), \end{aligned} \quad (\text{B.22})$$

which is clearly $o(t^{1-\frac{Y}{2}})$. Moreover,

$$J_{22}^{(1)}(t, y) = \tilde{\mathbb{E}}\left(\int_0^{\tilde{U}_t} (1-e^{-w}) dw \mathbf{1}_{\{Z_t \leq -t^{\frac{1}{2}}y, \tilde{U}_t \geq 0\}}\right) = \int_0^\infty (1-e^{-w}) \tilde{\mathbb{P}}\left(\tilde{U}_t \geq w, Z_t \leq -t^{\frac{1}{2}}y\right) dw \leq \int_0^\infty (1-e^{-w}) \tilde{\mathbb{P}}\left(\tilde{U}_t \geq w\right) dw.$$

Using (B.10), $t^{-1/2}(1-e^{-\sqrt{t}y}) \leq y$, $y > 0$, and by the dominated convergence theorem,

$$0 \leq t^{\frac{Y}{2}-1}e^{-\tilde{\eta}t} \int_0^\infty J_{22}^{(1)}(t, y) \frac{1-e^{-\sqrt{t}y}}{\sqrt{t}} \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy \leq t^{\frac{Y}{2}} \int_0^\infty (1-e^{-w}) w^{-Y} dw \cdot \int_0^\infty y \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy \rightarrow 0, \quad (\text{B.23})$$

as $t \rightarrow 0$. Combining (B.18), (B.20), (B.22) and (B.23) lead to

$$\lim_{t \rightarrow 0} t^{\frac{Y}{2}-1} I_2(t) = \frac{C_-}{Y} \int_0^\infty y^{1-Y} \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy. \quad (\text{B.24})$$

Step 3. To finish, let us study the behavior of $I_3(t)$. Note that

$$\begin{aligned} I_3(t) &= e^{-\tilde{\eta}t} \int_0^\infty \tilde{\mathbb{E}}\left(e^{-\tilde{U}_t} \mathbf{1}_{\{Z_t \geq \sqrt{t}y\}} \frac{1-e^{\sqrt{t}y}e^{-Z_t}}{\sqrt{t}}\right) \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy \\ &= e^{-\tilde{\eta}t} \int_0^\infty \tilde{\mathbb{E}}\left(e^{-\tilde{U}_t} \mathbf{1}_{\{Z_t \geq \sqrt{t}y\}}\right) \frac{1-e^{\sqrt{t}y}e^{-Z_t}}{\sqrt{t}} \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy + e^{-\tilde{\eta}t} \int_0^\infty \tilde{\mathbb{E}}\left(e^{-\tilde{U}_t} \mathbf{1}_{\{Z_t \geq \sqrt{t}y\}} \frac{1-e^{-Z_t}}{\sqrt{t}}\right) e^{\sqrt{t}y} \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy. \end{aligned} \quad (\text{B.25})$$

First, decompose $J_{31}(t, y) := \tilde{\mathbb{E}}\left(e^{-\tilde{U}_t} \mathbf{1}_{\{Z_t \geq \sqrt{t}y\}}\right)$ as:

$$J_{31}(t, y) = \tilde{\mathbb{E}}\left[\left(e^{-\tilde{U}_t} - 1\right) \mathbf{1}_{\{Z_t \geq \sqrt{t}y\}}\right] + \tilde{\mathbb{P}}\left(Z_t \geq \sqrt{t}y\right) =: J_{31}^{(1)}(t, y) + J_{31}^{(2)}(t, y). \quad (\text{B.26})$$

By (2.19), it is easy to see that $J_{31}^{(2)}(t, y) \leq \kappa t^{1-\frac{Y}{2}} y^{-Y}$, for any $0 < t \leq 1$ and $y \geq 0$. Hence, the dominated convergence theorem together with (2.17) lead to

$$\lim_{t \rightarrow 0} e^{-\tilde{\eta}t} t^{\frac{Y}{2}-1} \int_0^\infty J_{31}^{(2)}(t, y) \frac{1 - e^{\sqrt{t}y}}{\sqrt{t}} \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy = -\frac{C_+}{Y} \int_0^\infty y^{1-Y} \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy. \quad (\text{B.27})$$

As before, $J_{31}^{(1)}(t, y)$ can be further decomposed as

$$J_{31}^{(1)}(t, y) = \tilde{\mathbb{E}} \left[\left(e^{-\tilde{U}_t} - 1 \right) \mathbf{1}_{\{Z_t \geq \sqrt{t}y, \tilde{U}_t \geq 0\}} \right] + \int_0^\infty e^u \tilde{\mathbb{P}} \left(Z_t \geq \sqrt{t}y, \tilde{U}_t \leq -u \right) du.$$

Now, for $u > 0$, $y > 0$ and $t > 0$, by Markov's inequality and (3.2-ii), there exist $0 < \tilde{\kappa} < \infty$ and $0 < t_0 < 1$, such that

$$\tilde{\mathbb{P}} \left(Z_t \geq \sqrt{t}y, \tilde{U}_t \leq -u \right) \leq \tilde{\mathbb{P}} \left(\tilde{U}_t \leq -u \right) \leq \tilde{\mathbb{E}} \left(e^{-t^{-\frac{1}{Y}} \tilde{U}_t} \right) e^{-t^{-\frac{1}{Y}} u} \leq \tilde{\kappa} e^{-t^{-\frac{1}{Y}} u}, \quad (\text{B.28})$$

for all $0 < t < t_0$. Also, as done in (B.22), there exist $0 < K < \infty$ and $0 < t_0 < 1$ such that

$$0 \leq \tilde{\mathbb{E}} \left[\left(1 - e^{-\tilde{U}_t} \right) \mathbf{1}_{\{Z_t \geq \sqrt{t}y, \tilde{U}_t \geq 0\}} \right] \leq \tilde{\mathbb{E}} \left(\tilde{U}_t^+ \right) \leq K t^{\frac{1}{Y}}, \quad (\text{B.29})$$

for all $0 < t < t_0$. Hence, for any $y \geq 0$ and $0 < t < t_0 < 1$, and since $1 - Y/2 < 1/2 < 1/Y$, for $1 < Y < 2$,

$$t^{\frac{Y}{2}-1} \left| J_{31}^{(1)}(t, y) \right| \leq \tilde{\kappa} t^{\frac{Y}{2}-1} \int_0^\infty e^{-u(t^{-\frac{1}{Y}}-1)} du + K t^{\frac{1}{Y}+\frac{Y}{2}-1} = \tilde{\kappa} t^{\frac{Y}{2}-1} \frac{t^{\frac{1}{Y}}}{1-t^{\frac{1}{Y}}} + K t^{\frac{1}{Y}+\frac{Y}{2}-1} \rightarrow 0, \quad \text{as } t \rightarrow 0. \quad (\text{B.30})$$

Since both control functions in (B.28) and (B.29) are independent of y , combining (B.27) and (B.30), and by the dominated convergence theorem,

$$\lim_{t \rightarrow 0} e^{-\tilde{\eta}t} t^{\frac{Y}{2}-2} \int_0^\infty J_{31}(t, y) \frac{1 - e^{\sqrt{t}y}}{\sqrt{t}} \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy = -\frac{C_+}{Y} \int_0^\infty y^{1-Y} \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy. \quad (\text{B.31})$$

Next, by the self-similarity of $(Z_t)_{t \geq 0}$, the term $J_{32}(t, y) := \tilde{\mathbb{E}} \left(t^{-1/2} e^{-\tilde{U}_t} \mathbf{1}_{\{Z_t \geq t^{\frac{1}{2}} y\}} (1 - e^{-Z_t}) \right)$ appearing in (B.25) is decomposed as:

$$\begin{aligned} J_{32}(t, y) &= \tilde{\mathbb{E}} \left[\mathbf{1}_{\{Z_t \geq \sqrt{t}y\}} \left(\frac{e^{-\tilde{U}_t} - e^{-(Z_t + \tilde{U}_t)}}{\sqrt{t}} - t^{-\frac{1}{2}} Z_t \right) \right] + \tilde{\mathbb{E}} \left(t^{\frac{1}{Y}-\frac{1}{2}} Z_1 \mathbf{1}_{\{Z_1 \geq t^{\frac{1}{2}-\frac{1}{Y}} y\}} \right) \\ &=: J_{32}^{(1)}(t, y) + J_{32}^{(2)}(t, y). \end{aligned} \quad (\text{B.32})$$

Note that $J_{32}^{(2)}(t, y)$ is quite similar to $J_{12}(t, y)$ in (B.4) and, thus, the corresponding integral has an asymptotic behavior similar to (B.6). Concretely,

$$\lim_{t \rightarrow 0} t^{\frac{Y}{2}-1} e^{-\tilde{\eta}t} \int_0^\infty \tilde{\mathbb{E}} \left(t^{\frac{1}{Y}-\frac{1}{2}} Z_1 \mathbf{1}_{\{Z_1 \geq t^{\frac{1}{2}-\frac{1}{Y}} y\}} \right) e^{\sqrt{t}y} \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy = \frac{C_+}{Y-1} \int_0^\infty y^{1-Y} \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy. \quad (\text{B.33})$$

Next, decompose $J_{32}^{(1)}(t, y)$ as:

$$\begin{aligned} J_{32}^{(1)}(t, y) &= t^{-\frac{1}{2}} \tilde{\mathbb{E}} \left(\mathbf{1}_{\{Z_t \geq t^{\frac{1}{2}} y\}} \int_{\tilde{U}_t}^{Z_t + \tilde{U}_t} (e^{-x} - 1) dx \right) \\ &= t^{-\frac{1}{2}} \int_{-\infty}^0 (e^{-x} - 1) \tilde{\mathbb{P}} \left(Z_t \geq \sqrt{t}y, \tilde{U}_t \leq x \leq Z_t + \tilde{U}_t \right) dx + t^{-\frac{1}{2}} \int_0^\infty (e^{-x} - 1) \tilde{\mathbb{P}} \left(Z_t \geq \sqrt{t}y, \tilde{U}_t \leq x \leq Z_t + \tilde{U}_t \right) dx. \end{aligned}$$

Note that for $x > 0$,

$$\tilde{\mathbb{P}}\left(Z_t \geq \sqrt{t}y, \tilde{U}_t \leq x \leq Z_t + \tilde{U}_t\right) \leq \tilde{\mathbb{P}}\left(x \leq Z_t + \tilde{U}_t\right),$$

while for $x < 0$,

$$\tilde{\mathbb{P}}\left(Z_t \geq \sqrt{t}y, \tilde{U}_t \leq x \leq Z_t + \tilde{U}_t\right) \leq \tilde{\mathbb{P}}\left(\tilde{U}_t \leq x\right).$$

Using the estimates (B.8) and (B.12), arguments as in getting (B.9) and (B.13) give

$$\lim_{t \rightarrow 0} e^{-\tilde{\eta}t} t^{\frac{Y}{2}-1} \int_0^\infty J_{32}^{(1)}(t, y) e^{\sqrt{t}y} \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy = 0. \quad (\text{B.34})$$

Combining (B.31), (B.33) and (B.34) leads to

$$\lim_{t \rightarrow 0} t^{\frac{Y}{2}-1} I_3(t) = \frac{C_+}{Y(Y-1)} \int_0^\infty y^{1-Y} \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy. \quad (\text{B.35})$$

Finally, from (B.2), (B.16), (B.24) and (B.35), and since $1 - Y/2 < 1/2$, for $1 < y < 2$, (4.1) is obtained. \square

Proof of Corollary 4.3.

When the diffusion component is nonzero, (Tankov, 2010, Proposition 5) implies that $\hat{\sigma}(t) \rightarrow \sigma$ as $t \rightarrow 0$. In particular, $\hat{\sigma}(t)t^{1/2} \rightarrow 0$ as $t \rightarrow 0$ and, thus, (A.27) above remains true. Let $\tilde{\sigma}(t) := \hat{\sigma}(t) - \sigma$, then $\tilde{\sigma}(t) \rightarrow 0$ as $t \rightarrow 0$, and (A.27) can be written as

$$\begin{aligned} C_{BS}(t, \hat{\sigma}(t)) &= \frac{\sigma}{\sqrt{2\pi}} t^{\frac{1}{2}} + \frac{\tilde{\sigma}(t)}{\sqrt{2\pi}} t^{\frac{1}{2}} - \frac{\hat{\sigma}(t)^3}{24\sqrt{2\pi}} t^{\frac{3}{2}} + O\left(\left(\hat{\sigma}(t)t^{\frac{1}{2}}\right)^5\right) \\ &= \frac{\sigma}{\sqrt{2\pi}} t^{\frac{1}{2}} + \frac{\tilde{\sigma}(t)}{\sqrt{2\pi}} t^{\frac{1}{2}} + O\left(t^{\frac{3}{2}}\right). \end{aligned} \quad (\text{B.36})$$

Comparing (4.2)-(4.3) and (B.36) gives

$$\frac{(C_+ + C_-)2^{-\frac{1+Y}{2}}\sigma^{1-Y}}{Y(Y-1)\sqrt{\pi}} \Gamma\left(1 - \frac{Y}{2}\right) t^{\frac{3-Y}{2}} \sim \frac{\tilde{\sigma}(t)}{\sqrt{2\pi}} \sqrt{t}, \quad t \rightarrow 0,$$

and, therefore,

$$\tilde{\sigma}(t) \sim \frac{(C_+ + C_-)2^{-\frac{Y}{2}}\sigma^{1-Y}}{Y(Y-1)} \Gamma\left(1 - \frac{Y}{2}\right) t^{1-\frac{Y}{2}}, \quad t \rightarrow 0.$$

The proof is now complete. \square

C Additional Proofs

Verification of (2.19).

It suffices to show that, for some constant $0 < \kappa < \infty$ and any $v > 0$ and $0 < t \leq 1$,

$$\tilde{\mathbb{P}}\left(Z_1 \geq t^{-\frac{1}{Y}}v\right) \leq \kappa t|v|^{-Y}.$$

Indeed, the same argument for $-Z_1$ shows an analogous bound for $\tilde{\mathbb{P}}\left(Z_1 \leq t^{-1/Y}v\right)$, with $v < 0$, which in turn will imply (2.19). First, in light of (2.16), there exist $0 < R < \infty$ and $H > 0$, such that for any $u \geq H$,

$$p_Z(u) \leq Ru^{-Y-1}. \quad (\text{C.1})$$

Thus, whenever $t^{-1/Y}v \geq H$,

$$\tilde{\mathbb{P}}\left(Z_1 \geq t^{-\frac{1}{Y}}v\right) \leq R \int_{t^{-\frac{1}{Y}}v}^{\infty} u^{-Y-1} du = \frac{R}{Y} \left(t^{-\frac{1}{Y}}v\right)^{-Y} = \frac{R}{Y} tv^{-Y}.$$

Since $\tilde{\mathbb{P}}\left(Z_1 \geq t^{-1/Y}v\right) \leq 1 < H^Y tv^{-Y}$, when $t^{-1/Y}v < H$,

$$\begin{aligned} \tilde{\mathbb{P}}\left(Z_1 \geq t^{-\frac{1}{Y}}v\right) &\leq \mathbf{1}_{\{t^{-\frac{1}{Y}}v \geq H\}} \tilde{\mathbb{P}}\left(Z_1 \geq t^{-\frac{1}{Y}}v\right) + \mathbf{1}_{\{t^{-\frac{1}{Y}}v < H\}} \tilde{\mathbb{P}}\left(Z_1 \geq t^{-\frac{1}{Y}}v\right) \\ &\leq \mathbf{1}_{\{t^{-\frac{1}{Y}}v \geq H\}} \frac{R}{Y} tv^{-Y} + \mathbf{1}_{\{t^{-\frac{1}{Y}}v < H\}} H^Y tv^{-Y} \\ &\leq tv^{-Y} \left(\frac{R}{Y} + H^Y\right), \end{aligned}$$

for all $0 < t \leq 1$ and $v > 0$. □

Verification of (B.5)

This is similar to the proof of (2.19). Using (C.1), for any $t > 0$ and $y > 0$,

$$\begin{aligned} t^{\frac{Y}{2}-1} J_{12}(t, y) &= t^{\frac{Y}{2}-1} t^{\frac{1}{Y}-\frac{1}{2}} \int_{t^{\frac{1}{2}-\frac{1}{Y}}y}^{\infty} up_Z(u) du \\ &= t^{(\frac{1}{Y}-\frac{1}{2})(1-Y)} \mathbf{1}_{\{t^{\frac{1}{2}-\frac{1}{Y}}y \geq H\}} \int_{t^{\frac{1}{2}-\frac{1}{Y}}y}^{\infty} up_Z(u) du \\ &\quad + t^{(\frac{1}{Y}-\frac{1}{2})(1-Y)} \mathbf{1}_{\{t^{\frac{1}{2}-\frac{1}{Y}}y \leq H\}} \left(\int_H^{\infty} up_Z(u) du + \int_{t^{\frac{1}{2}-\frac{1}{Y}}y}^H up_Z(u) du \right) \\ &\leq t^{(\frac{1}{Y}-\frac{1}{2})(1-Y)} \mathbf{1}_{\{t^{\frac{1}{2}-\frac{1}{Y}}y \geq H\}} \int_{t^{\frac{1}{2}-\frac{1}{Y}}y}^{\infty} Ru^{-Y} du \\ &\quad + t^{(\frac{1}{Y}-\frac{1}{2})(1-Y)} \mathbf{1}_{\{t^{\frac{1}{2}-\frac{1}{Y}}y \leq H\}} \left(\int_H^{\infty} Ru^{-Y} du + H \tilde{\mathbb{P}}\left(Z_1 \geq t^{\frac{1}{2}-\frac{1}{Y}}y\right) \right). \end{aligned}$$

Then, using (2.19),

$$\begin{aligned} t^{\frac{Y}{2}-1} J_{12}(t, y) &\leq t^{(\frac{1}{Y}-\frac{1}{2})(1-Y)} R(Y-1)^{-1} y^{1-Y} t^{(\frac{1}{2}-\frac{1}{Y})(1-Y)} \\ &\quad + t^{(\frac{1}{Y}-\frac{1}{2})(1-Y)} \mathbf{1}_{\{t^{\frac{1}{2}-\frac{1}{Y}}y \leq H\}} \left(RH^{1-Y} + H \left(\frac{H}{t^{\frac{1}{2}-\frac{1}{Y}}y}\right)^{Y-1} \right) \\ &\leq R(Y-1)^{-1} y^{1-Y} + t^{(\frac{1}{Y}-\frac{1}{2})(1-Y)} Rt^{(\frac{1}{2}-\frac{1}{Y})(1-Y)} y^{1-Y} + H^Y y^{1-Y}, \end{aligned}$$

and (B.5) follows with $\lambda := RY/(Y-1) + H^Y$. □

References

- Ait-Sahalia, Y., and J. Jacod (2009): Estimating the degree of activity of jumps in high-frequency data. *Annals of Statistics* 37(5A), 2202–2244.
- Applebaum, D. (2004): *Lévy Processes and Stochastic Calculus*, Cambridge University Press, Cambridge.
- Belomestny, D. (2010): Spectral estimation of the fractional order of a Lévy process, *Annals of Statistics* 38(1), 317–351.
- Barndorff-Nielsen, O. E. (1997): Processes of normal inverse Gaussian type, *Finance and Stochastics*, 2, 41–68.

- Berestycki, H., J. Busca, and I. Florent (2002): Asymptotics and calibration of local volatility models, *Quantitative Finance*, 2, 61-69.
- Berestycki, H., J. Busca, and I. Florent (2004): Computing the implied volatility in stochastic volatility models, *Communications on Pure and Applied Mathematics*, Vol LVII, 1352-1373.
- Bertoin, J. (1996): Lévy Processes, Cambridge University Press, Cambridge
- Boyarchenko S.I., and S.Z. Levendorskii (2002): Non-Gaussian Merton-Black-Scholes theory, *Adv. Ser. Stat. Sci. Appl. Probab.* 9. World Scientific Publishing Co., Inc., River Edge, NJ.
- Carr, P., H. Geman, D. Madan, and M. Yor (2002): The fine structure of asset returns: an empirical investigation, *Journal of Business*, 75, 305-332.
- Carr, P., and D. Madan (2009): Saddle point methods for option pricing, *The Journal of Computational Finance*, 13(1), 49-61.
- Cont, R., J. Bouchaud, and M. Potters (1997): Scaling in financial data: stable laws and beyond. In "Scale Invariance and Beyond", B. Durrleman, F. Granger and D. Sornette, eds..
- Cont, R., and P. Tankov (2004): *Financial modelling with jump processes*, Chapman & Hall.
- Carr, R., and L. Wu (2003): What type of process underlies options? A simple robust test, *Journal of Finance*, 58(6), 2581-2610.
- Eberlein, E., U. Keller, and K. Prause (1998): New insights into smile, mispricing and value at risk, *Journal of Business*, 71, 371-406.
- Feng, J., M. Forde, and J.P. Fouque (2010): Short maturity asymptotics for a fast mean reverting Heston stochastic volatility model, *SIAM Journal on Financial Mathematics*, 1, 126-141.
- Feng, J., J.P. Fouque, and R. Kumar (2012): Small-time asymptotics for fast mean-reverting stochastic volatility models, *Annals of Applied Probability*, 22(4), 1541-1575.
- Figuerola-López, J.E., and M. Forde (2012): The small-maturity smile for exponential Lévy models, *SIAM Journal on Financial Mathematics* 3(1), 33-65.
- Figuerola-López, J.E., R. Gong, and C. Houdré (2012): Small-time expansions of the distributions, densities, and option prices under stochastic volatility models with Lévy jumps, *Stochastic Processes and their Applications*, 122, 1808-1839.
- Figuerola-López, J.E., R. Gong, and C. Houdré (2011). High-order short-time expansions for ATM option prices under the CGMY model, *Preprint. Available at arXiv:1112.3111v1 [q-fin.CP]*.
- Figuerola-López, J.E., R. Gong, and C. Houdré (2013): A note on high-order short-time expansions for ATM option prices under the CGMY model, *Preprint. Available at arXiv:1305.4719 [q-fin.CP]*.
- Figuerola-López, J.E., and S. Ólafsson (2014): Short-time expansions for close-to-the-money options under a Lévy jump model with stochastic volatility, *Preprint. Available at arXiv:1404.0601 [q-fin.PR]*.
- Forde, M., and A. Jacquier (2009): Small-time asymptotics for implied volatility under the Heston model, *Int. J. Theor. Appl. Finance*, 12(6), 861-876.
- Forde, M., and A. Jacquier (2011): Small time asymptotics for an uncorrelated local-stochastic volatility model, *Applied Mathematical Finance*, 18(6), 517-535.
- Forde, M., A. Jacquier, and R. Lee (2012): The small-time smile and term structure for implied volatility under the Heston model, *SIAM Journal on Financial Mathematics*, 3:690-708.
- Gao, K., and R. Lee (2013): Asymptotics of implied volatility to arbitrary order, *Finance and Stochastics*, Forthcoming. Available at SSRN: <http://ssrn.com/abstract=1768383>.

- Gatheral, J., E. Hsu, P. Laurence, C. Ouyang, and T-H. Wang (2012): Asymptotics of implied volatility in local volatility models, *Mathematical Finance*, 22:591-620.
- Henry-Labordère, P. (2009): Analysis, geometry, and modeling in finance: advanced methods in option pricing, *Chapman & Hall*.
- Houdré, C. (2002): Remarks on deviation inequalities for functions of infinitely divisible random vectors, *Annals of Probability*, 30(3), 1223-1237.
- Jacod, J. (2007): Asymptotic properties of power variations of Lévy processes. *ESAIM:P&S* 11, 173–196.
- Kallenberg, O. (1997): *Foundations of Modern Probability*. Springer-Verlag, Berlin, New York, Heidelberg.
- Koponen, I. (1995): Analytic approach to the problem of convergence of truncated Lévy flights towards the Gaussian stochastic process, *Physical Review E*, 52, 1197-1199.
- Kou, S. (2002): A jump-diffusion model for option pricing, *Management Science*, 48, 1086-1101.
- Kyprianou, A., W. Schoutens, and P. Wilmott (2005): Exotic Option Pricing and Advanced Lévy Models, John Wiley & Sons, West Sussex, England.
- Madan, D., P. Carr, and E. Chang (1998): The variance gamma process and option pricing, *European Finance Review*, 2, 79-105.
- Madan, D., and F. Milne (1991): Option pricing with VG martingale components, *Mathematical Finance*, 1, 39-56.
- Madan, D., and E. Seneta (1990): The variance gamma (VG) model for share market returns, *Journal of Business*, 63, 511-524.
- Mandelbrot, B. (1963): The variation of certain speculative prices. *The Journal of Business*, 36:394–419.
- Matacz, A. (2000): Financial modeling and option theory with the truncated Lévy process, *Int. J. Theor. Appl. Finance*, 3:143–160.
- Medvedev, A., and O. Scaillet (2007): Approximation and calibration of short-term implied volatility under jump-diffusion stochastic volatility, *The review of Financial Studies*, 20(2):427-459.
- Merton, R. (1976): Option pricing when underlying stock returns are discontinuous, *Journal of Financial Economics*, 3, 125-144.
- Muhle-Karbe, J., and M. Nutz (2011): Small-time asymptotics of option prices and first absolute moments, *Journal of Applied Probability*, 48(4), 1003-1020.
- Paulot, L. (2009): Asymptotic implied volatility at the second order with application to the SABR model, *Preprint*, 2009.
- Press, S.J. (1967): A compound event model for security prices. *The Journal of Business*, 40:317–335.
- Roper, M. (2009): Implied volatility: small time to expiry asymptotics in exponential Lévy models, *Thesis, University of New South Wales*.
- Roper, M., and M. Rutkowski (2007): A note on the behaviour of the Black-Scholes implied volatility close to expiry, *Tech. Report, UNSW*.
- Rosenbaum, M., and P. Tankov (2011): Asymptotic results for time-changed Lévy processes sampled at hitting times. *Stochastic processes and their applications*, 121:1607–1633.
- Rüschendorf, L., and J. Woerner (2002): Expansion of transition distributions of Lévy processes in small time. *Bernoulli*, 8, 81-96.
- Rosiński, J. (2007): Tempering stable processes. *Stochastic processes and their applications*, 117:677–707.

- Samorodnitsky, G., and M. Taqqu (1994): *Stable non-Gaussian random processes*, Chapman & Hall, New York.
- Sato, K. (1999): *Lévy processes and infinitely divisible distributions*, Cambridge University Press.
- Tankov, P. (2010): Pricing and hedging in exponential Lévy models: review of recent results, *Paris-Princeton Lecture Notes in Mathematical Finance*, Springer.
- Zolotarev, V.M. (1986): *One-dimensional stable distributions*, Amer. Math. Soc., Providence, R.I.