Optimal Placement of a Small Order Under a Diffusive Limit Order Book Model

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Outline

1. Introduction
2. Model and Assumptions
3. Main Results
4. Concluding Remarks and Future Work
LOB Markets: Market Order vs. Limit Order

Market Order (MO)
- pros: immediate execution (i.e., no execution risk),
- cons: pay spread and an additional fee ($f$).

Limit Order (LO)
- pros: get “discount" or better price (when executed) and an additional rebate ($r$),
- cons: execution risk (the order may never be executed),
- discount and execution risk are related to the book depth and flow.
Optimal Placement Problem

Find the level in the LOB (say, buy side) to place a limit order for one share so that to minimizing the "expected cost" during a fixed time horizon $t$:

\textit{Cost is understood as the price paid for the share (taking into account rebate or fee) minus the initial ask price.}

The optimal placement is aim to give the best tradeoff between execution risk and discount.
Discrete-time model for the best ask price as a symmetric CRW:

\[ S_t = S_0 + \sum_{i=1}^{[t/\delta]} X_i, \quad X_i \in \{-\varepsilon, \varepsilon\}, \quad \delta > 0, \]

\[ P(X_1 = \varepsilon) = 1 - P(X_1 = -\varepsilon) = \bar{p}, \]

\[ P(X_i = \varepsilon | X_{i-1} = \varepsilon) = P(X_i = -\varepsilon | X_{i-1} = -\varepsilon) = p < 1/2. \]

Spread is always constant to one-tick \( \varepsilon \)

Constant probability \( q \in (0, 1] \) that a limit order (of size 1) is executed when sitting at the best bid.

Order placement (either market or limit order) takes place at time 0 and there is no intermediate cancellation.

If the order is not executed by the time horizon \( t \), this is cancelled and changed to a market buy order.
Key Result:

The optimal placement strategy is one of the following:

(i) placement at the best bid $S_0 - \varepsilon$,
(ii) placement at the second best bid $S_0 - 2\varepsilon$,
(iii) placement of a market order at time 0.

Drawbacks:

- No consideration of the initial state of the LOB at time 0; however, everything else the same, placement at a level with a large queue should be less desirable;
- Model does not incorporate any local “drift" or momentum (e.g., say, $P(X_i = \varepsilon | X_{i-1} = \varepsilon) < P(X_i = -\varepsilon | X_{i-1} = -\varepsilon)$);
Our work

- We consider a variation of the problem which incorporates the initial state of the LOB and some assumptions about the order flow during the specified time horizon \([0, t]\);
- We analyze the problem when the time changes are frequent enough so that the dynamics of the best ask price can be well approximated by a diffusive process. This will be the case if, for instance, the flow intensity of market orders at level I is high enough.
- Find condition under which a nontrivial optimal placement (different from the level I or II) exists.
Notation

1. $\tilde{S}_u := \tilde{S}_u^{(\delta, \varepsilon)}$ denotes the best ask price at time $u \geq 0$, when tick size is $\varepsilon$ and time step is $\delta$ (or another parameter such that average time between price changes goes to 0 when $\delta \to 0$);

2. $\tilde{C}_{\delta, \varepsilon}(x, t)$: expected cost when the time horizon is $t$ and the limit order is placed at $-x$ lower than the best ask price;

3. $\tilde{Y}_t = \inf\{\tilde{S}_u : u \leq t\}$:

4. $\rho = \rho(t, x)$:

   the probability that a limit order placed at level $\tilde{S}_0 - x$ is executed, before time $t$, during the first time when this is possible (i.e., when $\tilde{S}_u = \tilde{S}_0 - x + \varepsilon$), conditional on the latter event to occur.

   In general, $\rho(t, x)$ would depend on the initial queue size $Q_x(0)$ at the level $\tilde{S}_0 - x$ as we expect that $\rho(t, x) \downarrow$ when $Q_x(0) \nearrow$. 

J.E. Figueroa-López (WUSTL)
Expected Cost at $\tilde{S}_0 - x$: Strategy I

We place a buy LO at level $\tilde{S}_0 - x$:

Case 1: $\bar{Y}_t > \tilde{S}_0 - x + \varepsilon$
- Order is not executed, buy MO at $\tilde{S}_t \implies \text{cost} = \tilde{S}_t - \tilde{S}_0 + f$.

Case 2: The first time $\tilde{S}_u$ reaches $\tilde{S}_0 - x + \varepsilon$, $\tau$, happens before $t$:
- If order is executed before or at $\tau + \delta \leq t$: $\implies \text{cost} = -x - r$.
  This happens with probability $\rho := \rho(x, t) \in (0, 1]$
- If order is not executed before $\tau + \delta \leq t$:
  Cancel LO and buy with MO at $\tilde{S}_0 - x + 2\varepsilon \implies \text{cost} = -x + f + 2\varepsilon$.

Expected Cost:

$$\tilde{C}_{\delta,\varepsilon}(x, t) = E \left[ \tilde{S}_t - \tilde{S}_0 \mid \bar{Y}_t > \tilde{S}_0 - x + \varepsilon \right] P(\bar{Y}_t > \tilde{S}_0 - x + \varepsilon)$$
$$+ P(\bar{Y}_t \leq \tilde{S}_0 - x + \varepsilon) [-x - (r + f)\rho(x, t)] + f$$
$$+ 2\varepsilon P(\bar{Y}_t \leq \tilde{S}_0 - x + \varepsilon)(1 - \rho(x, t)).$$
Expected Cost at $\tilde{S}_0 - x$: Strategy II

Case 1: $\tilde{Y}_t > \tilde{S}_0 - x + \varepsilon$
- Order not executed, buy MO at $\tilde{S}_t \implies$ cost $= \tilde{S}_t - \tilde{S}_0 + f$.

Case 2: If $\tilde{S}_u$ reaches $\tilde{S}_0 - x + \varepsilon$ before $t$:
- The order is executed before time $t$: $\implies$ cost $= -x - r$.
  This happens with a conditional probability $\tilde{\rho} := \tilde{\rho}(x, t) \in (0, 1]$.
- The order is not executed before $t$:
  Cancel LO and buy with MO at $\tilde{S}_t \implies$ cost $= \tilde{S}_t - \tilde{S}_0 + f$.

Expected Cost:

$$\tilde{C}_{\delta, \varepsilon}(x, t) = E \left[ \tilde{S}_t - \tilde{S}_0 \mid \tilde{Y}_t > \tilde{S}_0 - x + \varepsilon \right] P \left( \tilde{Y}_t > \tilde{S}_0 - x + \varepsilon \right)$$
$$+ P(\tilde{Y}_t \leq \tilde{S}_0 - x + \varepsilon) [ -x - r - f ] \tilde{\rho}(x, t) + f$$
$$+ E \left[ 1_{E_t^c} (\tilde{S}_t - \tilde{S}_0) \mid \tilde{Y}_t \leq S_0 - x + \varepsilon \right] P \left( \tilde{Y}_t \leq \tilde{S}_0 - x + \varepsilon \right),$$

where $E_t$ is the event that the order is executed by time $t$ and

$$\tilde{\rho}(x, t) = P(E_t \mid \tilde{Y}_t \leq \tilde{S}_0 - x + \varepsilon).$$
Expected Cost in Continuous-Time

Motivation:

- Several LOB models (including, symmetric and some asymmetric CRW) have been shown to admit a diffusive limit \( \{ S_u \}_{u \geq 0} \) when \( \delta \to 0 \) and \( \varepsilon \to 0 \);

- In the case of the second strategy, it is also natural to assume that \( \tilde{\rho}(x, t) = P(\text{order gets executed} | \bar{Y}_t \leq \bar{S}_0 - x + \varepsilon) \to 1 \), as \( \delta, \varepsilon \to 0 \).

It is natural to consider the following analog continuous time problem:

**Expected Cost, Continuous Case.**

\[
C(x, t) = E \left[ S_t - S_0 \mid Y_t > S_0 - x \right] P(Y_t > S_0 - x) \\
+ P(Y_t \leq S_0 - x) \left(-x - \rho(x, t)(r + f)\right) + f.
\]

where \( \{ S_t \}_t \) is a suitable continuous time process, \( Y_t := \min_{u \leq t} S_u \) and, as for the second strategy, \( \rho(x, t) \) could be 1.
Price Models for $S$

- Brownian Motion with Drift (Bachelier Model):
  \[ dS_u = \mu du + \sigma dW_u, \]
  (reasonable approximation for intermediate intraday time horizons)

- Geometric Brownian Motion (Black Scholes Model)
  \[ dS_u = \mu S_u du + \sigma S_u dW_u, \]
  (better model for asset price movement at longer time periods)
Modeling $\rho(x, t)$

This can be estimated from

- the queue size $Q_x^b(0)$ at level $S_0 - x$.
- the cancellation order flow at each level as prescribed by the counting process $N_x^b(s)$ of cancellation.
- assumed order flow at level I: $\alpha_u(i, j)$ is the probability of a price decrease before time $u$ when there are $i$ and $j$ orders at the best ask and bid queues.
- $f^a$ distribution of the best ask queue size after a mid price decrease;

A reasonable formula is given by:

$$
\rho(x, t) := \sum_{i=1}^{\infty} \sum_{j=0}^{Q_x^b(0)} f^a(i) \int_0^t f_\tau(s \mid 0 < \tau < t) P(N_x^b(s) = j) \alpha_{t-s}(i, Q_x^b(0) - j + 1) \, ds,
$$

where $\tau$ is the first time the best bid price hits the level $S_0 - x$. 
Figure 2: Left: Distribution of best ask queue size after a price decrease from MSFT data from April 17th to April 28th (8 days). The unit of queue size is a batch (100 stocks). Right: Order Flow Intensity Rates per second in number of batches.

<table>
<thead>
<tr>
<th></th>
<th>Description</th>
<th>Rate</th>
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<tr>
<td>$\mu_a + \theta_a,1$</td>
<td>Depletion rate of best ask queue</td>
<td>19.32</td>
</tr>
<tr>
<td>$\lambda_a$</td>
<td>Addition rate of best ask queue</td>
<td>21.78</td>
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<tr>
<td>$\mu_b + \theta_b,1$</td>
<td>Depletion rate of best bid queue</td>
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</tr>
<tr>
<td>$\lambda_b$</td>
<td>Addition rate of best bid queue</td>
<td>21.98</td>
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Figure 3: Left Panel: Initial LOB profile $i \rightarrow Q^b_{i\epsilon}(0)$; Right Panel: Graphs of $i \rightarrow \rho(\epsilon i, t)$ with $\epsilon = 0.01$ and $t = 30$ sec (black line), $t = 60$ sec (red dashed line), and $t = 90$ sec (blue dotted line).
Figure 4: $t \rightarrow \rho(0.1, t)$ under the Bachelier model (left) and the Black-Scholes Model (right) for $Q^b_x(0) = 0$ (black), $Q^b_x(0) = 1$ (red), $Q^b_x(0) = 10$ (blue), $Q^b_x(0) = 38$ (magenta), $Q^b_x(0) = 50$ (green), and $Q^b_x(0) = 100$ (yellow). Queue size is in batches (each of size 100 shares). Black line is almost overlapping with red line.
Lemma (F-L, Lee, & Pasupathy, 2017)

For the BM model and placement at level \(S_0 - x\),

\[
C(x, t) = \mu t N\left(\frac{x + \mu t}{\sigma \sqrt{t}}\right) + e^{-2x\mu/\sigma^2} (2x - \mu t) N\left(\frac{-x + \mu t}{\sigma \sqrt{t}}\right) + f
\]
\[
+ \left\{ N\left(\frac{-x - \mu t}{\sigma \sqrt{t}}\right) + e^{-2x\mu/\sigma^2} N\left(\frac{-x + \mu t}{\sigma \sqrt{t}}\right) \right\} (-x - \rho(x, t)(r + f)).
\]

For the GBM model and placement at level \(S_0 e^{-y}\) (\(y > 0\)) (i.e., \(x = S_0 - S_0 e^{-y}\)),

\[
C(y, t) = S_0 e^{\mu t} \left[ N\left(\frac{y + \alpha_{+} t}{\sigma \sqrt{t}}\right) - e^{-2y\alpha_{+}/\sigma^2} N\left(\frac{-y + \alpha_{+} t}{\sigma \sqrt{t}}\right) \right] + f - S_0
\]
\[
+ \left\{ N\left(\frac{-y - \alpha_{-} t}{\sigma \sqrt{t}}\right) + e^{-2y\alpha_{-}/\sigma^2} N\left(\frac{-y + \alpha_{-} t}{\sigma \sqrt{t}}\right) \right\} (S_0 e^{-y} - \rho(y, t)(r + f)),
\]

where \(\alpha_{\pm} := \mu \pm \sigma^2/2\).
Expected Cost for GBM: $\mu > 0$ vs. $\mu < 0$

Figure 5: $C(y, t)$ against $y$ with $\rho(r + f) = 0.01$, $\sigma = 0.2$, $S_0 = 10$, $\mu = 0.1$, $t = 0.1$

Figure 6: $C(y, t)$ against $y$ with $\rho(r + f) = 0.01$, $\sigma = 0.2$, $S_0 = 10$, $\mu = -0.1$, $t = 0.1$
Expected Cost for GBM with $\mu < 0$: large vs. small $t$

**Figure 7:** $C(y, t)$ against $y$ with $\rho(r + f) = 0.01, \sigma = 0.2, S_0 = 10, \mu = -0.05, t = 0.02$

**Figure 8:** $C(y, t)$ against $y$ with $\rho(r + f) = 0.01, \sigma = 0.2, S_0 = 10, \mu = -0.05, t = 0.002$. 
Optimal Placement Solution

Simplifying Assumption

\( \rho(x, t) \) is independent of \( t \);

Theorem (F-L, Lee, & Pasupathy, 2017)

Let \( x^*(t) \in [0, \infty] \) be such that

\[
x^*(t) = \arg \inf_{x>0} C(x, t),
\]

where \( x^*(t) = 0 \) (resp., \( x^*(t) = \infty \)) means that \( C(0^+, t) < C(x, t) \) (resp., \( C(\infty, t) < C(x, t) \)), for all \( x > 0 \).

1. Suppose that \( \mu \geq 0 \) and \( x \rightarrow \rho(x) \) is decreasing. Then, \( x \rightarrow C(x, t) \) is strictly increasing and, thus, \( x^*(t) = 0 \);

2. Suppose that \( \mu < 0 \) and \( \rho'(0^+) > 0 \). Then, there exists a \( t_0 > 0 \) such that \( x^*(t) \in (0, \infty) \), for all \( t > t_0 \).
$0^+$ in $C(0^+, t)$ can be interpreted as an order placed at the best bid; The case $\mu = 0$ is the analog of Guo et al. (2016) result.

As it turns out, in the case of $\mu < 0$, $\frac{\partial^2 C}{\partial t \partial x}(0^+, t) < 0$ and $t_0$ is such that

$$
\frac{\partial C}{\partial x}(0^+, t_0) = 0, \quad \frac{\partial C}{\partial x}(0^+, t) > 0, \quad t < t_0, \quad \frac{\partial C}{\partial x}(0^+, t) < 0, \quad t > t_0.
$$
Asymptotic Behavior of the Optimal Placement Problem

Goals:

- Approximation of critical horizon \((t_0)\) as \(r + f \to 0\),

- optimal placement \((x^*(t)\) or \(y^*(t)\)) behavior when \(t \to t_0\),

- optimal placement behavior in the low volatility regime \((\sigma \to 0)\),

- optimal placement behavior when \(t\) is large enough.

\(^1\)e.g., the fee and rebate for NYSE are 0.003 and 0.0014, respectively.
Critical Horizon \((t_0)\) Behavior

**Theorem (F-L, Lee, & Pasupathy, 2017)**

Let \(\mu < 0\). Then, as \(r + f \to 0\),

\[
BM: \quad t_0 \sim \frac{\rho(0^+)(r + f)}{2|\mu|}, \quad GBM: \quad t_0 \sim \frac{\rho(0^+)(r + f)}{2|\mu|S_0}.
\]

Figure 9: \(t_0\) (black) for GBM vs. \(\hat{t}_0 := \frac{\rho(0^+)(r + f)}{2|\mu|S_0}\) (red) against \(S_0\) with \(\rho(r + f) = 0.01, \sigma = 0.2, \mu = -0.1\).

Figure 10: \(t_0\) (black) for GBM vs. \(\hat{t}_0 := \frac{\rho(0^+)(r + f)}{2|\mu|S_0}\) (red) against \(S_0\) near \(t=1\) day (same parameters).
Theorem (F-L, Lee, & Pasupathy, 2017)

Suppose that $\mu < 0$ and let $t_0$ be defined as above. Then, as $t \searrow t_0$,

$$y^*(t) = \kappa_1(t - t_0) + \kappa_2(t - t_0)^2 + o((t - t_0)^2),$$

where

$$\kappa_1 := -\frac{\partial^2 C}{\partial t \partial y} (0, t_0), \quad \kappa_2 := \frac{1}{2} \frac{\partial^3 C}{\partial y^3} (0, t_0) \kappa_1^2 + \frac{\partial^3 C}{\partial t \partial y^2} (0, t_0) \kappa_1 + \frac{1}{2} \frac{\partial^3 C}{\partial y \partial t^2} (0, t_0) \kappa_2 + \frac{1}{2} \frac{\partial^4 C}{\partial y^4} (0, t_0).$$
Optimal Placement behavior near $t_0$: example

Figure 11: $y^*(t)$ (black) vs. $\kappa_1(\hat{t}_0)(t - \hat{t}_0)$ (red) against $t$ with $\rho(0^+)(r + f) = 0.01$, $\sigma = 0.2$, $\mu = -0.1$, $S_0 = 50$. Here, $\hat{t}_0 := \rho(0^+)(r + f)/2|\mu|S_0$.

Figure 12: $y^*(1\text{ day})$ (black) and $\kappa_1(\hat{t}_0)(1\text{ day} - \hat{t}_0)$ (red) against $S_0$ with $\rho(0^+)(r + f) = 0.01$, $\sigma = 0.2$, $\mu = -0.1$, $S_0 = 50$. 
Figure 13: Approximations against $t$(days) when $\rho(0^+)(r + f) = 0.006, \sigma = 0.2, \mu = -0.1, S_0 = 50, \hat{t}_0 = \rho(r + f)/(2|\mu|S_0)$. 
Conclusions

BM and GBM Model

- nontrivial optimal placement solution exists when $t$ is larger than a critical value $t_0$
- accurate and simple estimation of threshold horizon, $t_0$,
- behavior of optimal placement solution when:
  - time horizon is near the threshold,
  - volatility is low,
  - time horizon is long enough.

Future and ongoing work:

- introduce robust optimization (unknown $\mu$ and $\sigma$),
- consider other price dynamics (e.g., stochastic volatility model, Lévy process)
- Multistep and “large” order problem