Third-Order Short-Time Expansions for Close-to-the-Money Option Prices Under the CGMY Model

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Abstract

A third-order approximation for close-to-the-money European option prices under an infinite-variation CGMY Lévy model is derived, and is then extended to a model with an additional independent Brownian component. The asymptotic regime considered, in which the strike is made to converge to the spot stock price as the maturity approaches zero, is relevant in applications since the most liquid options have strikes that are close to the spot price. Our results shed new light on the connection between both the volatility of the continuous component and the jump parameters and the behavior of option prices near expiration when the strike is close to the spot price. In particular, a new type of transition phenomenon is uncovered in which the third order term exhibits two distinct asymptotic regimes depending on whether $Y \in (1, 3/2)$ or $Y \in (3/2, 2)$. Unlike second order approximations, the expansions herein are shown to be remarkably accurate so that they can actually be used for calibrating some model parameters. For illustration, we calibrate the volatility $\sigma$ of the Brownian component and the jump intensity $C$ of the CGMY model to actual option prices.

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1 Introduction

Stemming in part from its importance for model testing and calibration, small-time asymptotics of option prices have received a lot of attention in recent years (see, e.g., [3], [5], [10], [16], [17], [18], [19], [23], and references therein). The fact that option prices and implied volatilities exhibit sharply different behaviors under different model assumptions provides a natural tool to test the suitability of these assumptions, as already exploited by the seminal work of Carr and Wu [5]. Hence, for instance, close-to-the-money implied volatilities are expected to stabilize towards a positive value (the spot volatility) near expiration under the presence of a Brownian-like component, while, in contrast, they are expected to vanish near expiration, under a pure-jump model. In both cases, the rates of convergence toward their respective steady limits are determined by the jump activity parameter $Y$, a fact that can potentially allow to assess suitable values for this parameter. Besides testing, it is important to determine what are the most important parameters driving the behavior of option prices near expiration within a class of models. For instance, within the CGMY framework in the presence of a continuous component, the most important parameter is the spot volatility, and the second most (equally) important parameters are $C$ and $Y$. However, nothing was known related to the relevance of $G$ or $M$, before this work.

In this paper, we study the small-time behavior of close-to-the-money European call option prices

$$
E \left( (S_t - S_0 e^{X_t})^+ \right) = S_0 E \left( (e^{X_t} - e^{X_0})^+ \right), \quad t \geq 0,
$$

(1.1)
where \( t \to \kappa_t \) is a deterministic function such that \( \kappa_t \to 0 \) as \( t \to 0 \), and for an exponential Lévy model of the form:
\[
S_t := S_0 e^{X_t}, \quad \text{with} \quad X_t := L_t + \sigma W_t, \quad t \geq 0.
\]
Here, \( L = (L_t)_{t \geq 0} \) is a CGMY Lévy process (cf. [6]), while \( W = (W_t)_{t \geq 0} \) is an independent standard Brownian motion (as usual, \( x^+ := x 1_{\{x > 0\}} \) and \( x^- := x 1_{\{x < 0\}} \) denote the positive and negative parts of a real \( x \)).

The asymptotic behavior of (1.1) is known to change radically depending on whether the parameter \( Y \) of the process \( L \) is smaller or larger than 1 (cf. [23]). We focus here on the latter case, which arguably is more relevant for financial applications, in light of some recent empirical evidence based on high-frequency data supporting this assumption (cf. [1], [4], and the references therein). In the pure-jump CGMY case \((\sigma = 0)\), it is known (cf. [11]) that the short-time second-order asymptotic behavior of the ATM call option price is of the form
\[
\frac{1}{S_0} \mathbb{E} \left( (S_t - S_0)^+ \right) = d_1 t^{\frac{1}{2}} + d_2 t + o(t), \quad t \to 0,
\]
while in the case of a non-zero independent Brownian component \((\sigma \neq 0)\),
\[
\frac{1}{S_0} \mathbb{E} \left( (S_t - S_0)^+ \right) = d_1 t^{\frac{1}{2}} + d_2 t^{\frac{2}{1-Y}} + o \left( t^{\frac{2}{1-Y}} \right), \quad t \to 0,
\]
for (different) constants \( d_1 \) and \( d_2 \), which are explicitly given in the sequel. For extensions of these results to a more general class of processes, we refer the reader to [12] and [13].

In this paper, we derive the third-order asymptotic behavior for the close-to-the-money option prices (1.1) in the CGMY model both with and without an independent Brownian component, when the log-moneyness \( \kappa_t \) converges to 0 at a suitable rate, as the maturity \( t \) goes to 0. Our motivations for considering these expansions are twofold. First, though being a significant improvement over the first-order expansion, in some cases the second-order expansion might not be that accurate unless \( t \) is extremely small (see the numerical examples provided in [12, Section 6] and also in Section 4 herein). This is particularly true in the presence of an independent Brownian component. As shown in the sequel, the third-order expansions, derived here, can dramatically improve the approximation’s accuracy, even for maturities as large as a few years (see, e.g., the left panel of Figure 5 in Section 4). As shown herein, this improvement in accuracy allows to provide adequate calibration of some model parameters such as the volatility \( \sigma \) of the Brownian component and the jump intensity \( C \) of the CGMY model. Second, the expansions developed here shed a new light on the effects of both the volatility of the continuous component and the jump parameters in the behavior of option prices, near expiration, when the strike is close to the spot price. In particular, in the same way as the asymptotic behavior of the leading term substantially changes when \( Y \) transitions at 1, we uncover a similar phenomenon for the third-order term, but this time when \( Y \) transitions at 3/2. This identifies the value of \( Y = 3/2 \) as another transition point for the asymptotic behavior of ATM option prices.

As in [12], an important ingredient in our approach is a change of probability measure, under which \((L_t)_{t \geq 0}\) becomes a stable Lévy process, enabling us to exploit high-order asymptotics for the transition densities of such processes. However, the extension from the second-order to the third-order asymptotics for option prices is quite intricate and requires the development of some new techniques beyond those used in [12]. For instance, as it turns out, an important step in obtaining the asymptotic expansion in the presence of an independent Brownian component is to determine the short-time asymptotics of the following quantity:
\[
R_t^{(k)} := \int_0^\infty \mathbb{E} \left( (\sigma W_1)^k 1_{\{0 \leq \sigma W_1 \leq tz\}} \right) z^{1-k} (p_Z(z) - C z^{-Y-1}) dz, \quad \text{for} \quad k = 0, 1,
\]
where \( p_Z \) is the density of a symmetric stable random variable \( Z \) with a Lévy density \( C |x|^{-Y-1} \) so that
\[
p_Z(z) = C z^{-Y-1} + C' z^{-2Y-1} + o (z^{-2Y-1}), \quad z \to \infty,
\]
for an appropriate constant \( C' \) (see (2.9) below for details). A natural idea to analyze (1.4) is then to plug (1.5) in (1.4) and change variables to \( u = tz \) to get
\[
R_t^{(k)} \sim C' t^{2Y+k-1} \int_0^\infty \mathbb{E} \left( (\sigma W_1)^k 1_{\{0 \leq \sigma W_1 \leq u\}} \right) u^{-2Y-k} du = -C' \frac{\sigma^{1-2Y}}{2(2Y+k-1)} t^{2Y+k-1} \mathbb{E} \left( |W_1|^{1-2Y} \right),
\]
where, in the last equality, Fubini’s theorem and the symmetry of \(W_1\) were used. However, when \(Y > 1, 1-2Y < -1\), and the last expectation is infinite, which shows that the above heuristic argument is false. Instead, in this work, we make use of Fourier analysis techniques for tempered distributions to handle (1.4). This method, interesting on its own, is new and differs from the arguments developed in our earlier work in [12].

The remaining of the paper is organized as follows. Section 2 contains preliminary results on the CGMY model, some probability measure transformations, and asymptotic results for stable Lévy processes, which will be needed throughout the paper. Section 3 establishes the third-order asymptotics for close-to-the-money call option prices, as some probability measure transformations, and asymptotic results for stable Lévy processes, which will be needed in this section to illustrate the high performance of our asymptotic expansions, together with an actual calibration exercise with real option data. The proofs of our main results are deferred to the Appendix.

2 Setup and Preliminary Results

Throughout, \(W := (W_t)_{t \geq 0}\) and \(L := (L_t)_{t \geq 0}\) respectively stand for a standard Brownian motion and a pure-jump CGMY Lévy process independent of each other (cf. [6]) defined on a complete filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\). As usual, we denote the parameters of \(L\) by \(C, G, M > 0\), and \(Y \in (0, 2)\) so that the Lévy measure of \(L\) is given by \(\nu(dx) = C|x|^{-Y-1}(e^{-Mx}1_{x>0} + e^{Gx}1_{x<0})dx\). Hereafter, we assume \(Y \in (1, 2), M > 1\), zero interest rate, and that \(\mathbb{P}\) is a martingale measure for the exponential Lévy model (1.2) with the log-return process \(X_t := L_t + \sigma W_t, t \geq 0\). The following notation is also used in what follows:

\[
M^* := M - 1, \quad G^* := G + 1, \quad \varphi(x) := M^*x 1_{x>0} - G^*x 1_{x<0}, \quad \nu^* (dx) = e^x \nu(dx).
\]

We will make use of two density transformations of the Lévy process (cf. [21, Definition 33.4]). Hereafter, \(\mathbb{P}^*\) and \(\bar{\mathbb{P}}\) are probability measures on \((\Omega, \mathcal{F})\) such that, for any \(t \geq 0\),

\[
\ln \left( \frac{d\mathbb{P}^*}{d\mathbb{P}} \right)_{\mathcal{F}_t} = X_t, \quad \ln \left( \frac{d\bar{\mathbb{P}}}{d\mathbb{P}} \right)_{\mathcal{F}_t} = U_t := \lim_{\epsilon \to 0} \left( \int_0^t \int_{|x|>\epsilon} \varphi(x) N(ds, dx) - t \int_{|x|>\epsilon} (e^{\varphi(x)} - 1) \nu^*(dx) \right),
\]

where \(N(dt, dx) := \#\{ (s, \Delta X_s) \in dt \times dx \} \) is the jump measure of \((X_t)_{t \geq 0}\). Throughout, \(\mathbb{E}^*\) and \(\mathbb{E}\) denote the expectations under \(\mathbb{P}^*\) and \(\bar{\mathbb{P}}\), respectively.

From the density transformation and the Lévy-Itô decomposition of a Lévy process (cf. [21, Theorems 19.2 and Theorem 33.1]), \(X_t = L_t^* + \sigma W_t\), \(t \geq 0\), where, under \(\mathbb{P}^*\), \((W_t^*)_{t \geq 0}\) is again a Wiener process while \((L_t^*)_{t \geq 0}\) is still a CGMY Lévy process, independent of \(W^*\), but with parameters \(C, Y, M = M^*\) and \(G = G^*\). The Lévy triplet of \((X_t)_{t \geq 0}\) under \(\mathbb{P}^*\) is given by \((b^*, (\sigma^*)^2, \nu^*)\) with \(\sigma^* := \sigma, \nu^*(dx) = C|x|^{-Y-1}(e^{-M^*x}1_{x>0} + e^{G^*x}1_{x<0})dx\), and

\[
b^* := -C \Gamma(-Y) [(M^*)^Y + (G^*)^Y - M^Y - G^Y] + \frac{\sigma^2}{2} - \int_{|x|>1} x \nu^*(dx) - C \gamma(-Y) [(M^*)^{-1} - (G^*)^{-1}].
\]

Under the measure \(\bar{\mathbb{P}}\), the process \((L_t^*)_{t \geq 0}\) becomes a stable Lévy process while \((W_t^*)_{t \geq 0}\) remains a Wiener process independent of \(L^*\). Concretely, setting \(\tilde{\nu}(dx) := C|x|^{-Y-1}dx\) and \(\tilde{b} = b^* + \int_{|x| \leq 1} x (\tilde{\nu} - \nu^*)(dx)\), \((X_t)_{t \geq 0}\) is a Lévy process with Lévy triplet \((\tilde{b}, \sigma^2, \tilde{\nu})\), under \(\bar{\mathbb{P}}\). In particular, letting

\[
\tilde{\gamma} := \mathbb{E}(X_t) = -C \Gamma(-Y) [(M - 1)^Y + (G + 1)^Y - M^Y - G^Y] + \frac{\sigma^2}{2},
\]

the centered process \(Z_t := L_t - t \tilde{\gamma}\) is symmetric and strictly \(Y\)-stable under \(\bar{\mathbb{P}}\).

It will be convenient to express the process \((U_t)_{t \geq 0}\) defined in (2.1) in terms of the compensated measure \(\bar{N}(dt, dx) := N(dt, dx) - \tilde{\nu}(dx)dt\) (under \(\bar{\mathbb{P}}\), namely,

\[
U_t = M^* \bar{U}^{(p)}_t - G^* \bar{U}^{(n)}_t + \eta t =: \bar{U}_t + \eta t, \quad t \geq 0,
\]
where
\[ \bar{U}_t^{(p)} := \int_0^t \int_0^\infty x \tilde{N}(ds, dx), \quad \bar{U}_t^{(n)} := \int_0^t \int_{-\infty}^0 x \tilde{N}(ds, dx), \quad \eta := \Gamma(Y) [(M-1)^Y + (G+1)^Y]. \] (2.4)

Note that, under \( \bar{P} \), \((\bar{U}_t^{(p)})_{t \geq 0}\) and \((\bar{U}_t^{(n)})_{t \geq 0}\) are independent and identically distributed one-sided \( Y \)-stable processes with scale, skewness, and location parameters given by \( C|\cos(\pi Y/2)|\Gamma(Y) \), 1, and 0, respectively. Finally, let us further note the following decomposition of the process \( X \) in terms of the processes previously defined:
\[ X_t = Z_t + \bar{t} \gamma + \sigma W^*_t = \bar{U}_t^{(p)} + \bar{U}_t^{(n)} + \bar{t} \gamma + \sigma W^*_t, \quad t \geq 0. \] (2.5)

To conclude this section, we recall some well-known results on the transition densities of stable processes. The following second-order approximation of the density \( p_U(1, x) \) of \( \bar{U}_1^{(p)} \equiv \bar{U}_1^{(n)} \) is well-known (cf. (14.34) in [21]):
\[ p_U(1, x) = Cx^{-Y-1} - \frac{C^2}{2\pi} \sin(2\pi Y) \Gamma(2Y + 1) \Gamma^2(-Y)x^{-2Y-1} + o(x^{-2Y-1}), \quad x \to \infty. \] (2.6)

In particular,
\[ \bar{P}\left( \bar{U}_1^{(p)} \geq x \right) = \bar{P}\left( -\bar{U}_1^{(n)} \geq x \right) = \frac{C}{\sqrt{\pi Y}} x^{-Y} - \frac{C^2}{2\pi} \sin(2\pi Y) \Gamma(2Y + 1) \Gamma^2(-Y)x^{-2Y-1} + o(x^{-2Y-1}), \quad x \to \infty. \] (2.7)
The following result sharpens (2.6) and (2.7). The proof of (2.8-i) is given in [12], while (2.8-ii) is presented in the Appendix B.

**Lemma 2.1.** There exist constants \( 0 < K_1, K_2 < \infty \) such that, for any \( x > 0 \),
\[ (i) \bar{P}\left( \bar{U}_1^{(p)} \geq x \right) = \bar{P}\left( -\bar{U}_1^{(n)} \geq x \right) \leq K_1 x^{-Y}, \quad (ii) \left| \bar{P}\left( \bar{U}_1^{(p)} \geq x \right) - \frac{C}{\sqrt{\pi Y}} x^{-Y} \right| = \left| \bar{P}\left( -\bar{U}_1^{(n)} \geq x \right) - \frac{C}{\sqrt{\pi Y}} x^{-Y} \right| \leq K_2 x^{-2Y}. \] (2.8)

Similarly, the tail distribution and the probability density of \( Z_1 \), hereafter denoted by \( p_Z(1, z) \), admit the following asymptotic behavior (cf. (14.34) in [21]):
\[ \bar{P}(Z_1 \geq z) = \frac{C}{Y} z^{-Y} - \frac{C^2}{\pi Y} \sin(\pi Y) \cos^2 \left( \frac{\pi Y}{2} \right) \Gamma(2Y + 1) \Gamma^2(-Y)z^{-2Y-1} + o(z^{-2Y}), \quad z \to \infty, \]
\[ p_Z(1, z) = Cz^{-Y-1} - \frac{2C^2}{\pi} \sin(\pi Y) \cos^2 \left( \frac{\pi Y}{2} \right) \Gamma(2Y + 1) \Gamma^2(-Y)z^{-2Y-1} + o(z^{-2Y-1}), \quad z \to \infty. \] (2.9)

### 3 The Main Results

In this section, we give the third-order asymptotic behavior for near at-the-money call option prices and implied volatilities in both the pure-jump (\( \sigma = 0 \)) and the mixed (\( \sigma \neq 0 \)) models. We first consider the expansion for the latter case since it is more explicit and of greater use for financial application in view of some empirical evidence, based on high-frequency data, which tends to support a mixed model over either a pure-jump or a purely continuous one (cf. [2]). The results for the pure-jump case are given at the end of the section. The proofs of the main results are deferred to Appendix A.

For a mixed CGMY model with the addition of an independent Brownian component, it was shown in [12, Section 5] that, the second-order correction term for the ATM European call option price is given by (1.3) with
\[ d_1 := \mathbb{E}^* \left( (\sigma W^*_t)^+ \right) = \frac{\sigma}{\sqrt{2\pi}} \quad d_2 := \frac{C \sigma^{1-Y}}{Y(Y-1)} \mathbb{E} \left( |W^*_t|^{1-Y} \right) = \frac{C^{1-Y} \sigma^{1-Y}}{\sqrt{\pi Y}(Y-1) \Gamma \left( \frac{1-Y}{2} \right)}. \] (3.1)

As observed from these expressions, the first-order term only synthesizes the information about the continuous volatility parameter \( \sigma \), while the second-order term also incorporates the information on the degree of jump activity

\[ \text{in terms of the parametrization in [21, Definition 14.16], } (\alpha, \beta, \tau, c) \text{ of } Z_1 \text{ therein is } (Y, 0, 0, 2C|\cos(\pi Y/2)|\Gamma(-Y)). \]
Remark 3.3. Let $\kappa$ the CGMY model (1.2) with an independent Brownian component, as described in the forthcoming theorem.

As mentioned in the introduction, the following preliminary result will play a crucial role in the proof of Theorem 3.2.

**Lemma 3.1.** Let $R_t^{(k)}$ be as in (1.4). Then, $R_t^{(k)} \sim t^{2Y+k-1} E^{(k)}$, where

$$E^{(0)} = -\frac{2C^2Y \cos(\frac{\pi Y}{2}) \Gamma^2(-Y)}{\sqrt{2\pi \sigma^{2Y-1}}} \mathbb{E}(|W_1|^{2Y-2}), \quad E^{(1)} = -\frac{C^2(2Y-1) \cos(\frac{\pi Y}{2}) \Gamma^2(-Y)}{\sqrt{2\pi \sigma^{2Y-1}}} \mathbb{E}(|W_1|^{2Y-2}).$$

**Theorem 3.2.** Let $t \mapsto \kappa_t$, $t \geq 0$, be a deterministic function such that $\kappa_t = o(1)$ as $t \to 0$. Let also

$$d_{31} := \frac{CT(-Y)}{2} [(M-1)^Y - M Y - (G+1)^Y + G^Y],$$

$$d_{32} := -\frac{1}{\pi} \sigma^{1-2Y} C^2 \cos^2(\frac{\pi Y}{2}) \Gamma^2(-Y) 2^{Y-\frac{3}{2}} \Gamma(Y - \frac{1}{2}),$$

$$c_{\kappa,\sigma}(t) := \kappa \int_0^t \mathbb{P}(\sigma W_1 \geq \kappa w) \, dw.$$

Then, under the exponential CGMY model (1.2) with an independent Brownian component,

$$e^{-\kappa_t} S_0 \mathbb{E} \left[ (S_t - S_0 e^{\kappa_t})^+ \right] + c_{\kappa,\sigma}(t) = d_1 t^{\frac{3}{2}} + d_2 t^{\frac{5}{2} - Y} + d_{31} t + d_{32} t^{\frac{5}{2} - Y} + o(\kappa_t) + o(t) + o(t^{\frac{5}{2} - Y}), \quad t \to 0,$$

where $d_1$ and $d_2$ are given as in (3.1) and the terms $o(t)$ and $o(t^{\frac{5}{2} - Y})$ do not depend on $\kappa_t$.

**Remark 3.3.** The form of the asymptotic expansion (3.5), which might, at first, appear unconventional, is chosen with the aim to partially disentangle the effect of the log-moneyness $\kappa_t$, whose value is actually observed, from the option price. If we further assume that $\kappa_t = o(\sqrt{t})$, the expansion becomes

$$e^{-\kappa_t} S_0 \mathbb{E} \left[ (S_t - S_0 e^{\kappa_t})^+ \right] + \frac{\kappa_t}{2} = d_1 t^{\frac{3}{2}} + d_2 t^{\frac{5}{2} - Y} + d_{31} t + d_{32} t^{\frac{5}{2} - Y} + o(t) + o(t^{\frac{5}{2} - Y}) + o(\kappa_t), \quad t \to 0.$$

The quantity appearing on the left-hand side of the above equation, is then called the log-moneyness adjusted price. This quantity can be easily computed since $\kappa_t$ is known. If $Y \in (1, 3/2)$ and $\kappa_t = O(t)$, the third-order term of the log-moneyness adjusted price is $d_{31} t$; if $Y \in (3/2, 2)$ and $\kappa_t = O(t^{5/2-Y})$, the third-order term is $d_{32} t^{\frac{5}{2} - Y}$; finally, if $Y = 3/2$ and $\kappa_t = O(t)$, the third-order term is $(d_{31} + d_{32}) t$.

Our next proposition gives the small-time asymptotic behavior for the close-to-the-money Black-Scholes implied volatility, hereafter denoted by $\sigma$, corresponding to the option prices of Theorem 3.2. The proof is similar to that of [12, Corollary 4.3] and is therefore omitted.

**Proposition 3.4.** Let $d_1$, $d_2$, $d_{31}$, and $d_{32}$ be respectively given by (3.1), (3.3), and (3.4), and let $d_3 = d_{31} 1_{Y \leq 3/2} + d_{32} 1_{Y \geq 3/2}$. Suppose that the log-moneyness $\kappa_t$ is such that $\kappa_t = o(\sqrt{t})$, as $t \to 0$. Then, under the exponential CGMY model (1.2) with an independent Brownian component, as $t \to 0$,

$$\frac{1}{\sqrt{2\pi}} \sigma(t) + \frac{\kappa_t}{2\sqrt{t}} = \begin{cases} 
    d_1 + d_2 t^{1-\frac{Y}{2}} + d_3 t^{\frac{5}{2} - Y} + o(t^{\frac{5}{2} - Y}), & \text{if } 1 < Y \leq 3/2, \\
    d_1 + d_2 t^{1-\frac{Y}{2}} + d_3 t^{2-Y} + o(t^{2-Y}), & \text{if } 3/2 < Y < 2.
\end{cases}$$

Note that the coefficients of the log-moneyness expansion are not needed in (3.6) and the left-hand side of that equation, which we could call the log-moneyness adjusted implied volatility, can actually be computed since we typically know the value of $\kappa_t$.

We now analyze the case of a pure-jump CGMY model.
Theorem 3.5. Let \( t \to \kappa_t \) be a deterministic function such that \( \kappa_t = o(1) \) as \( t \to 0 \). Let also

\[
d_1 := \tilde{E}(Z^+_t) = \frac{1}{\pi} \Gamma \left( 1 - \frac{1}{Y} \right) \left( 2C\Gamma(-Y) \cos \left( \frac{\pi Y}{2} \right) \right)^{\frac{1}{2}},
\]

\[
d_2 := C\Gamma(-Y) \left[ (M - 1)^Y - M^Y - (G + 1)^Y + G^Y \right],
\]

\[
d_{31} := \frac{\tilde{\gamma}^2}{2} p_z(1,0) = \frac{\tilde{\gamma}^2}{2\pi} \Gamma \left( 1 + \frac{1}{Y} \right) \left( -2C\Gamma(-Y) \cos \left( \frac{\pi Y}{2} \right) \right)^{\frac{1}{2}},
\]

\[
d_{32} := -\frac{1}{2\pi} \left[ (Z^+_t + \bar{U}_t)^2 1_{\{Z^+_t + \bar{U}_t \leq 0\}} \right] - \int_{0}^{\infty} w \left( \tilde{P} \left( Z^+_t + \bar{U}_t \geq w \right) - C\Gamma(Y) \frac{Y}{Y^Y} - C(G + 1)^Y \right) dw,
\]

\[
\tilde{\kappa}_t(t) := \kappa_t \int_{0}^{1} \tilde{P}(Z_t \geq \kappa_t w) dw.
\]

Then, under the exponential CGMY model (1.2) without a Brownian component,

\[
\frac{e^{-\kappa_t}}{S_0} \tilde{E} \left( (S_t - S_0 e^{\kappa_t})^+ \right) + \tilde{\kappa}_t(t) = d_1 t^{\frac{1}{2}} + d_2 t + d_{31} t^{\frac{1}{4}} + d_{32} t^{\frac{1}{4}} + o(\kappa_t) + o \left( t^{\frac{1}{4}} \right) + o \left( t^{\frac{1}{2}} \right), \quad t \to 0,
\]

where the terms \( o(t^{2-1/Y}) \) and \( o(t^{2/Y}) \) do not depend on \( \kappa_t \).

Remark 3.6. In the pure-jump case, if we further assume that \( \kappa_t = o(t^{1/Y}) \), then (3.11) becomes

\[
\frac{e^{-\kappa_t}}{S_0} \tilde{E} \left( (S_t - S_0 e^{\kappa_t})^+ \right) + \frac{\kappa_t}{2} = d_1 t^{\frac{1}{2}} + d_2 t + d_{31} t^{\frac{1}{4}} + d_{32} t^{\frac{1}{2}} + o(\kappa_t) + o \left( t^{\frac{1}{4}} \right) + o \left( t^{\frac{1}{2}} \right), \quad t \to 0.
\]

In particular, if \( Y \in (1,3/2) \) and \( \kappa_t = O(t^{2-1/Y}) \), the third-order term of the log-moneyness adjusted price is \( d_{31} t^{3-1/Y} \); if \( Y \in (3/2,2) \) and \( \kappa_t = O(t^{2/Y}) \), the third-order term is \( d_{32} t^{3/Y} \); and finally, if \( Y = 3/2 \) and \( \kappa_t = O(t^{4/3}) \), the third-order term is \( (d_{31} + d_{32}) t^{4/3} \).

We conclude the section by stating the following small-time asymptotic expansion of the close-to-the-money Black-Scholes implied volatility, denoted again by \( \tilde{\sigma} \), corresponding to the option prices of Theorem 3.5. The proof is similar to that of [12, Corollary 3.7] and is therefore omitted.

Proposition 3.7. Let \( d_1, d_2, d_{31}, \) and \( d_{32} \) be respectively given as in (3.7)-(3.10), and let \( d_3 = d_{31} 1_{\{Y \leq 3/2\}} + d_{32} 1_{\{Y \geq 3/2\}} \). Suppose that the log-moneyness \( \kappa_t \) is such that \( \kappa_t = o(t^{1/Y}) \), as \( t \to 0 \). Then, under the exponential CGMY model (1.2) without a Brownian component, as \( t \to 0 \),

\[
\frac{1}{\sqrt{2\pi}} \tilde{\sigma}(t) = \frac{\kappa_t}{2\sqrt{t}} = \begin{cases}
  d_1 t^{\frac{1}{2} - \frac{1}{Y}} + d_2 t^{\frac{1}{2}} + d_3 t^{\frac{1}{4} - \frac{1}{Y}} + o \left( t^{\frac{1}{4} - \frac{1}{Y}} \right), & \text{if } 1 < Y \leq 3/2, \\
  d_1 t^{\frac{1}{2} - \frac{1}{Y}} + d_2 t^{\frac{1}{2}} + d_3 t^{\frac{1}{2} - \frac{1}{Y}} + o \left( t^{\frac{1}{2} - \frac{1}{Y}} \right), & \text{if } 3/2 < Y < 2.
\end{cases}
\]

4 Numerical Examples

4.1 Performance of Approximations

This section is devoted to assess the performance of the previous approximations. For simplicity, we assume \( S_0 = 1 \) and zero interest rate throughout this section. There are two popular numerical methods to evaluate the option prices of parametric Lévy models: Inverse Fourier Transform (IFT) and Monte Carlo (MC) Methods. As illustrated in [12], the IFT method is less accurate than MC method when computing close-to-the-money option prices with short maturities. For instance, consider the IFT method described in [8, Section 11.1.3]), which is based on the formula

\[
z_t(\kappa) := C(\kappa) - C^S_{BS}(\kappa) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iv\kappa} \frac{\varphi_1(v-i) - \varphi^{BS, S}_1(v-i)}{iv(1+iv)} dv =: \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iv\kappa} \zeta_1(v) dv, \quad (4.1)
\]
where $C(\kappa)$ denotes the call option price at the log-moneyness $\kappa = \ln K$ that we wish to compute and $C_{BS}^{2}(\kappa)$ denotes the corresponding call option price at the log-moneyness $\kappa = \ln K$. Above, $\varphi_{t}^{BS,\Sigma} = \exp\left( -\frac{\Sigma^{2}t}{2} (\nu^{2} + iv) \right)$ denotes the characteristic function corresponding to the Black-Scholes model with the volatility $\Sigma$ and $\varphi_{t}$ is the characteristic function of the log-return, under the mixed CGMY model with an independent Brownian component. To explain where the issues in applying (4.1) come from, let us note that, for a close-to-the-money regime, where $\kappa$ is close to zero, the integrand in (4.1) approximately reduces to $\zeta$, which is not easy to integrate numerically for small $t$, since in that case $\varphi_{t}$ and $\varphi_{t}^{BS,\Sigma}$ are quite flat in a large domain of the integration variable $v$ (see [12, Section 10] for numerical results using Simpson’s rule).

In this work, we apply a MC method to compute the option prices under a CGMY model. This is based on the option price representation under the probability measure $\bar{\mathbb{P}}$ (see also [12, Section 6.1]). Using (2.3)-(2.5), we have

$$
\mathbb{E}\left[ (e^{X_{t}} - e^{\kappa_{t}})^{+} \right] = \mathbb{E}^{\ast}\left[ e^{-X_{t}} (e^{X_{t}} - e^{\kappa_{t}})^{+} \right] = \mathbb{E} \left[ e^{-U_{t}} (1 - e^{\kappa_{t}-X_{t}})^{+} \right] = \mathbb{E} \left[ e^{-M^{*}U_{t}^{(p)} + G^{*}U_{t}^{(n)} - \eta t} (1 - e^{\kappa_{t}-U_{t}^{(p)} - U_{t}^{(n)} - \bar{\zeta}t - \sigma W_{t}^{*})^{+} \right],
$$

which can be easily computed by the MC method using the fact that, under $\bar{\mathbb{P}}$, $U_{t}^{(p)}$ and $-U_{t}^{(n)}$ are independent $Y$-stable random variables with scale, skewness and location parameters $(tC|\cos(xY/2)|^{1/Y}, 1$ and 0, respectively. We use the simulation method of [7] to generate the stable random variables $U_{t}^{(p)}$ and $U_{t}^{(n)}$.

Our parameter settings are motivated by the studies in [3] and [22]. Concretely, in [3], a mixed exponential Lévy model with Lévy measure

$$
\nu(dx) = \left( \frac{C_{+} e^{-Mx}}{x^{1+Y}} 1_{\{x>0\}} + \frac{C_{-} e^{Gx}}{|x|^{1+Y}} 1_{\{x<0\}} \right) dx,
$$

was considered. The calibrated parameters were given as follows (see Table 5 therein) for two different stocks:

<table>
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<tr>
<th></th>
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<th>$C_{-}$</th>
<th>$G$</th>
<th>$M$</th>
<th>$Y$</th>
<th>$\sigma$</th>
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<tr>
<td>Stock1</td>
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<td>0.41</td>
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<td>1.5</td>
<td>0.0</td>
</tr>
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<td>0.41</td>
<td>1.93</td>
<td>1.5</td>
<td>0.1</td>
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</tbody>
</table>

In the sequel, we simply take $C := (C_{+} + C_{-})/2$. In [22], the CGMY model was considered, and the calibrated parameters were given as (see Table 6.3 therein):

<table>
<thead>
<tr>
<th></th>
<th>$C$</th>
<th>$G$</th>
<th>$M$</th>
<th>$Y$</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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<td>0.0765</td>
<td>7.5515</td>
<td>1.2945</td>
<td>0.0</td>
</tr>
</tbody>
</table>

We use 100,000 samples to simulate each of the MC-based option prices. The Figures 1-3 compare the first-, second- and third-order approximations, as given in Theorem 3.5 and Theorem 3.2, to the prices based on the MC method introduced above, for both the pure-jump CGMY model and the mixed CGMY model. In all cases, the third-order approximation is much more accurate than the first- and the second-order approximations, for a time $t$ as large as one month.

Moreover, Table 1 summarizes the elapsed time, in seconds, in simulating the MC-based prices as well as the first-, second- and third-order approximations in all cases. As expected, our asymptotic approximations are much more efficient than MC simulations, since all coefficients in our approximations are only made of simple algebraic computations of model parameters, except for $d_{32}$ in the pure-jump case.

<table>
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<tr>
<th>Prices Models</th>
<th>Figure 1 Left</th>
<th>Figure 1 Right</th>
<th>Figure 2 Left</th>
<th>Figure 2 Right</th>
<th>Figure 3 Left</th>
<th>Figure 3 Right</th>
</tr>
</thead>
<tbody>
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<td>182.83</td>
<td>182.81</td>
<td>150.33</td>
<td>150.48</td>
</tr>
<tr>
<td>1st-order Approx</td>
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<td>$7.21 \times 10^{-4}$</td>
<td>$3.61 \times 10^{-4}$</td>
<td>$4.41 \times 10^{-4}$</td>
<td>$1.24 \times 10^{-4}$</td>
<td>$1.46 \times 10^{-4}$</td>
</tr>
<tr>
<td>2nd-order Approx</td>
<td>$4.35 \times 10^{-4}$</td>
<td>$1.12 \times 10^{-3}$</td>
<td>$4.02 \times 10^{-4}$</td>
<td>$4.83 \times 10^{-3}$</td>
<td>$9.06 \times 10^{-4}$</td>
<td>$9.42 \times 10^{-4}$</td>
</tr>
<tr>
<td>3rd-order Approx</td>
<td>$8.19 \times 10^{-4}$</td>
<td>$2.11 \times 10^{-3}$</td>
<td>$0.97$</td>
<td>$0.98$</td>
<td>$1.90 \times 10^{-3}$</td>
<td>$1.94 \times 10^{-3}$</td>
</tr>
</tbody>
</table>
Figure 1: Comparisons of CGMY ATM and close-to-the-money call option prices with the first-, second- and third-order approximations. In both panels, \( C = 0.0244, G = 0.0765, M = 7.5515, Y = 1.2945, \sigma = 0, \) and \( \kappa_t = e_1 t + e_2 t^{2-1/Y}. \) In the left panel, \( e_1 = e_2 = 0, \) while in the right panel, \( e_1 = 0.1 \) and \( e_2 = -0.1. \)

Table 1: Comparisons of the elapsed time (in seconds) of MC-based ATM and close-to-the-money call option prices with the first-, second- and third-order approximations under both the pure-jump CGMY model and the mixed CGMY model.

The computation of \( d_{32}, \) as given in (3.10), involves a double integral, which is numerically unstable if we first compute the tail probability as a function of \( w \) using the MC method, and then evaluate the integral with respect to \( w. \) Instead, we will apply a two-dimensional MC method to evaluate the double integral term in \( d_{32}, \) which is denoted as \( \tilde{d}_{32}. \) More precisely, let

\[
g(u, w) = w \left( 1_{u \geq w} - \frac{CM^Y}{Yw^Y} - \frac{C(G + 1)^Y}{Yw^Y} \right), \quad u \in \mathbb{R}, \quad w \geq 0,
\]

and let \( V \) be an absolutely continuous random variable, supported on \([0, \infty)\) and with density \( f \) under \( \tilde{P}, \) which is independent of \( U := Z_1 + \tilde{U}_1. \) Then,

\[
\tilde{d}_{32} = \mathbb{E} \left( g(U, V) f(V) \right).
\]

The choice of the random variable \( V \) will affect the efficiency and the stability of the MC method. Here we choose \( V \) to be a standard half-normal random variable, and simulate \( \tilde{d}_{32} \) using \( 1000^2 \) pairs of samples \((U, V)\). As shown in Figure 2 as well as in the fourth and fifth column of Table 1, the third-order approximations are almost identical to the MC-based prices, for \( t \) as large as one month, while the corresponding elapsed time is negligible compared to that of the MC-based prices.

4.2 Relevance in Calibration

As indicated in the introduction, a main application of short-time asymptotics is to the calibration of the model’s parameters. In practice this is complicated by the fact that option prices generally exhibit errors and that only certain maturities \( t \) and log-moneyness values \( k \) are available. Hence, at any given date, we can expect the observed option prices \( \Pi^*(t_i, \kappa_j) \) to be given by

\[
\Pi^*(t_i, \kappa_j) = \Pi(t_i, \kappa_j) + \varepsilon_{ij} = S_0 \mathbb{E} \left( e^{X_{t_i}} - e^{\kappa_j} \right) + \varepsilon_{ij}, \quad i = 1, \ldots, I, \quad j = 1, \ldots, J,
\]

for some random errors \( \varepsilon_{ij}. \) The goal in this section is to show that, in spite of the obstacles just mentioned, the approximations herein can be applied to calibrate some of the model’s parameters. For illustration purposes, we
Figure 2: Comparisons of CGMY ATM and close-to-the-money call option prices with the first-, second- and third-order approximations. In both panels, $C = 0.0066$, $G = 0.41$, $M = 1.932$, $Y = 1.5$, $\sigma = 0$, and $\kappa_t = e_1 t + e_2 t^{2-1/Y}$. In the left panel, $e_1 = e_2 = 0$, while in the right panel, $e_1 = 0.1$ and $e_2 = -0.1$.

focus on the volatility $\sigma$ and the parameter $C$. As we shall see, this is not only feasible but, moreover, the higher-order approximations developed here are crucial for this endeavor, and the first- and second-order approximations are not enough to give these results.

Let us assume that, at a given date, we have at our hand ATM option prices $\Pi_i^* := \Pi^*(t_i,0)$ at maturities $t_i (i = 1, \ldots, I)$ and an estimate of the index $Y$, say $\hat{Y}$. Then, the basic idea to estimate $\sigma$ consists in fitting the linear models below to the data:

\begin{align*}
\Pi_i^* := d_1 t_i^{1/2} + \varepsilon_i, & \quad \text{(First-Order),} \\
\Pi_i^* := d_1 t_i^{1/2} + d_2 t_i^{3-Y} + \varepsilon_i, & \quad \text{(Second-Order),} \\
\Pi_i^* := d_1 t_i^{1/2} + d_2 t_i^{3-Y} + d_3 t_i + d_4 t_i^{-1} + \varepsilon_i, & \quad \text{(Higher-Order).}
\end{align*}

Let us denote the resulting least-squares error estimates of $d_1$ based on the three models above by $\hat{d}_1^{(1)}$, $\hat{d}_1^{(2)}$, $\hat{d}_1^{(3)}$. Since, theoretically, $d_1 = \sigma/\sqrt{2\pi}$, the natural estimates for $\sigma$ are then given by

$$\hat{\sigma}^{(\ell)} = \sqrt{2\pi \hat{d}_1^{(\ell)}}, \quad \ell = 1, 2, 3.$$  

We need to keep in mind that these estimates are based on short-time asymptotics for the option prices, which suggest to consider only “small” $t_i$. However, these will reduce the sample size, making the estimates more sensitive to errors.

Let us now show some numerical assessment of the estimates above. We first need to decide on some suitable maturities $t_i$. Based on the closing bid and ask prices for S&P 500 index options on January 2nd, 2014, we find close-to-the-money call option prices for the following maturities (in years):

$$\{t_i\}_{i=1,\ldots,15} \in \{0.021, 0.043, 0.060, 0.079, 0.140, 0.217, 0.241, 0.293, 0.467, 0.491, 0.717, 0.744, 0.967, 1.043, 1.467\}.$$  

The data was obtained from the website HistoricalOptionData.com and is shown on Figure 4. We then simulate ATM option prices at the above maturities using the parameter setting in (4.2) (borrowed from [3]). The option prices and the approximations in (4.3)-(4.5) with $\hat{Y} = 1.5$ are shown on the left panel of Figure 5 (note that the

\footnote{In addition to traditional S&P 500 index options (SPX), the data includes SPXQ (quarterly) and SPXW (weekly) options. The latter class was first introduced in 2005, and, by the end of 2014, it accounted for over 40% of the overall trading of S&P 500 options on the CBOE.}
high-order approximation almost overlaps with the option price). Now, since we do not really have the exact value of $Y$ at our disposal, we compute the estimates of $\hat{\sigma}^{(l)}$ for the range of values $\hat{Y} \in [1.1, 1.8]$ to assess their sensitivity to the value $\hat{Y}$. The results are shown in Figure 6. As shown therein, the estimate $\hat{\sigma}^{(3)}$ is relatively accurate for a large range of values of $\hat{Y}$ (it ranges from 0.0968 to 0.1053), while the first- and second-order estimates are not. This fact would allow us to determine good estimates $\sigma$ even if our estimate $\hat{Y}$ is not very accurate.

We can also apply a similar idea to estimate the parameter $C$ based on the estimate of $\hat{d}_2$, which theoretically is equal to

$$d_2 := \frac{C^2}{\sqrt{\pi Y (Y - 1)}} \eta^{1-Y} \left( 1 - \frac{Y}{2} \right) m_Y.$$  

Concretely, if $\hat{d}_2^{(2)}$ and $\hat{d}_2^{(3)}$ are the estimated values $d_2$ based on the regressions models (4.4)-(4.5), then we set

$$\hat{C}^{(l)} = d_2^{(l)} \left( \hat{\sigma}^{(l)} \right)^{-1} m_Y^{1-l}, \quad \ell = 2, 3. \quad (4.7)$$

The resulting estimates applied to the simulated data of the left panel in Figure 5 are shown on the left panel of Figure 7. Unlike the estimate of $\sigma^{(3)}$, the estimate $\hat{C}^{(3)}$ is more sensitive to the estimate $\hat{Y}$, but when $\hat{Y} \approx 1.5$, the estimate seems accurate, while the estimate $\hat{C}^{(2)}$ based on the second-order approximation (4.3) grossly underestimate $C$ even for $\hat{Y} = 1.5$.

To finish this section, we apply the estimators above to the market option data shown on the table of Figure 4. For the first six maturities (1/10/14-3/22/14), the closest strike to the spot price $S_0 = 1831.98$ is $K = 1830$, while for the rest it is 1825. For the first 4 maturities we take the closest-to-the-money option prices (those with $K = 1830$), while for the rest we take the option prices corresponding to $K = 1825$, because either there is no option with strike $K = 1830$ or because the option with $K = 1825$ has larger volume than the one with $K = 1830$, as in the case of options with maturities 2/22/14 and 3/22/14. The mid prices of the selected data points against maturities are shown on the right panel of Figure 5. We also ran the estimates below taking the options with strike $K = 1825$ for all maturities and essentially obtained the same results. The estimates of $\hat{\sigma}^{(l)}$, $\ell = 1, 2, 3$, for different values of $\hat{Y}$ are shown on the right-panel of Figure 6. As seeing therein, the volatility estimates $\hat{\sigma}^{(3)}$ are relatively stable for $\hat{Y}$ values in the range $[1.1, 1.8]$ (they range from 0.0898 to 0.1121). The estimates based on the first-order approximation (4.3) are much higher, while those based on the second-order approximation (4.4) are more sensitive to the estimates of $\hat{Y}$. We also consider the estimates of $C$ defined in (4.7). The results are shown on the right-panel of Figure 7. As with the simulated results, the estimate $\hat{C}^{(3)}$ is more sensitive to the value $\hat{Y}$ (it ranges from 0.0013 to 0.0054), and it is expected that an accurate estimate of $\hat{Y}$ would be able to accurately estimate $C$. The estimate
\( \hat{C}^{(2)} \) based only on the second-order approximation is more stable but, from our simulations, it is expected to sharply underestimate \( C \). For completeness, we also show the analogous estimators based on the expansion (3.11) using only the \( d_1 \), \( d_2 \), and \( d_{31} \) terms, and the estimator using only the \( d_1 \), \( d_2 \), and \( d_{32} \) terms.

<table>
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<th>bid</th>
<th>ask</th>
<th>volume</th>
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Figure 4: Close-to-the-money SPX Call Option Data on Jan/2/2014 when the underlying was at 1831.98. The emphasized rows (in green) consists of those used in our calibration.

**A  Proofs of the Main Results**

For notational simplicity, throughout all the proofs, we fix \( S_0 = 1 \).

**Proof of Lemma 3.1.** We show the proof for \( R_t^{(0)} \) (the proof for \( R_t^{(1)} \) is similar). To start,

\[
R_t^{(0)} = \frac{1}{2} \int_0^1 \int_\mathbb{R} \frac{t}{\sqrt{2\pi \sigma^2}} e^{-\frac{z^2 + \frac{\lambda}{2t}}{2\sigma^2}} |z|^2 \left( p_Z(z) - C|z|^{-\lambda-1} \right) dz \, du.
\]

Next, denoting the characteristic function of \( Z_1 \) by \( \hat{p}_Z(x) \), note that

\[
p_Z(z) = \mathcal{F} \left( \frac{1}{\sqrt{2\pi}} \hat{p}_Z \right) (z), \quad z^2 p_Z(z) = \mathcal{F} \left( \frac{-1}{\sqrt{2\pi}} \hat{p}''_Z \right) (z),
\]
where $\mathcal{F}(h)(z) := \int_{\mathbb{R}} e^{-izv}h(v)dv/\sqrt{2\pi}$ denotes the Fourier transformation of $h \in L_1(\mathbb{R})$. Also, regarding $|x|^{Y-2}$ as a tempered distribution, it is known that $|z|^{1-Y} = \mathcal{F}(K^{-1}|x|^{Y-2})(z)$, with $K := -2\sin(\pi(Y-2)/2)\Gamma(Y-1)/\sqrt{2\pi}$.

In particular,

$$R_t^{(0)} = \frac{1}{2} \int_{0}^{1} \int_{\mathbb{R}} \mathcal{F} \left( \frac{t}{\sqrt{2\pi}\sigma^2} e^{-\frac{x^2}{2\sigma^2}} \right) \left( -\frac{1}{\sqrt{2\pi}} \hat{p}_Z(x) - \frac{C}{K} |x|^{Y-2} \right) dx du$$

$$= -\frac{1}{2\sqrt{2\pi}} \int_{0}^{1} \int_{\mathbb{R}} u^{-1}e^{-\frac{u^2}{2\sigma^2}} \left( \frac{1}{\sqrt{2\pi}} \hat{p}_Z(x) + \frac{C}{K} |x|^{Y-2} \right) dx du.$$

Since $\hat{p}_Z(x) = e^{-c|x|^Y}$ with $c := 2C|\cos(\pi Y/2)|\Gamma(-Y)$ and $C/K = eY(Y-1)/\sqrt{2\pi}$, differentiation gives

$$R_t^{(0)} = -\frac{e^{-cY}}{2\pi} \int_{0}^{\infty} \int_{0}^{1} e^{-\frac{x^2}{2\sigma^2}} u e^{-cx} dx du + \frac{e^{-Y}(Y-1)}{2\pi} \int_{0}^{1} \int_{0}^{\infty} e^{-\frac{u^2}{2\sigma^2}} u (1-e^{-cx}) dx du. \quad (A.1)$$

For the first term in (A.1), which we denote by $R_t^{(01)}$, we change variables from $x$ to $v = \sigma x/tu$ to get

$$R_t^{(01)} = -\frac{e^{-cY}}{2\pi\sigma^2Y-1} t^{2Y-1} \int_{0}^{1} \left( \int_{0}^{\infty} e^{-\frac{v^2}{2\sigma^2Y-2}} u^{-1} \exp \left(-\frac{c(tu)v}{\sigma^2Y-1}\right) dv \right) du.$$

Hence, by the dominated convergence theorem,

$$\lim_{t \to 0} t^{1-2Y} R_t^{(01)}(t) = -\frac{e^{-cY}}{2\sqrt{2\pi}(2Y-1)} \mathbb{E} \left(|W_1|^{2Y-2}\right) = -\frac{2C^2Y^2 \cos^2 \left(\frac{\pi Y}{2}\right) \Gamma^2(-Y)}{\sqrt{2\pi}(2Y-1)\sigma^{2Y-1}} \mathbb{E} \left(|W_1|^{2Y-2}\right). \quad (A.2)$$

Similarly, the asymptotic behavior of the second term in (A.1), which we denote by $R_t^{(02)}$, is given by

$$\lim_{t \to 0} t^{1-2Y} R_t^{(02)}(t) = -\frac{e^{-Y}(Y-1)}{2\sqrt{2\pi}(2Y-1)} \mathbb{E} \left(|W_1|^{2Y-2}\right) = -\frac{2C^2Y(Y-1) \cos^2 \left(\frac{\pi Y}{2}\right) \Gamma^2(-Y)}{\sqrt{2\pi}(2Y-1)\sigma^{2Y-1}} \mathbb{E} \left(|W_1|^{2Y-2}\right). \quad (A.3)$$

Combining (A.2) and (A.3), we get that $R_t^{(0)} \sim t^{2Y+k-1} E^{(0)}$, with $E^{(0)}$ given in (3.2).

**Proof of Theorem 3.2.** Let $\tilde{X}_t := X_t - \kappa_t$. Then,

$$\mathbb{E} \left( (S_t - e^{\kappa_t})^+ \right) = \mathbb{E}^* \left( 1 - e^{-\tilde{X}_t^+} \right) = e^{\kappa_t} \int_{\kappa_t}^{\infty} e^{-v} \mathbb{P}^*(X_t \geq v) dv =: e^{\kappa_t} (G_2(t) - G_1(t)), \quad (A.4)$$
where
\[
G_1(t) := \int_0^\infty e^{-v} \mathbb{P}^{\ast}(X_t \geq v) \, dv, \quad G_2(t) := \int_0^\infty e^{-v} \mathbb{P}^{\ast}(X_t \geq v) \, dv = \sqrt{t} \int_0^\infty e^{-\sqrt{t}w} \mathbb{P}^{\ast}(X_t \geq t^{-1/2}w) \, dw. \tag{A.5}
\]

For \(G_1(t)\), note that, if \(\kappa_\ell \neq 0\), changing variables to \(w = v/\kappa_\ell\),
\[
G_1(t) = \kappa_\ell \int_0^1 e^{-\kappa_\ell w} \mathbb{P}^{\ast}( \{ t^{-1/2}X_t \geq t^{-1/2}\kappa_\ell w \} ) \, dw = \kappa_\ell \int_0^1 \mathbb{P}^{\ast}(\sigma W_1 \geq t^{-1/2}\kappa_\ell w) \, dw + o(\kappa_\ell), \quad t \to 0,
\]

where we used the well-know fact that \(t^{-1/2}X_t \overset{D}{\to} \sigma W_1\), as \(t \to 0\), and the fact that pointwise convergence of a sequence of distribution functions to a continuous distribution function implies uniform convergence. Indeed, under \(\mathbb{P}^{\ast}\), \(t^{-1/2}X_t = \sigma W_1^t + t^{(2-Y)/2} \Gamma_t^* \) and \(t^{-1/2}L_t^*\) converges in distribution to a symmetric strictly \(Y\)-stable random variable under \(\mathbb{P}^{\ast}\) (cf. [20, Theorem 3.1]). Now, to handle \(G_2(t)\), fix \(\tilde{\gamma}_t := t^{1/2}\tilde{\gamma}\) and consider
\[
\Delta_0(t) := \frac{1}{\sqrt{t}} G_2(t) - d_1 = \int_{-\tilde{\gamma}_t}^\infty e^{-\sqrt{\tilde{\gamma}_t}y} \mathbb{P}^{\ast}(\sigma W_1^t \geq y - t^{-\frac{1}{2}}Z_t) \, dy - \int_0^\infty \mathbb{P}^{\ast}(\sigma W_1^t \geq y) \, dy.
\]

By changing the probability measure \(\mathbb{P}^{\ast}\) to \(\overline{\mathbb{P}}^{\ast}\) and using (2.3) as well as the self-similarity of \(((Z_t, U_t))_{t \geq 0}\),
\[
\Delta_0(t) = e^{-\frac{1}{2}t^{1/2}} \int_0^\infty e^{-\sqrt{\tilde{\gamma}_t}y} \left[ \mathbb{E} \left( e^{-\frac{1}{2}t^{1/2}U_t} \mathbf{1}_{\{ \sigma W_1^t \geq y - t^{-\frac{1}{2}}Z_t \}} \right) \right. - \mathbb{E} \left( e^{-\frac{1}{2}t^{1/2}U_t} \mathbf{1}_{\{ \sigma W_1^t \geq y \}} \right) \left. \right] \, dy
+ \int_0^\infty \left( e^{-\sqrt{\tilde{\gamma}_t}y} - 1 \right) \mathbb{P}^{\ast}(\sigma W_1^t \geq y) \, dy + e^{-\sqrt{\tilde{\gamma}_t}y} \mathbb{E} \left( e^{-\frac{1}{2}t^{1/2}U_t - \frac{1}{2}t^{1/2}y} \mathbf{1}_{\{ \sigma W_1^t \geq y - t^{-\frac{1}{2}}Z_t \}} \right) \, dy
=: A_1(t) + A_2(t) + A_3(t). \tag{A.6}
\]

It is not hard to see that
\[
A_2(t) \sim -\frac{\sigma^2}{4} t^{1/2}, \quad A_3(t) = \frac{\tilde{\gamma}_t^{1/2}}{2} + o \left( t^{1/2} \right), \quad t \to 0.
\]
To analyze Step 1.

Each of the above terms is now analyzed individually in three subsequent steps.

**Step 1.** First, by the change of variable $u = t^{1/2-1/Y} y - \sigma t^{1/2-1/Y} W_1^* + \tilde{U}_1$, Fubini’s theorem, the independence of $W_1^*$ and $(Z_1, \tilde{U}_1)$, and the symmetry of $Z_1$, we have

$$
B_1(t) = t^{\frac{1}{T}} \mathbb{E} \left( e^{-\sqrt{T} W_1^*} 1_{\{W_1^* \geq 0\}} \right) \int_{\mathbb{R}} \left( e^{-t^{\frac{1}{2}} u} - 1 \right) \mathbb{P} \left( Z_1 \geq 0, \tilde{U}_1 \leq u \leq \tilde{U}_1 + Z_1 \right) du - t^{\frac{1}{T}} \int_{-\infty}^{\infty} \left( \int_{\mathbb{R}} \left( e^{-t^{\frac{1}{2}} u} - 1 \right) \mathbb{P} \left( t^{\frac{1}{2}} - \frac{1}{2} w \leq Z_1 \leq 0, \tilde{U}_1 + Z_1 \leq u \leq \tilde{U}_1 \right) dw \right) e^{-\sqrt{T} w} \frac{e^{-\frac{w^2}{2\sigma^2}}}{\sqrt{2\pi \sigma^2}} dw + t^{\frac{1}{T}} \mathbb{E} \left( Z_1 1_{\{W_1^* \geq 0, Z_1 \geq t^{\frac{1}{2}} \sigma W_1^* \}} e^{-\sqrt{T} W_1^*} \right) =: B_{11}(t) - B_{12}(t) + B_{13}(t).
$$

(A.7)

Each of the above terms is now analyzed individually in three subsequent steps.

To study the asymptotic behavior of $A_1(t)$, we first decompose it as:

$$
A_1(t) = e^{-(\eta t + \sqrt{T} \eta t)} \mathbb{E} \left( e^{-t^{\frac{1}{2}} \tilde{U}_1} 1_{\{W_1^* \geq 0, \sigma W_1^* + t^{\frac{1}{2}} - \frac{1}{2} Z_1 \geq 0\}} \int_{\sigma W_1^*}^{\sigma W_1^* + t^{\frac{1}{2}} - \frac{1}{2} Z_1} e^{-\sqrt{T} y} dy \right)
$$

$$
- e^{-(\eta t + \sqrt{T} \eta t)} \mathbb{E} \left( e^{-t^{\frac{1}{2}} \tilde{U}_1} 1_{\{0 \leq \sigma W_1^* \leq -t^{\frac{1}{2}} + \frac{1}{2} Z_1\}} \int_{0}^{\sigma W_1^*} e^{-\sqrt{T} y} dy \right)
$$

$$
+ e^{-(\eta t + \sqrt{T} \eta t)} \mathbb{E} \left( e^{-t^{\frac{1}{2}} \tilde{U}_1} 1_{\{0 \leq -\sigma W_1^* \leq t^{\frac{1}{2}} - \frac{1}{2} Z_1\}} \int_{0}^{\sigma W_1^* + t^{\frac{1}{2}} - \frac{1}{2} Z_1} e^{-\sqrt{T} y} dy \right) =: e^{-(\eta t + \sqrt{T} \eta t)} \left( B_1(t) - B_2(t) + B_3(t) \right).
$$

(A.8)

To analyze $B_{11}(t)$, we use arguments similar to those used to obtain (A.28) and (A.30) to get

$$
\int_{-\infty}^{0} \left( e^{-t^{\frac{1}{2}} u} - 1 \right) \mathbb{P} \left( Z_1 \geq 0, \tilde{U}_1 \leq u \leq \tilde{U}_1 + Z_1 \right) du = t^{\frac{1}{T}} \int_{0}^{0} \left( -u \right) \mathbb{P} \left( Z_1 \geq 0, \tilde{U}_1 \leq u \leq \tilde{U}_1 + Z_1 \right) du + o(t^{\frac{1}{T}}),
$$

$$
\int_{0}^{\infty} \left( e^{-t^{\frac{1}{2}} u} - 1 \right) \mathbb{P} \left( Z_1 \geq 0, \tilde{U}_1 \leq u \leq \tilde{U}_1 + Z_1 \right) du = -t^{1 - \frac{1}{T}} CT(-Y) \left[ M^Y - (M^*)^Y \right] + O(t^{\frac{1}{T}}),
$$

where $C$ is the constant from (4.3).
which imply that
\[ B_{11}(t) = -\frac{1}{2} t^\frac{1}{2} CT(-Y) \left[ M^Y - (M^*)^Y \right] + O \left( t^\frac{1}{2} \right), \quad t \to 0. \]  
(A.9)

It turns out that (see Appendix B),
\[ B_{12}(t) = o(t^\frac{1}{2}), \quad t \to 0. \]  
(A.10)

Finally, we deal with \( B_{13}(t) \), for which we study the asymptotic behavior of
\[ \tilde{B}_{13}(t) := t^\frac{Y}{2} B_{13}(t) - \frac{C \sigma_1^{-1} Y}{2(Y - 1)} \cdot \mathbb{E} \left[ |W_1|^1^{-Y} \right] \]
\[ = t^\frac{Y}{2} + \frac{1}{2} \mathbb{E} \left[ e^{-\sqrt{\sigma} w_1} - 1 \right] 1\{w_1 \geq 0\} \int_{t^\frac{1}{2} + \sqrt{\sigma} w_1}^{\infty} z p_Z(1, z) \, dz \]
\[ + t^\frac{Y}{2} + \frac{1}{2} \mathbb{E} \left[ 1\{w_2 \geq 0\} \int_{t^\frac{1}{2} + \sqrt{\sigma} w_1}^{\infty} z (p_Z(1, z) - C z^{-Y - 1}) \, dz \right]. \]  
(A.11)

We claim that the first term in (A.12) is of order \( O(\sqrt{t}) \). Indeed, by (2.9), there exists \( H_1 > 0 \) such that, for any \( z \geq H_1, p_Z(1, z) \leq 2Cz^{-Y - 1} \). Hence, for any \( w > 0 \),
\[ \int_{t^\frac{1}{2} + \sqrt{\sigma} w}^{\infty} z p_Z(1, z) \, dz \leq \int_{t^\frac{1}{2} + \sqrt{\sigma} w}^{\infty} 2Cw^Y \, dw + 1_{\{t^\frac{1}{2} + \sqrt{\sigma} w < H_1\}} H_1 \mathbb{P} \left( Z_1 \geq t^\frac{1}{2} + \sqrt{\sigma} w \right) \leq t^\frac{1}{2} + \sqrt{\sigma} \left( \frac{2Cw^Y}{Y - 1} + H_1 w^{Y - 1} \right), \]
where in the last inequality we used that \( \mathbb{P}(Z_1 \geq t^{1/2-1/Y} w) \leq (H_1/t^{1/2-1/Y} w)^{Y-1} \), when \( t^{1/2-1/Y} w < H_1 \). Now, the second term in (A.12) is nothing else than \( t^{Y/2+1/Y-3/2}R_{(t^{-2/2})/\sqrt{y}} \) and, thus, we can apply Lemma 3.1 to get
\[ \lim_{t \to 0} t^\frac{Y}{2-1} \tilde{B}_{13}(t) = -\frac{2C^2Y \cos^2 \left( \frac{Y}{2} \right) \Gamma^2(-Y)}{\sqrt{2\pi Y} - 1} \mathbb{E} \left[ |W_1|^2Y^{-2} \right] =: d_3'. \]  
(A.13)

Combining (A.8), (A.9)-(A.11), and (A.13), and setting \( d_3' := CT(-Y)|M^Y - (M^*)^Y| \), lead to
\[ B_1(t) = -\frac{1}{2} CT(-Y) \left[ M^Y - (M^*)^Y \right] t^\frac{1}{2} + t^\frac{1}{2} \left( t^{1+Y} d_3 + o(t^{1+Y}) \right) + \frac{C \sigma_1^{-1} Y}{2(Y - 1)} \mathbb{E} \left[ |W_1|^1^{-Y} \right] + o(t^\frac{1}{2}) \]
\[ = -\frac{1}{2} d_3' t^\frac{1}{2} + \frac{C \sigma_1^{-1} Y}{2(Y - 1)} \mathbb{E} \left[ |W_1|^1^{-Y} \right] t^{1-\frac{Y}{2}} + d_3' t^{2-Y} + o(t^\frac{1}{2}), \quad t \to 0. \]  
(A.14)

**Step 2.** Next, we tackle \( B_2(t) \) by decomposing it as
\[ B_2(t) = \int_0^{\infty} \mathbb{E} \left( e^{-t^\frac{1}{2} \hat{v}_1} - 1 \right) 1\{w_1 \leq -t^\frac{1}{2} + \sqrt{\sigma} w\} \left( 1 - e^{-\sqrt{t} w} \right) \frac{e^{-\frac{w^2}{2\sigma}}}{{\sqrt{2\pi\sigma}}} \, dw + \int_0^{\infty} \mathbb{P} \left( Z_1 \leq -t^\frac{1}{2} + \sqrt{\sigma} w \right) \left( 1 - e^{-\sqrt{t} w} \right) \frac{e^{-\frac{w^2}{2\sigma}}}{{\sqrt{2\pi\sigma}}} \, dw \]
\[ =: B_{21}(t) + B_{22}(t). \]  
(A.15)

We begin with proving that \( B_{21}(t) = o(t^{1/2}) \). To this end, set
\[ B_{21}^{(1)}(t) := \int_0^{\infty} b_{21}^{(1)}(t; w) \, dw, \quad b_{21}^{(1)}(t; w) := \mathbb{E} \left( e^{-t^\frac{1}{2} \hat{v}_1} - 1 \right) 1\{w_1 \leq -t^\frac{1}{2} + \sqrt{\sigma} w, \hat{v}_1 < 0\}. \]

By (A.25), for any \( 0 < t < 1 \) and \( w > 0 \),
\[ 0 \leq t^\frac{1}{2} b_{21}^{(1)}(t; w) = t^\frac{1}{2} \mathbb{E} \left( e^{-t^\frac{1}{2} \hat{v}_1} - 1 \right) 1\{w_1 \leq -t^\frac{1}{2} + \sqrt{\sigma} w, \hat{v}_1 < 0\} \int_0^0 1\{t^\frac{1}{2} \hat{v}_1 \leq u \leq 0\} \, du \]
\[ \leq t^\frac{1}{2} \int_{-\infty}^0 e^{-u} \mathbb{P} \left( \hat{U}_1 \leq -t^\frac{1}{2} u \right) \, du \leq \mathbb{E} \left( e^{-\hat{v}_1} \right) \int_{-\infty}^0 e^{-u \left( 1 - t^\frac{1}{2} \right)} \, du = e^{\frac{1}{2} \left( 1 - t^\frac{1}{2} \right)}. \]
Since $Y \in (1, 2)$, dominated convergence implies that $B_{21}^{(1)}(t) = o(t^{1/2})$, as $t \to 0$. Next, consider

$$B_{21}^{(2)}(t) := \int_0^\infty b_{21}^{(2)}(t; w) \frac{1 - e^{-\sqrt{t} w}}{\sqrt{t}} e^{-\frac{w^2}{2\pi \sigma^2}} dw, \quad b_{21}^{(2)}(t; w) := \mathbb{E} \left( \left( e^{-t^{1/2} \tilde{U}_1} - 1 \right) \mathbbm{1}_{\{Z_1 \leq -t^{1/2} + w, \tilde{U}_1 \geq 0\}} \right),$$

and further consider the decomposition

$$b_{21}^{(2)}(t; w) = \mathbb{E} \left( \left( e^{-t^{1/2} \tilde{U}_1} - 1 + t^{1/2} \tilde{U}_1 \right) \mathbbm{1}_{\{Z_1 \leq -t^{1/2} + w, \tilde{U}_1 \geq 0\}} \right) - t^{1/2} \mathbb{E} \left( \tilde{U}_1 \mathbbm{1}_{\{Z_1 \leq -t^{1/2} + w, \tilde{U}_1 \geq 0\}} \right).$$

Note that, as $t \to 0$,

$$0 \leq t^{-1/2} \int_0^\infty t^{1/2} \mathbb{E} \left( \left( e^{-t^{1/2} \tilde{U}_1} - 1 + t^{1/2} \tilde{U}_1 \right) \mathbbm{1}_{\{Z_1 \leq -t^{1/2} + w, \tilde{U}_1 \geq 0\}} \right) \frac{1 - e^{-\sqrt{t} w}}{\sqrt{t}} e^{-\frac{w^2}{2\pi \sigma^2}} dw \leq t^{-1/2} \mathbb{E} \left( \left| \tilde{U}_1 \right| \right) \int_0^\infty \frac{w e^{-\frac{w^2}{2\pi \sigma^2}}}{\sqrt{2\pi \sigma^2}} dw \to 0.$$

Moreover, by (2.8) and the decomposition $\tilde{U}_1 = M^* \tilde{U}_1^{(p)} - G^* \tilde{U}_1^{(m)}$, for any $t > 0$ and $w > 0$,

$$0 \leq \mathbb{E} \left( \left( e^{-t^{1/2} \tilde{U}_1} - 1 + t^{1/2} \tilde{U}_1 \right) \mathbbm{1}_{\{Z_1 \leq -t^{1/2} + w, \tilde{U}_1 \geq 0\}} \right) = \int_0^\infty \left( 1 - e^{-u} \right) \mathbb{P} \left( \tilde{U}_1 \geq t^{-1/2} u, Z_1 \leq -t^{1/2} + w \right) du \leq 2^{Y+1} K_1 \left[ (M^*)^Y + (G^*)^Y \right] t \int_0^\infty \left( 1 - e^{-u} \right) u^{-Y} du.$$

Hence, by the dominated convergence theorem,

$$t^{-1/2} \int_0^\infty \mathbb{E} \left( \left( e^{-t^{1/2} \tilde{U}_1} - 1 + t^{1/2} \tilde{U}_1 \right) \mathbbm{1}_{\{Z_1 \leq -t^{1/2} + w, \tilde{U}_1 \geq 0\}} \right) \frac{1 - e^{-\sqrt{t} w}}{\sqrt{t}} e^{-\frac{w^2}{2\pi \sigma^2}} dw \to 0, \quad t \to 0.$$

We then conclude that $B_{21}(t) = o(t^{1/2})$. To finish, we analyze $B_{22}(t)$ defined via (A.15). To this end, let

$$B_{22}(t) := t^{Y-1} B_{22}(t) - \frac{C \sigma^{1-Y}}{2Y} \mathbb{E} \left( |W_1^*|^{1-Y} \right) = t^{Y-1} \int_0^\infty \mathbb{E} \left( \frac{1 - e^{-\sqrt{\sigma \tilde{w}}}}{\sqrt{t}} - \sigma \sqrt{t} W_1^* \right) \mathbb{P} \left( \tilde{W}_1^* \leq t^{1/2} \right) p_Z(1, z) dz + t^{Y-1} \int_0^\infty \mathbb{E} \left( \sigma W_1^* \mathbbm{1}_{\{0 \leq \tilde{W}_1^* < t^{1/2} \}} \right) p_Z(1, z) dz - \frac{C \sigma^{1-Y}}{2Y} \mathbb{E} \left( |W_1^*|^{2Y-2} \right) =: d'_2. \quad (A.16)$$

From the inequality $0 \leq e^{-\sqrt{\sigma \tilde{w}}} - 1 + \sigma \sqrt{t} W_1^* \leq \sigma^2 t(W_1^*)^2/2$, valid when $W_1^* \geq 0$, and the estimate (2.9), it is easy to see that the first term in (A.16) is of order $O(t^{1/2})$. The second term in (A.16) is just $t^{1/2 - 1} R_{1(t^{1/2}-1/2)}^{(1)}$ and, thus, applying Lemma 3.1, we conclude that

$$\lim_{t \to 0} t^{Y-1} B_{22}(t) = \frac{C^2 (2Y - 1) \cos^2 \left( \frac{\pi Y}{2} \right) \Gamma^2(-Y)}{\sqrt{2\pi \sigma^2}^{Y-1}} \mathbb{E} \left( |W_1^*|^{2Y-2} \right) =: d'_2. \quad (A.17)$$

Thus, by combining (A.15) and (A.17),

$$B_2(t) = \frac{C \sigma^{1-Y}}{2Y} \mathbb{E} \left( |W_1^*|^{1-Y} \right) t^{1-Y} + d'_2 2^{2Y} + o(t^{2-Y}) + o(t^{1/2} + o(t^{2-Y}), \quad t \to 0. \quad (A.18)$$

Step 3. Finally, we study the behavior of $B_3(t)$ by further decomposing it as

$$B_3(t) = \int_0^\infty \mathbb{E} \left( \left( e^{-t^{1/2} \tilde{U}_1} - 1 \right) \mathbbm{1}_{\{Z_1 \geq t^{1/2} + w\}} \right) \frac{1 - e^{-\sqrt{t} w}}{\sqrt{t}} e^{-\frac{w^2}{2\pi \sigma^2}} dw + \int_0^\infty \mathbb{E} \left( Z_1 \leq t^{1/2} + w \right) \frac{1 - e^{-\sqrt{t} w}}{\sqrt{t}} e^{-\frac{w^2}{2\pi \sigma^2}} dw + \int_0^\infty \mathbb{E} \left( \left( e^{-t^{1/2} \tilde{U}_1} - e^{-t^{1/2} \tilde{U}_1} \tilde{Z} + \tilde{U}_1 \tilde{Z} \right) - t^{1/2} \tilde{Z} \right) \right) e^{-\sqrt{t} w} e^{-\frac{w^2}{2\pi \sigma^2}} dw + \int_0^\infty \mathbb{E} \left( Z_1 \mathbbm{1}_{\{Z_1 \geq t^{1/2} + w\}} \right) e^{-\sqrt{t} w} e^{-\frac{w^2}{2\pi \sigma^2}} dw =: B_{31}(t) + B_{32}(t) + B_{33}(t) + B_{34}(t).$$
First, $B_{32}(t)$ is similar to $B_{22}(t)$ in (A.15) and, thus, arguments similar to those leading to (A.17) imply that
\[ B_{32}(t) = -\frac{C \sigma^{1-Y}}{2Y} \mathbb{E} \left( |W_1^*|^{1-Y} \right) t^{1-Y} - d_{32}' t^{2-Y} + o(t^{2-Y}), \quad t \to 0. \]

Similarly, $B_{34}(t)$ is similar to the term $B_{13}(t)$ introduced in (A.8) and, thus, using arguments similar to those leading to (A.13), we have
\[ B_{34}(t) = \frac{C \sigma^{1-Y}}{2(Y-1)} \mathbb{E} \left( |W_1^*|^{1-Y} \right) t^{1-Y} + d_{31}' t^{2-Y} + o(t^{2-Y}), \quad t \to 0. \]

Since $(-\bar{U}_1^{(n)}, -\bar{U}_1^{(p)}) \overset{D}{=} (\bar{U}_1^{(p)}, \bar{U}_1^{(n)})$, $B_{33}(t)$ has a form similar to $B_{21}(t)$ defined in (A.15), but with the role of the parameters $M^*$ and $G^*$ reversed and $e^{-\sqrt{t}w}$ replaced by $e^{\sqrt{t}w}$. Therefore, as for $B_{21}(t)$, we have that $B_{31}(t) = o(t^{1/2})$, as $t \to 0$. To finish, we further decompose $B_{33}(t)$ as follows:
\[ B_{33}(t) = t^{-\frac{1}{2}} \int_0^\infty \left[ \left( \int_{-\infty}^0 + \int_0^{\infty} \right) (e^{-x} - 1) P_t(w, x) \, dx \right] e^{\sqrt{t}w} \frac{e^{-\frac{w^2}{2\pi \sigma^2}}}{\sqrt{2\pi \sigma^2}} \, dw, \quad (A.19) \]
where
\[ P_t(w, x) := \mathbb{P} \left( Z_1 \geq t^{\frac{1}{2}} + w, \bar{U}_1 \leq t^{\frac{1}{2}} x \leq Z_1 + \bar{U}_1 \right). \quad (A.20) \]

When $x < 0$ and $w > 0$, by (A.25), $P_t(w, x) \leq e^w e^{-t^{1/Y} x}$ and, thus, for $0 < t < 1$ and some constant $K > 0$,
\[ 0 \leq \frac{1}{t} \int_0^\infty \left( \int_{-\infty}^0 (e^{-x} - 1) P_t(w, x) \, dx \right) e^{\sqrt{t}w} \frac{e^{-\frac{w^2}{2\pi \sigma^2}}}{\sqrt{2\pi \sigma^2}} \, dw = K e^{-1} e^w \int_0^\infty e^{w^2} \frac{e^{-\frac{w^2}{2\pi \sigma^2}}}{\sqrt{2\pi \sigma^2}} \, dw \to 0, \quad \text{as} \ t \to 0. \quad (A.21) \]

For the integral $\int_0^\infty \int_0^\infty$ in (A.19), we show below (see Appendix B) that
\[ \lim_{t \to 0} \frac{1}{t} P_t(w, x) = \frac{C}{Y} \left[ M^Y - (M^*)^Y \right] x^{-Y}. \quad (A.22) \]

Moreover, by arguments similar to those leading to (2.8), $t^{-1/2} \mathbb{P}(t^{-1/Y} x \leq Z_1 + \bar{U}_1) \leq \lambda x^{-Y}$, for any $x > 0$ and some constant $\lambda > 0$, and thus, by the dominated convergence theorem,
\[ \lim_{t \to 0} t^{1/2} \int_0^\infty \left( \int_{-\infty}^0 (e^{-x} - 1) P_t(w, x) \, dx \right) e^{\sqrt{t}w} \frac{e^{-\frac{w^2}{2\pi \sigma^2}}}{\sqrt{2\pi \sigma^2}} \, dw = - \frac{C T(-Y)}{2} \left[ M^Y - (M^*)^Y \right] =: d_3. \]

The above limit, together with (A.21), implies that $B_{33}(t) = t^{1/2} d_3' + o(t^{1/2})$ and, thus,
\[ B_3(t) = d_3' t^{\frac{1}{2}} + \frac{C \sigma^{1-Y}}{2Y(Y-1)} \mathbb{E} \left( |W_1^*|^{1-Y} \right) t^{1-Y} + (d_{31}' - d_{32}') t^{2-Y} + o(t^{1/2}) + o(t^{2-Y}), \quad t \to 0. \quad (A.23) \]

Finally, combining (A.6), (A.7), (A.14), (A.18), and (A.23), we establish that
\[ \Delta_0(t) = \left( \frac{\gamma}{2} - \frac{\sigma^2}{4} + 2d_3' \right) t^{\frac{1}{2}} + \frac{C \sigma^{1-Y}}{Y(Y-1)} \mathbb{E} \left( |W_1^*|^{1-Y} \right) t^{1-Y} + 2 (d_{31}' - d_{32}') t^{2-Y} + o(t^{1/2}) + o(t^{2-Y}), \]
which yields (3.5), by noting that the coefficient of the first term above reduces to the expression $d_{31}$ in (3.3) and that $d_{32} = 2(d_{31}' - d_{32}')$. \hfill \Box

**Proof of Theorem 3.5.** In the pure-jump case, the decomposition (A.4) still holds. For $G_1(t)$ given by (A.5), when $\kappa_t \neq 0$, changing variables to $w = v/\kappa_t$ leads to
\[ G_1(t) = \kappa_t \int_0^1 e^{-\kappa_t w} \mathbb{P}^* \left( v^{-1/Y} X_t \geq t^{-1/Y} \kappa_t w \right) \, dw = \kappa_t \int_0^1 \mathbb{P}^* \left( Z \geq t^{-1/Y} \kappa_t w \right) \, dw + o(\kappa_t), \quad t \to 0, \]
where \(Z\) is a symmetric strictly \(Y\)-stable random variable under \(P^*\). In the last equality above, we have used the fact (cf. [20, Theorem 3.1]) that \(t^{-1/Y}X_t\) converges in distribution to a symmetric strictly \(Y\)-stable random variable under \(P^*\), and that pointwise convergence of a sequence of distribution functions to a continuous distribution functions implies uniform convergence.

Now, we study the asymptotic behavior of \(G_2(t)\) given by (A.5). Set \(\tilde{\gamma}_t := t^{1-1/Y}\tilde{\gamma}\) and \(\vartheta := -C\Gamma(-Y)[M^Y + (G^*)^Y]\) and note that, in view of (2.2) and (2.4), \(d_2 = \vartheta + \eta + \tilde{\gamma}/2\). For future reference, it is also convenient to write \(\vartheta\) as

\[
\vartheta = C \frac{1}{Y} \left[ M^Y + (G^*)^Y \right] \int_0^\infty \frac{e^{-t^{1/Y}v} - 1}{t^{1-\frac{1}{Y}}} \nu(Y) dv,
\]

which follows from the well-known identity \(\Gamma(1-Y) = \int_0^\infty (e^{-y} - 1) y^{-Y} dy\) (see (14.18) in [21]). Also, note that

\[
\bar{E} \left( e^{-tU_i} \right) = \bar{E} \left( e^{-t^{1/Y}U_i} \right) = \bar{E} \left( e^{-t^{1/Y}M^YU_i^{(Y)}} \right) \bar{E} \left( e^{t^{1/Y}(G^*)^YU_i^{(Y)}} \right) = e^{\vartheta t}, \quad t \geq 0.
\]

From (2.1), (2.3), (2.5), (A.5), and (A.25), we have

\[
G_2(t) = E^* \left( 1 - e^{-X_i^+} \right) = e^{-\eta t} \bar{E} \left( e^{-\tilde{U}_i} \left( 1 - e^{-X_i^+} \right) \right) = 1 - e^{-\eta t} \bar{E} \left( e^{-\tilde{U}_i - X_i^+} \right).
\]

Set

\[
\Delta_1(t) := t^{\frac{1}{Y}} \bar{E} \left( 1 - e^{-tU_i + X_i^+} \right) - \left( \bar{U}_i + X_i^+ \right), \quad \Delta_2(t) := t^{\frac{1}{Y}} \left( X_i^+ - \bar{E} \left( Z_i^+ \right) \right).
\]

Then, recalling that \(\bar{E} \left( \bar{U}_i \right) = 0\) and \(\bar{E} \left( Z_i^+ \right) = t^{1/Y} \bar{E} \left( Z_i^+ \right)\), we have the decomposition

\[
A(t) := t^{\frac{1}{Y}} \left( X_i^+ - \bar{E} \left( Z_i^+ \right) \right) - d_2
\]

\[
= \left( t^{\frac{1}{Y}} \Delta_1(t) - \vartheta \right) + \left( t^{\frac{1}{Y}} \Delta_2(t) - \frac{\gamma}{2} \right) - \frac{\nu t^{1-1/Y}}{Y} \bar{E} \left( e^{-\tilde{U}_i - X_i^+} \right) - \eta t^{1/Y} \Delta_1(t) - \eta \bar{E} \left( X_i^+ \right).
\]

We will prove that \(A_1(t) = O(t^{2/Y-1})\) (and so \(t^{1/Y} \Delta_1(t) = O(t)\)), and that \(A_2 = O(t^{1-1/Y})\). These results, in turn, imply that \(A_i(t) = O(t) = o(t^{2/Y-1}) = o(t^{1-1/Y})\), \(i \geq 3, 4\), and that \(A_5(t) = O(t^{1/Y}) = o(t^{2/Y-1}) = o(t^{1-1/Y})\), since \(\kappa t^{1-1/Y} \to 0\). So, it remains to analyze the asymptotic behavior of \(A_1(t) = A_2(t)\). This is done in two steps.

**Step 1.** By Fubini’s theorem and a change of variables,

\[
A_1(t) = \left( \int_0^\infty \frac{e^{-t^{1/Y}v} - 1}{t^{1-\frac{1}{Y}}} \bar{E} \left( (Z_i^+ + \bar{U}_i)^+ \right) v \geq \nu(Y) dv - \vartheta \right) - \int_0^\infty \frac{e^{-t^{1/Y}v} - 1}{t^{1-\frac{1}{Y}}} \bar{E} \left( (Z_i^+ + \bar{U}_i)^+ \right) \leq -\nu(Y) dv
\]

\[
=: B_1(t) - B_2(t).
\]

For \(B_2(t)\), using the decompositions (2.3)-(2.5) as well as the self-similarity of \((Z_i, \bar{U}_i)_{t \geq 0}\),

\[
\lim_{t \to 0} t^{-\frac{1}{Y}} B_2(t) = \lim_{t \to 0} \int_0^\infty \frac{e^{t^{1/Y}v} - 1}{t^{1/Y}} \bar{E} \left( (Z_i^+ + \tilde{\gamma}_t)^+ + \bar{U}_1 \leq -v \right) dv = \int_0^\infty v \bar{E} \left( (Z_i^+ + \bar{U}_1 \leq -v \right) dv,
\]

where the second equality follows from the dominated convergence theorem, which applies in view of the following direct consequences of (A.25):

\[
\frac{e^{t^{1/Y}v} - 1}{t^{1/Y}} \bar{E} \left( (Z_i^+ + \tilde{\gamma}_t)^+ + \bar{U}_1 \leq -v \right) \leq ve^{t^{1/Y}v} e^{-v} \bar{E} \left( e^{-\tilde{U}_1} \right) = e^{\vartheta Y} e^{(1-Y)^v} \leq e^{\vartheta Y} e^{-v/2}.
\]

To analyze \(B_1(t)\), we again use the decompositions (2.3)-(2.5) as well as the self-similarity of \((Z_i, \bar{U}_i)_{t \geq 0}\) to get

\[
t^{-\frac{1}{Y}} B_1(t) = \int_0^\infty \frac{e^{-t^{1/Y}v} - 1}{t^{1/Y}} \left( \bar{E} \left( Z_i^+ + \tilde{\gamma}_t > 0, Z_i + \tilde{\gamma}_t + \bar{U}_1 \geq -v \right) - \frac{CM^Y}{Y v^Y} \right) dv
\]

\[
+ \int_0^\infty \frac{e^{-t^{1/Y}v} - 1}{t^{1/Y}} \left( \bar{E} \left( Z_i + \tilde{\gamma}_t \leq 0, \bar{U}_1 \geq v \right) - \frac{C(G^*)^Y}{Y v^Y} \right) dv,
\]

(A.29)
where we have used (A.24). As suggested from the previous decomposition, the limit of each of the terms therein can be obtained by passing \( \lim_{t \to 0} \) into the various integrals to get

\[
\lim_{t \to 0} t^{\frac{1}{2}} R_t(t) = -\int_0^\infty v \left( \mathbb{P} \left( Z_t^+ + \bar{U}_t \geq v \right) - \frac{CM^Y}{Yv^Y} + \frac{C(G^*)^Y}{Yv^Y} \right) dv. \tag{A.30}
\]

For the sake of a more streamlined proof, we defer the justification of the latter operation to Appendix B. Combining (A.27), (A.28), and (A.30), we obtain that

\[
\lim_{t \to 0} t^{\frac{1}{2}} A_1(t) = -\int_0^\infty v \left( \mathbb{P} \left( Z_t^+ + \bar{U}_t \geq v \right) - \frac{CM^Y}{Yv^Y} - \frac{C(G^*)^Y}{Yv^Y} \right) dv - \frac{1}{2} \mathbb{E} \left( (Z_t^+ + \bar{U}_t)^2 \right) \tag{A.31}
\]

**Step 2.** Now, we analyze the behavior of \( A_2 = t^{1/Y-1} \Delta_2(t) - \frac{\gamma}{2} \). By the self-similarity of \( (Z_t)_{t \geq 0} \),

\[
\Delta_2(t) = \mathbb{E} \left( (Z_{\gamma t} + \gamma \bar{U}) - Z_t^+ \right) = \int_0^\infty \left( \mathbb{P}(Z_1 \geq u - \gamma \bar{U}) - \mathbb{P}(Z_1 \geq u) \right) du = \int_0^\infty \int_{\gamma u}^\infty p_Z(w) dw \, du,
\]

where for simplicity we wrote \( p_Z(u) \) for the density \( p_Z(1,u) \) of \( Z_1 \). From the symmetry of \( Z_1 \), \( \frac{\gamma}{2} = \gamma \int_0^\infty p_Z(u) \, du \) and, thus, recalling that \( \gamma := t^{-1/Y} \gamma \),

\[
\lim_{t \to 0} t^{\frac{1}{2}} A_2(t) = \lim_{t \to 0} t^{\frac{1}{2}} - \left( t^{1/Y-1} \int_0^\infty \left( \frac{1}{\gamma u} \int_{\gamma u}^\infty p_Z(w) \, dw \right) du \right)
\]

\[
= \lim_{t \to 0} t^{2-2\gamma^2} \int_0^\infty v \left( \int_0^1 \int_{u-\gamma u}^\infty p_Z(u + \beta \nu) \, d\beta \right) dv
\]

\[
= -\lim_{t \to 0} t^{2-2\gamma^2} \int_0^\infty v \left( \int_0^1 p_Z(\beta \nu t^{1-\gamma}) \, d\beta \right) dv = \frac{\gamma^2 p_Z(0)}{2} \tag{A.32}
\]

The expression for \( d_{31} \) as given in (3.9) is obtained from the power series representation of \( p_Z \) around \( z = 0 \) shown, for example, in (14.30) of [21]. Finally, combining (A.31) and (A.32) with (A.26), we obtain (3.11). \( \square \)

**B Further Proofs**

**Proof of Lemma 2.1.** From the leading term in the expansion (2.7), there exists \( N > 0 \) such that, for any \( x > 0 \),

\[
\mathbb{P} \left( U_1^{(p)} \geq x \right) = \mathbb{P} \left( U_1^{(p)} \geq x \right) \left( 1_{(x > N)} + 1_{(x < N)} \right) \leq \frac{2C}{Y} x^{-Y} 1_{(x > N)} + \frac{N^Y}{x^Y} 1_{(x < N)} \leq (2C_Y^{-1} + N_Y) x^{-Y},
\]

and the first relationship in (2.8) follows by setting \( K_1 = 2C_Y^{-1} + N_Y \). Similarly, from (2.7), there exists \( N > 0 \) such that, for any \( x > 0 \),

\[
\left| \mathbb{P} \left( U_1^{(p)} \geq x \right) - \frac{C}{Y} x^{-Y} \right| \leq \frac{C^2}{Y} \Gamma(2Y) \Gamma(1/Y) (x > N) \left( 1_{(x > N)} + 1_{(x < N)} \right) \leq \left( \frac{C^2}{Y} \right) \Gamma(2Y) \Gamma(1/Y) (x > N) \leq \left( \frac{C^2}{Y} \right) \Gamma(2Y) \Gamma(1/Y) (x > N) \leq \left( \frac{C^2}{Y} \right) \Gamma(2Y) \Gamma(1/Y) (x > N).
\]

The second identity in (2.8) follows by setting \( K_2 = C^2 \left( \left| \sin(2\pi Y) \right| \Gamma(2Y) \Gamma(1/Y) \right) \Gamma(2Y) \Gamma(1/Y) \). \( \square \)

**Proof of (A.30).** We begin with \( B_{11}(t) \). Using (2.3) and (2.5), leads to the decomposition

\[
\mathbb{P} \left( Z_t^+ + \bar{U} \geq v \right) = \mathbb{P} \left( U_1^{(p)} \geq v \right) + \mathbb{P} \left( U_1^{(p)} \geq v + \frac{M^* \bar{U}}{M + G} \right) + \mathbb{P} \left( U_1^{(p)} \geq v + \frac{G \bar{U}_1^{(n)}}{M} - \frac{\gamma}{2} \right) - \frac{CM^Y}{Yv^Y}
\]

\[
= b_1^{(1)}(t; v) + b_1^{(2)}(t; v).
\]

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For any \( v > 0 \) and \( t \) small enough (so that \( G^*|\gamma_t| < 1 \) and \( M^*|\gamma_t| < 1 \),
\[
b_{11}^{(1)}(t; v) \leq v \beta \left( \frac{v + M^* \tilde{\gamma}_t}{M + G} - \tilde{\gamma}_t \right) \bar{P} \left( \frac{v + M^* \tilde{\gamma}_t}{M + G} \right) \leq v 1_{\{v \leq 1\}} + v 1_{\{v > 1\}} \min \left( 1, K_1^2 (M + G)^2 v^{-2} \right),
\]
where \( K_1 \in (0, \infty) \) is given as in (2.8). We now consider \( b_{11}^{(2)}(t; v) \). It suffices to consider \( v > 1 \), since \( |b_{11}^{(2)}(t; v)| \leq v (1 + CY^{-1} M^* v^{-1}) \), which is integrable on \( \{v \leq 1\} \). We also let \( t \) be small enough, so that \( |\gamma_t| < 1 \), \( G^*|\gamma_t| < 1 \), and \( M^*|\gamma_t| < 1 \). Then, for any \( v > 1 \),
\[
|b_{11}^{(2)}(t; v)| \leq v \int_{-\infty}^{\frac{v + M^* \tilde{\gamma}_t}{M + G}} p_{U(t, y)}(1, y) \bar{P} \left( U^{(p)}_1 \geq \frac{v - G y - \tilde{\gamma}_t}{M} \right) - \frac{CM^Y}{Y (v - G y - \tilde{\gamma}_t) Y} dy
+ v \int_{-\infty}^{\frac{v + M^* \tilde{\gamma}_t}{M + G}} p_{U(t, y)}(1, y) \frac{CM^Y}{Y} \left( (v - G y - \tilde{\gamma}_t)^{-Y} - v^{-Y} \right) dy + \frac{CM^Y}{Y v^{-1}} \bar{P} \left( U^{(p)}_1 \geq \frac{v + M^* \tilde{\gamma}_t}{M + G} \right)
=: D_1^{(1)}(v) + D_1^{(2)}(v) + D_1^{(3)}(v).
\]
Next, by (2.8), we have \( D_1^{(1)}(v) \leq K_2 (M + G)^2 v^{-1} e^{-2} \), for any \( v > 1 \). Using the convexity and monotonicity of the function \( f(x) = x^{-Y} \) on \((0, \infty)\), \( D_1^{(2)}(v) \leq CM^Y v^{-Y} (G^* U^{(p)}_1 + 1) \). Finally, again by (2.8), we have \( D_1^{(3)}(v) \leq K_1 C M^Y Y^{-1} v^{-1} e^{-2} \), for any \( v > 1 \). Combining the previous estimates, it is now clear that we can apply the dominated convergence theorem to the first integral in (A.29) to obtain its limit as \( t \to 0 \). One can apply similar arguments to justify passing the limit in the second integral in (A.29).

**Proof of (A.10).** First, change variables, \( x = t^{1/Y} u \), in the integral of the term \( B_{12}(t) \) defined in (A.8), so that
\[
B_{12}(t) = t^{-\frac{1}{Y}} e^{- (u + \sqrt{\gamma_t})} \int_0^\infty \left( \int_R (e^{-x} - 1) \bar{P} \left( t^{\frac{1}{Y}} x \leq Z_1 \leq 0, \bar{U}_1 + Z_1 \leq t^{\frac{1}{Y}} x \leq \bar{U}_1 \right) \right) e^{-\sqrt{u}w} e^{-\frac{w^2}{2\pi\sigma^2}} dw.
\]
We next prove that \( B_{12}(t) = o(t^{1/2}) \) as \( t \to 0 \). To this end, let
\[
B_{12}^{(1)}(t) = \int_0^\infty \left[ \int_0^\infty (1 - e^{-x}) \bar{P} \left( t^{\frac{1}{Y}} x \leq Z_1 \leq 0, \bar{U}_1 + Z_1 \leq t^{\frac{1}{Y}} x \leq \bar{U}_1 \right) \right] e^{-\sqrt{u}w} e^{-\frac{w^2}{2\pi\sigma^2}} dw,
\]
\[
B_{12}^{(2)}(t) = \int_0^\infty \left( e^{-x} - 1 \right) \bar{P} \left( t^{\frac{1}{Y}} x \leq Z_1 \leq 0, \bar{U}_1 + Z_1 \leq t^{\frac{1}{Y}} x \leq \bar{U}_1 \right) \right) \right] e^{-\sqrt{u}w} e^{-\frac{w^2}{2\pi\sigma^2}} dw.
\]
For any \( t > 0 \), \( w > 0 \) and \( x > 0 \), by (2.8) and (A.20),
\[
\frac{1}{t} P_t(x, w) = \bar{P} \left( t^{\frac{1}{Y}} x \leq U^{(p)}_1 + U^{(n)}_1 \leq 0, (M^* + 1) \bar{U}_1^{(p)} - (G^* - 1) \bar{U}_1^{(n)} \leq t^{\frac{1}{Y}} x \leq M^* U^{(p)}_1 - G^* U^{(n)}_1 \right)
\leq \frac{1}{t} \bar{P} \left( t^{\frac{1}{Y}} x \leq \bar{U}_1^{(n)} \leq \bar{U}_1^{(p)} \right)
\leq \frac{1}{t} \bar{P} \left( t^{\frac{1}{Y}} x \leq M^* U^{(p)}_1 - G^* U^{(n)}_1 \right)
\leq K_1 (M^* + G^*)^{-Y} \bar{P} \left( \frac{t^{\frac{1}{Y}} x}{M^* + G^*} \leq U^{(p)}_1 \right) \to 0, \quad t \to 0,
\]
while for \( t > 0 \), \( w > 0 \) and \( x < 0 \),
\[
\frac{1}{t} P_t(x, w) \leq \frac{2}{t} \bar{P} \left( \bar{U}_1^{(p)} \leq \frac{t^{\frac{1}{Y}} x}{2 (M^* + G^*)} \right) \leq \frac{2}{t} \bar{P} \left( e^{-U^{(n)}_1} \right) \exp \left( \frac{t^{\frac{1}{Y}} x}{2 (M^* + G^*)} \right) \to 0, \quad t \to 0.
\]
It follows from dominated convergence that \( B_{12}^{(1)}(t) = o(t) \) and \( B_{12}^{(2)}(t) = o(t) \), which in turn implies (A.10).

**Proof of (A.22).** First, for any \( t > 0 \), \( x > 0 \) and \( w > 0 \), by (A.20),
\[
\frac{1}{t} P_t(x, w) = \frac{1}{t} \bar{P} \left( \bar{U}_1^{(p)} + U^{(n)}_1 \geq t^{\frac{1}{Y}} x + u, M^* \bar{U}_1^{(p)} - G^* \bar{U}_1^{(n)} \leq t^{\frac{1}{Y}} x \leq M \bar{U}_1^{(p)} - G \bar{U}_1^{(n)} \right)
= \frac{1}{t} \int_R \left( \bar{U}_1^{(p)} \geq t^{\frac{1}{Y}} x + u, \frac{t^{\frac{1}{Y}} x - G u}{M} \leq \bar{U}_1^{(p)} \leq \frac{t^{\frac{1}{Y}} x - G^* u}{M^*} \right) p_U(1, u) du.
\]
Note that
\[
\frac{t^{-\frac{1}{
u}} x - Gu}{M} \leq \frac{t^{-\frac{1}{
u}} x - G^* u}{M^*} \iff u \leq \frac{t^{-\frac{1}{
u}} x}{M + G}, \quad t^{-\frac{1}{
u}} w + u \leq \frac{t^{-\frac{1}{
u}} x - G^* u}{M^*} \iff u \leq \frac{t^{-\frac{1}{
u}} x - M^* t^{rac{1}{
u}} - \frac{1}{
u} w}{M + G},
\]

Hence,
\[
\frac{1}{t} P_t(w, x) = 1 \int_{-\infty}^{\frac{t^{-\frac{1}{
u}} x - M^* t^{rac{1}{
u}} - \frac{1}{
u} w}{M + G}} \mathbb{P} \left( \frac{t^{-\frac{1}{
u}} x - Gu}{M} \leq \tilde{U}_1^{(p)} \leq \frac{t^{-\frac{1}{
u}} x - G^* u}{M^*} \right) p_U(1, u) \, du
\]
\[
+ \int_{-\infty}^{\frac{t^{-\frac{1}{
u}} x - M^* t^{rac{1}{
u}} - \frac{1}{
u} w}{M + G}} \mathbb{P} \left( t^{-\frac{1}{
u}} w + u \leq \tilde{U}_1^{(p)} \leq \frac{t^{-\frac{1}{
u}} x - G^* u}{M^*} \right) p_U(1, u) \, du
\]
\[
=: I_1(t; w, x) + I_2(t; w, x).
\]

For $I_1(t; w, x)$, note that for any $t > 0, x > 0$ and $w > 0$,
\[
u \leq \frac{t^{-\frac{1}{
u}} x - M^* t^{rac{1}{
u}} - \frac{1}{
u} w}{M + G} < \frac{t^{-\frac{1}{
u}} x}{M + G} < \frac{t^{-\frac{1}{
u}} x}{G} \Rightarrow t^{-\frac{1}{
u}} x - Gu > 0, \quad \frac{x - Mw \sqrt{t}}{M + G} > 0 \iff t < \frac{x^2}{M^2 w^2}.
\]

Hence, by (2.7) and the dominated convergence theorem, for any $x > 0, w > 0$ and $u \leq t^{-1/y} (x - Mw)/(M + G),$
\[
\lim_{t \to 0} I_1(t; w, x) = \int_{\mathbb{R}} p_U(1, u) \left( \lim_{t \to 0} \int_{-\infty}^{t^{-\frac{1}{
u}} x - G^* u}{M^*} \right) \left( 1_{\left\{ t^{-\frac{1}{
u}} x - G^* u}{M^*} \leq \tilde{U}_1^{(p)} \leq \frac{t^{-\frac{1}{
u}} x - G^* u}{M^*} \right) \right) du
\]
\[
= \int_{\mathbb{R}} p_U(1, u) \left( \lim_{t \to 0} \mathbb{P} \left( \tilde{U}_1^{(p)} \geq \frac{t^{-\frac{1}{
u}} x - Gu}{M} \right) \right) du - \int_{\mathbb{R}} p_U(1, u) \left( \lim_{t \to 0} \mathbb{P} \left( \tilde{U}_1^{(p)} \geq \frac{t^{-\frac{1}{
u}} x - G^* u}{M^*} \right) \right) du
\]
\[
= C \left[ M^{-Y} - (M^*)^{-Y} \right] x^{-Y}.
\]

For $I_2(t; w, x)$, since for any $x > 0$ and $w > 0$, $t^{-1/y} x - M^* t^{rac{1}{
u}} - \frac{1}{
u} w > 0$ is equivalent to $t < w^2/(M^2 w^2),$
\[
0 \leq \frac{1}{t} \int_{t^{-\frac{1}{
u}} x - M^* t^{rac{1}{
u}} - \frac{1}{
u} w} \mathbb{P} \left( t^{-\frac{1}{
u}} w + u \leq \tilde{U}_1^{(p)} \leq \frac{t^{-\frac{1}{
u}} x - G^* u}{M^*} \right) p_U(1, u) \, du
\]
\[
\leq \frac{1}{t} \mathbb{P} \left( \tilde{U}_1^{(p)} \geq \frac{t^{-\frac{1}{
u}} x - M^* t^{rac{1}{
u}} - \frac{1}{
u} w}{M + G} \right) \to 0, \quad t \to 0,
\]

which completes the proof. \(\square\)

References


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