Third-Order Short-Time Expansions for Close-to-the-Money Option Prices Under the CGMY Model

José E. Figueroa-López∗ Ruoting Gong† Christian Houdré‡

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Abstract

The short-time asymptotic behavior of option prices for a variety of models with jumps has received much attention in recent years. In the present work, novel third-order approximations for close-to-the-money European option prices under an infinite-variation CGMY Lévy model are derived, and are then extended to a model with an additional independent Brownian component. The asymptotic regime considered, in which the strike is made to converge to the spot stock price as the maturity approaches zero, is relevant in applications since the most liquid options have strikes that are close to the spot price. Our results shed new light on the connection between both the volatility of the continuous component and the jump parameters and the behavior of option prices near expiration when the strike is close to the spot price. In particular, a new type of transition phenomenon is uncovered in which the third order term exhibits two distinct asymptotic regimes depending on whether $Y \in (1, 3/2)$ or $Y \in (3/2, 2)$.

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1 Introduction

Stemming in part from its importance for model testing and calibration, small-time asymptotics of option prices have received a lot of attention in recent years (see, e.g., [3], [5], [10], [11], [18], [19], [20], [22], [25], and references therein). The fact that option prices and implied volatilities exhibit sharply different behaviors under different model assumptions provides a natural tool to test the suitability of these assumptions, as already exploited by the seminal work of Carr and Wu [5]. Hence, for instance, close-to-the-money implied volatilities are expected to stabilize towards a positive value (the spot volatility) near expiration under the presence of a Brownian-like component, while, in contrast, they are expected to vanish near expiration, under a pure-jump model. In both cases, the rates of convergence toward their respective steady limits are determined by the jump activity parameter $Y$, a fact that can potentially allow to assess suitable values for this parameter. Besides testing, it is important to determine what are the most important parameters driving the behavior of option prices near expiration within a class of models. For instance, within the CGMY framework in the presence of a continuous component, the most important parameter is the spot volatility, and the second most (equally) important parameters are $C$ and $Y$. However, nothing was known related to the relevance of $G$ or $M$.

In the present paper, we revise and further extend the results in our preliminary report [12], which will stay unpublished, to consider third-order expansions for close-to-the-money option prices under an exponential CGMY model with or without an independent Brownian component. Concretely, we study the small-time behavior for close-to-the-money European call option prices

$$
\mathbb{E}\left[(S_t - S_0 e^{\kappa t})^+\right] = S_0 \mathbb{E}\left[(e^{X_t} - e^{\kappa t})^+\right], \quad t \geq 0,
$$

(1.1)

∗Department of Statistics, Purdue University, West Lafayette, IN 47907, USA (figueroa@purdue.edu). Research supported in part by the NSF Grants: DMS-0906919, DMS-1149692.
†Department of Applied Mathematics, Illinois Institute of Technology, Chicago, IL 60616, USA (rgong2@iit.edu).
‡School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332, USA (houdre@math.gatech.edu). Research supported in part by the grant #246283 from the Simons Foundation.
where \( t \to \kappa_t \) is a deterministic function such that \( \kappa_t \to 0 \) as \( t \to 0 \), under the exponential Lévy model

\[
S_t := S_0 e^{X_t}, \quad \text{with } X_t := L_t + \sigma W_t, \quad t \geq 0, \tag{1.2}
\]

where \( L = (L_t)_{t \geq 0} \) is a CGMY Lévy process (cf. [6]), while \( W = (W_t)_{t \geq 0} \) is an independent standard Brownian motion (as usual, \( x^+ := x 1_{\{x > 0\}} \) and \( x^- := x 1_{\{x < 0\}} \) denote the positive and negative parts of a real \( x \)). By the Markov property of the Lévy process \( (X_t)_{t \geq 0} \), the asymptotic behavior of (1.1), as \( t \downarrow 0 \), can be viewed as both the asymptotic behavior in small maturity (where \( t \) represents the maturity) and in time-to-maturity (where \( t \) represents the time-to-maturity for any given finite maturity) of option prices.

As is well known, the first-order asymptotic behavior of (1.1) changes radically depending on whether the parameter \( Y \) of the process \( L \) is smaller or larger than 1 (cf. [25]). We focus here on the latter case, which arguably is more relevant for financial applications, in light of some recent empirical evidence supporting this assumption, based on both high-frequency data (cf., [1], [4], [9]) and option data (cf., [3], [16], and [24]). For \( Y \in (1, 2) \), the short-time first-order asymptotic behavior for the at-the-money (ATM) option price (i.e., \( \kappa_t = 0 \) in (1.1) for all \( t \geq 0 \)) takes the form:

\[
\lim_{t \to 0} t^{-\frac{1}{\alpha}} \frac{1}{S_0} \mathbb{E} \left[ (S_t - S_0)^+ \right] = \mathbb{E} \left( Z^+ \right),
\]

where \( Z \) is a symmetric \( \alpha \)-stable random variable under \( \mathbb{P} \). When \( \sigma \neq 0 \), \( Z \sim \mathcal{N}(0, \sigma^2) \) (\( \alpha = 2 \)) and, thus, \( \mathbb{E}(Z^+) = \sigma/\sqrt{2\pi} \) (cf. [22] and [25]). When \( \sigma = 0 \), \( \alpha = Y \) and the characteristic function of \( Z \) is explicitly given (cf. [10] and [25]) by

\[
\mathbb{E} \left( e^{iuZ} \right) = \exp \left( -2C \Gamma(-Y) \left| \cos \left( \frac{\pi Y}{2} \right) \right| |u|^Y \right).
\]

In that case, (cf. (25.6) in [23]),

\[
\mathbb{E} \left( Z^+ \right) = \frac{1}{\pi} \Gamma \left( 1 - \frac{1}{Y} \right) \left( 2C \Gamma(-Y) \left| \cos \left( \frac{\pi Y}{2} \right) \right| \right) \frac{1}{Y}.
\]

Moreover (cf. [11]), in the pure-jump CGMY case (\( \sigma = 0 \)), the short-time second-order asymptotic behavior of the ATM call option price is of the form

\[
\frac{1}{S_0} \mathbb{E} \left[ (S_t - S_0)^+ \right] = d_1 t^{\frac{1}{2}} + d_2 t + o(t), \quad t \to 0,
\]

while in the case of a non-zero independent Brownian component (\( \sigma \neq 0 \)),

\[
\frac{1}{S_0} \mathbb{E} \left[ (S_t - S_0)^+ \right] = d_1 t^{\frac{1}{2}} + d_2 t^{\frac{2-Y}{2}} + o \left( t^{\frac{2-Y}{2}} \right), \quad t \to 0,
\]

for (different) constants \( d_1 \) and \( d_2 \), which are explicitly given in the sequel. For extensions of these results to a more general class of Lévy processes, we refer the reader to [13].

In this paper, we derive the third-order asymptotic behavior for the close-to-the-money option prices (1.1) in the CGMY model both with and without an independent Brownian component, when the log-moneyness \( \kappa_t \) converges to 0 (so that the strike \( K \) converges to \( S_0 \)) at a suitable rate, as the maturity \( t \) goes to 0. In particular, we obtain the third-order asymptotic behavior of the ATM option prices under both models. Moreover, these results are in line with the intuitive idea that if \( K \) is close enough to \( S_0 \) (relative to the time-to-expiration \( t \)), then the third-order expansion for ATM option prices is still accurate. See also [19] and [14] for other pieces of work where close-to-the-money asymptotics of first- and second-order are studied.

There are important reasons for considering the third-order expansions. As shown in the numerical examples provided in [13, Section 6], though being a significant improvement over the first-order expansion, in some cases the second-order expansion is not that accurate unless \( t \) is extremely small. This is particularly true in the presence of an independent Brownian component. As shown in the sequel (see the figures in Section 4 below), the third-order expansions, derived here, can significantly improve the approximation accuracy. Moreover, in the same way as the asymptotic behavior of the leading term substantially changes when \( Y \) transitions at 1, we uncover a similar
phenomenon for the third-order term, but this time when \( Y \) transitions at \( 3/2 \). This identifies the value of \( Y = 3/2 \) as another transition point for the asymptotic behavior of ATM option prices.

As in [13], an important ingredient in our approach is a change of probability measure under which \((L_t)_{t \geq 0}\) becomes a stable Lévy process\(^1\), which, in turn, enables us to exploit some high-order asymptotic results for the transition density and the tail distribution of such processes. However, it is important to emphasize that the generalization from the second- to the third-order is quite intricate and requires new results and techniques beyond those used in [13]. For instance, an important step in obtaining the asymptotic expansion in the presence of an independent Brownian component is to determine the short-time asymptotics of the following type of quantities:

\[
R_t^{(k)} := \int_0^\infty \mathbb{E} \left[ (\sigma W_1)^k 1_{\{0 \leq \sigma W_1 \leq t^2\}} \right] (p_Z(z) - C_1 z^{-Y-1}) \, dz, \quad \text{for } k = 0, 1,
\]

where \( p_Z \) is the density of a symmetric stable random variable \( Z \) with the Lévy density of the form \( C|x|^{-Y-1} \) so that (cf. (14.34) in [23])

\[
p_Z(z) = C_1 z^{-Y-1} + C_2 z^{-2Y-1} + o(z^{-2Y-1}), \quad z \to \infty,
\]

for appropriate constants \( C_1 \) and \( C_2 \) (see (2.10) below). A natural approach to tackle this problem would go along the following lines:

\[
R_t^{(k)} = t^{-1} \int_0^\infty \mathbb{E} \left[ (\sigma W_1)^k 1_{\{0 \leq \sigma W_1 \leq u\}} \right] \left[ p_Z(t^{-1}u) - C_1 (t^{-1}u)^{-Y-1} \right] du
\]

\[
\sim C_2 t^{2Y} \int_0^\infty \mathbb{E} \left[ (\sigma W_1)^k 1_{\{0 \leq \sigma W_1 \leq u\}} \right] u^{-2Y-1} du \quad (t \to 0)
\]

\[
= -C_2 \sigma^{k-2Y} \frac{4Y}{t^{2Y}} \mathbb{E} \left[ W_1^{k-2Y} \right],
\]

where in the last equality, Fubini’s theorem and the symmetry of \( W_1 \) would be used. However, since \( Y > 1 \), then \( k - 2Y < -1 \), and the last expectation is infinite, making the above argument useless. As it turns out, with Fourier analysis techniques for tempered distributions, we show that

\[
R_t^{(0)} \sim t^{2Y-1} E^{(0)}, \quad R_t^{(1)} \sim t^{2Y} E^{(1)}, \quad t \to 0,
\]

for some explicitly derived constants \( E^{(k)}, k = 0, 1 \).

Clearly, a method which is able to yield asymptotics of arbitrary orders for a relatively general model is desirable. This is indeed achievable for out-the-money option prices (cf. [10] and [11]), but asymptotics for ATM option prices are notoriously technically hard to obtain and attempts to extend the present results to, even, the fourth-order or to a more general tempered-stable-like process have yet to be successful.

The remaining of the paper is organized as follows. Section 2 contains preliminary results on the CGMY model, some probability measure transformations, and asymptotic results for stable Lévy processes which will be needed throughout the paper. Section 3 establishes the third-order asymptotics of close-to-the-money call option prices under both the pure-jump CGMY model (\( \sigma = 0 \)) and the CGMY model with an independent Brownian component (\( \sigma \neq 0 \)). The asymptotics for the Black-Scholes implied volatilities are also considered in this section. Section 4 provides several numerical examples that illustrate the high performance of our asymptotic expansions. The proofs of our main results are deferred to the Appendix.

### 2 Setup and Preliminary Results

Throughout, \( W := (W_t)_{t \geq 0} \) and \( L := (L_t)_{t \geq 0} \) respectively stand for a standard Brownian motion and a pure-jump CGMY Lévy process independent of each other (cf. [6]) defined on a complete filtered probability space

\(^1\)Such a probability measure transformation is well-known in the literature (see, e.g., [17]) and has been exploited for simulation purposes (cf. [21]).
\((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\). As usual, we denote the parameters of \(L\) by \(C, G, M > 0\), and \(Y \in (0, 2)\) so that the Lévy measure of \(L\) is given by
\[
\nu(dx) = \left( \frac{Ce^{-Mx}}{x^{1+Y}} 1_{\{x > 0\}} + \frac{Ce^{Gx}}{x^{1+Y}} 1_{\{x < 0\}} \right) dx.
\]

Hereafter, we assume \(Y \in (1, 2)\), \(M > 1\), zero interest rate, and that \(\mathbb{P}\) is a martingale measure for the exponential Lévy model \((1.2)\) with the log-return process \(X_t := L_t + \sigma W_t\), \(t \geq 0\). In particular, the characteristic function of \(X_t\), for any \(t \geq 0\), is given by
\[
\varphi_t(u) := \mathbb{E} \left( e^{iuX_t} \right) = \exp \left( -\frac{\sigma^2 u^2}{2} + CT(-Y) \left[ (M - iu)^Y + (G + iu)^Y - M^Y - G^Y \right] \right),
\]

with \(c := -CT(-Y)[(M-1)^Y + (G+1)^Y - M^Y - G^Y] - \sigma^2/2\). The following notation is used in what follows:
\[
M^* = M - 1, \quad G^* = G + 1, \quad e^* = c + \sigma^2, \quad \varphi(x) := M^* x 1_{\{x > 0\}} - G^* x 1_{\{x < 0\}}, \quad \nu^*(dx) = e^* \nu(dx).
\]

We will also make use of two density transformations of the Lévy process (cf. [23, Definition 33.4]). Hereafter, \(\mathbb{P}^*\) and \(\mathbb{P}\) are probability measures on \((\Omega, \mathcal{F})\) such that for any \(t \geq 0\):
\[
\frac{d\mathbb{P}^*|_{\mathcal{F}_t}}{d\mathbb{P}|_{\mathcal{F}_t}} = e^{U_t}, \quad \frac{d\mathbb{E}^*|_{\mathcal{F}_t}}{d\mathbb{E}|_{\mathcal{F}_t}} = e^{U_t},
\]

where
\[
U_t := \lim_{\varepsilon \to 0} \sum_{s \leq \varepsilon |\Delta X_s| > \varepsilon} \varphi(\Delta X_s) - \frac{t}{\varepsilon} \int_{|x| > \varepsilon} (e^{\varphi(x)} - 1) \nu^*(dx), \quad t \geq 0.
\]

Throughout, \(\mathbb{E}^*\) and \(\mathbb{E}\) denote the expectations under \(\mathbb{P}^*\) and \(\mathbb{P}\), respectively.

From the density transformation and the Lévy-Itô decomposition of a Lévy process (cf. [23, Theorems 19.2 and Theorem 33.1]), \((X_t)_{t \geq 0}\) can be written as
\[
X_t = L^*_t + \sigma W^*_t, \quad t \geq 0,
\]

where, under \(\mathbb{P}^*\), \((W^*_t)_{t \geq 0}\) is again a Wiener process while \((L^*_t)_{t \geq 0}\) is still a CGMY process, independent of \(W^*\), but with parameters \(C, Y, M = M^*\) and \(G = G^*\). The Lévy triplet of \((X_t)_{t \geq 0}\) under \(\mathbb{P}^*\) is given by \((b^*, (\sigma^*)^2, \nu^*)\) with \(\sigma^* := \sigma\) and
\[
b^* := c^* - \int_{|x| > 1} x \nu^*(dx) - CYT(-Y) \left[ (M^*)^{Y-1} - (G^*)^{Y-1} \right].
\]

Similarly, under the measure \(\mathbb{P}\), the process \((L^*_t)_{t \geq 0}\) becomes a stable Lévy process while \((W^*_t)_{t \geq 0}\) remains a Wiener process independent of \(L^*\). Concretely, setting
\[
\tilde{\nu}(dx) := C|x|^{-Y-1}dx, \quad \tilde{b} = b^* + \int_{|x| \leq 1} x (\tilde{\nu} - \nu^*) (dx),
\]

under \(\mathbb{P}\), \((X_t)_{t \geq 0}\) is a Lévy process with Lévy triplet \((\tilde{b}, (\sigma^2), \tilde{\nu})\). In particular, letting
\[
\tilde{\gamma} := \mathbb{E} (X_1) = -CT(-Y) \left[ (M - 1)^Y + (G + 1)^Y - M^Y - G^Y \right] + \frac{\sigma^2}{2},
\]

the centered process
\[
Z_t := L^*_t - t\tilde{\gamma}, \quad t \geq 0,
\]

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is symmetric and strictly $Y$-stable under $\tilde{P}$, and is thus self-similar; i.e., $(t^{-1/Y}Z_{ut})_{u \geq 0} \equiv (Z_u)_{u \geq 0}$, for any $t > 0$. As usual, $\equiv$ represents equality of the respective finite-dimensional distributions.

The process $(U_t)_{t \geq 0}$ can be expressed in terms of the jump-measure $N(dt, dx) := \{(s, \Delta X_s) \in dt \times dx\}$ of $(X_t)_{t \geq 0}$ and its compensator $\tilde{N}(dt, dx) := N(dt, dx) - \tilde{v}(dx)dt$ (under $\tilde{P}$), namely,

$$U_t := \tilde{U}_t + \eta t := M^*U^{(p)}_t - G^*U^{(n)}_t + \eta t, \quad t \geq 0,$$

where

$$\tilde{U}^{(p)}_t := \int_0^t \int_{0^+} \infty \times N(ds, dx), \quad \tilde{U}^{(n)}_t := \int_0^t \int_{-\infty}^0 x \tilde{N}(ds, dx),$$

$$\eta := C \int_0^\infty \left( e^{-M^*x} + M^*x \right) x^{-Y-1} dx + C \int_{-\infty}^0 \left( e^{G^*x} + G^*x \right) |x|^{-Y-1} dx = C \Gamma(-Y) \left[ (M-1)^Y + (G+1)^Y \right].$$

Finally, let us further note the following decomposition of the process $X$ in terms of the previously defined processes:

$$X_t = Z_t + \Theta \gamma + \sigma W^*_t = \tilde{U}^{(p)}_t + \tilde{U}^{(n)}_t + \Theta \gamma + \sigma W^*_t, \quad t \geq 0.$$  

(2.6)

To conclude this section, we recall some well-known results on stable Lévy processes needed in the sequel. First, under $\tilde{P}$, $(\tilde{U}^{(p)}_t)_{t \geq 0}$ and $(-\tilde{U}^{(n)}_t)_{t \geq 0}$ are independent and identically distributed one-sided $Y$-stable processes with scale, skewness, and location parameters given by $C \cos(\pi Y/2)\Gamma(-Y)$, $1$, and $0$, respectively. The common transition density of $\tilde{U}^{(p)}_t$ and $-\tilde{U}^{(n)}_t$ is hereafter denoted by $p_U(t, x, t \geq 0)$. The following second-order approximation of $p_U(1, x)$ is also well-known² (cf. (14.34) in [23]):

$$p_U(1, x) = C x^{-Y-1} - \frac{C^2}{2\pi} \sin(2\pi Y) \Gamma(2Y + 1) \Gamma^2(-Y) x^{-2Y-1} + o(x^{-2Y-1}), \quad x \to \infty.$$  

(2.7)

In particular,

$$\tilde{P}\left( \tilde{U}^{(p)}_1 \geq x \right) = \tilde{P}\left( -\tilde{U}^{(n)}_1 \geq x \right) = \frac{C}{Y} x^{-Y} - \frac{C^2}{2\pi} \sin(2\pi Y) \Gamma(2Y) \Gamma^2(-Y) x^{-2Y} + o(x^{-2Y}), \quad x \to \infty.$$  

(2.8)

The following result sharpens (2.7) and (2.8). The proof of (2.9-i) was given in [13], while the proof of (2.9-ii) is presented in Appendix B.

**Lemma 2.1.** There exist constants $0 < K_1, K_2 < \infty$ such that, for any $x > 0$,

(i) $\tilde{P}\left( \tilde{U}^{(p)}_1 \geq x \right) \leq K_1 x^{-Y}$, \quad (ii) $\left| \tilde{P}\left( \tilde{U}^{(p)}_1 \geq x \right) - \frac{C}{Y} x^{-Y} \right| \leq K_2 x^{-2Y}$.  

(2.9)

Similarly, the tail distribution and the probability density of $Z_1$, hereafter denoted by $p_Z(1, z)$, respectively admit the following asymptotic expansions³ (cf. (14.34) in [23]),

$$\tilde{P}(Z_1 \geq z) = \frac{C}{Y} z^{-Y} - \frac{C^2}{\pi Y} \sin(\pi Y) \cos^2 \left( \frac{\pi Y}{2} \right) \Gamma(2Y + 1) \Gamma^2(-Y) z^{-2Y} + o(z^{-2Y}), \quad z \to \infty,$n

$$p_Z(1, z) = C z^{-Y-1} - \frac{2C^2}{\pi} \sin(\pi Y) \cos^2 \left( \frac{\pi Y}{2} \right) \Gamma(2Y + 1) \Gamma^2(-Y) z^{-2Y-1} + o(z^{-2Y-1}), \quad z \to \infty.$$  

(2.10)

As in the proof of Lemma 2.1, there exists a constant $0 < K_3 < \infty$ such that, for any $z > 0$,

$$\tilde{P}(Z_1 \geq z) \leq K_3 z^{-Y}.$$  

(2.11)

Finally, the following identity will also be of use:

$$\mathbb{E} \left( e^{-\tilde{U}_t} \right) = \mathbb{E} \left( e^{-t^{1/Y} \tilde{U}_1} \right) = \mathbb{E} \left( e^{-t^{1/Y} M^* \tilde{U}^{(p)}_1} \right) \mathbb{E} \left( e^{t^{1/Y} G^* \tilde{U}^{(n)}_1} \right) = e^{\eta t}, \quad t \geq 0.$$  

(2.12)

²In terms of the parameterization $(\alpha, \beta, \gamma, c)$ introduced in [23, Definition 14.16], $(\alpha, \beta, \gamma, c)$ of $\tilde{U}^{(p)}_1$ and $-\tilde{U}^{(n)}_1$ is $(Y, 1, 0, C \cos(\pi Y/2)\Gamma(-Y))$.

³In terms of the parametrization in [23, Definition 14.16], $(\alpha, \beta, \gamma, c)$ of $Z_1$ therein is $(Y, 0, 0, 2C \cos(\pi Y/2)\Gamma(-Y))$. 

5
3 The Main Results

In this section, we obtain the third-order asymptotic behavior for near at-the-money call option prices in both the pure-jump ($\sigma = 0$) and the mixed ($\sigma \neq 0$) models. The expansion for the latter is more explicit and of greater use for financial application in view of some recent empirical evidence, based on high-frequency data, which tends to support a mixed model over either a pure-jump or a purely continuous one (cf. [2]). However, since the proof for the mixed model is more intricate, we first present the pure-jump case for easiness of exposition. The proofs of all the results are deferred to the Appendix.

**Theorem 3.1.** Let $t \to \kappa_t$ be a deterministic function such that

$$\kappa_t = e_1 t + e_2 t^{2-1/Y} + o(t^{2-1/Y}), \quad \text{as } t \to 0,$$

for some constants $e_1, e_2 \in \mathbb{R}$. Then, under the exponential CGMY model (1.2) without a Brownian component,

$$\Pi(t) := \frac{1}{S_0} \mathbb{E} \left[ (S_t - S_0 e^{\kappa_t})^+ \right] = d_1 t^{\frac{1}{Y}} + d_2 t + d_{31} t^{2-\frac{1}{Y}} + d_{32} t^2 + o \left( t^{2-\frac{1}{Y}} \right) + o \left( t^{\frac{1}{Y}} \right), \quad t \to 0, \quad \text{(3.2)}$$

where

$$d_1 := \mathbb{E} \left( Z_1^+ \right) = \frac{1}{\pi} \Gamma \left( 1 - \frac{1}{Y} \right) \left( 2CT(-Y) \left| \cos \left( \frac{\pi Y}{2} \right) \right| \right)^{\frac{1}{Y}},$$

$$d_2 := \frac{CT(-Y)}{2} \left[ [(M - 1)^Y - M^Y - (G + 1)^Y + G^Y] - \frac{e_1}{2} \right],$$

$$d_{31} := \frac{(\tilde{\gamma} - e_1)^2}{2} p_Z(1,0) - \frac{e_2}{2} = \left( \frac{\tilde{\gamma} - e_1}{2} \right)^2 \Gamma \left( 1 + \frac{1}{Y} \right) \left( -2CT(-Y) \cos \left( \frac{\pi Y}{2} \right) \right)^{-\frac{1}{Y}} - \frac{e_2}{2},$$

$$d_{32} := \frac{1}{\mathbb{E} \left( Z_1^+ + \bar{U}_1 \right)} \left[ Z_1^+ + \bar{U}_1 \right] 1_{\{Z_1^+ + \bar{U}_1 \leq 0\}} - \int_0^\infty w \left[ \tilde{\mathbb{E}} \left( Z_1^+ + \bar{U}_1 \geq w \right) - \frac{CM^Y}{Y w^Y} - \frac{C(G + 1)^Y}{Y w^Y} \right] dw. \quad \text{(3.6)}$$

In particular, if $Y \in (1,3/2)$, the third-order term is $d_{31} t^{2-\frac{1}{Y}}$, if $Y \in (3/2, 2)$, the third-order term is $d_{32} t^2$, and, finally, if $Y = 3/2$, the third-order term is $(d_{31} + d_{32}) t^{\frac{2}{Y}}$.

**Remark 3.2.** The finiteness of the integral appearing in (3.6) is not that clear but, follows from the decompositions for $\bar{U}_1$ and $Z_1$ in terms of $U_1^{(p)}$ and $U_1^{(n)}$ described in (2.4) and (2.6), respectively. More specifically,

$$\tilde{\mathbb{E}} \left( Z_1^+ + \bar{U}_1 \geq w \right) = \tilde{\mathbb{E}} \left( \bar{U}_1^{(p)} + \bar{U}_1^{(n)} \geq 0, M U_1^{(p)} - G U_1^{(n)} \geq w \right) + \tilde{\mathbb{E}} \left( \bar{U}_1^{(p)} + \bar{U}_1^{(n)} < 0, M^* U_1^{(p)} - G^* U_1^{(n)} \geq w \right).$$

Then, by arguments similar to those used in the proof of (A.7) given in Appendix B,

$$w \left( \tilde{\mathbb{E}} \left( \bar{U}_1^{(p)} + \bar{U}_1^{(n)} \geq 0, M U_1^{(p)} - G U_1^{(n)} \geq w \right) - \frac{CM^Y}{Y w^Y} \right) = O(w^{1-2Y}) + O(w^{-Y}), \quad w \to \infty. \quad \text{(3.7)}$$

Similarly,

$$w \left( \tilde{\mathbb{E}} \left( \bar{U}_1^{(p)} + \bar{U}_1^{(n)} < 0, M^* U_1^{(p)} - G^* U_1^{(n)} \geq w \right) - \frac{C(G + 1)^Y}{Y w^Y} \right) = O(w^{1-2Y}) + O(w^{-Y}), \quad w \to \infty. \quad \text{(3.8)}$$

Together (3.7) and (3.8) imply the finiteness of the integral appearing in (3.6).

The next result provides the asymptotic behavior of the corresponding ATM Black-Scholes implied volatility, which hereafter is denoted by $\hat{\sigma}$. The proof is similar to that of [13, Corollary 3.7] and is therefore omitted.

**Proposition 3.3.** Let $\kappa_t \equiv 0$ in (3.1). Let $d_1$, $d_2$, $d_{31}$ and $d_{32}$ be respectively given as in (3.3)-(3.6), with $e_1 = e_2 = 0$, and let $d_3 = d_{31} 1_{\{Y \leq 3/2\}} + d_{32} 1_{\{Y \geq 3/2\}}$. Then, under the exponential CGMY model (1.2) without a Brownian component, as $t \to 0$,

$$\frac{1}{\sqrt{2\pi}} \hat{\sigma}(t) = \begin{cases} 
    d_1 t^{\frac{1}{Y} - 1} + d_2 t^\frac{1}{Y} + d_{31} t^{\frac{2}{Y} - \frac{1}{Y}} + o \left( t^{\frac{2}{Y} - \frac{1}{Y}} \right), & \text{if } 1 < Y \leq \frac{3}{2}, \\
    d_1 t^{\frac{1}{Y} - 1} + d_2 t^\frac{1}{Y} + d_{32} t^{\frac{2}{Y} - \frac{1}{Y}} + o \left( t^{\frac{2}{Y} - \frac{1}{Y}} \right), & \text{if } \frac{3}{2} < Y < 2.
\end{cases} \quad \text{(3.9)}$$
We now analyze the case of a mixed CGMY model with the addition of an independent Brownian component. In that instance, it was shown in [13, Section 5] that, the second-order correction term for the ATM European call option price is given by
\[
\frac{1}{S_0} \mathbb{E} \left[ (S_t - S_0)^+ \right] = d_1 t^{\frac{1}{2}} + d_2 t^{\frac{3}{2} - Y} + o \left( t^{\frac{3}{2} - Y} \right), \quad t \to 0,
\]
with
\[
d_1 := \mathbb{E}^* \left[ (\sigma W_t^*)^+ \right] = \frac{\sigma}{\sqrt{2\pi}} , \quad d_2 := \frac{C \sigma^{1-Y}}{Y(Y-1)} \mathbb{E} \left[ W_t^{|1-Y} \right] = \frac{C^2 \sigma^{1-Y}}{\sqrt{\pi Y} (Y-1)} \Gamma \left( 1 - \frac{Y}{2} \right) .
\]
As observed from these expressions, the first-order term only synthesizes the information about the continuous volatility parameter \( \sigma \), while the second-order term also incorporates the information on the degree of jump activity \( Y \) and the overall jump-intensity parameter \( C \). However, these two terms do not reflect the relative intensities of the negative or positive jumps (as given by the parameters \( G \) and \( M \), respectively). This fact suggests the need of a higher-order approximation as described in the following theorem.

**Theorem 3.4.** Let \( t \to \kappa_t \) be a deterministic function such that
\[
\kappa_t = e_1 t + e_2 t^{\frac{3}{2} - Y} + o \left( t^{\frac{3}{2} - Y} \right), \quad as \ t \to 0 ,
\]
for some constants \( e_1, e_2 \in \mathbb{R} \). Let also
\[
d_{31} := -CT(-Y) \left[ (G + 1)^Y - G^Y \right] + \frac{\tilde{Y} - e_1}{2} ,
\]
\[
d_{32} := \frac{1}{\pi} \sigma^{1-2Y} C^2 \cos^2 \left( \frac{\pi Y}{2} \right) \Gamma^2(-Y) 2^{2Y-\frac{3}{2}} \Gamma \left( Y - \frac{1}{2} \right) - \frac{e_2}{2} .
\]
Then, under the exponential CGMY model (1.2) with an independent Brownian component,
\[
\Pi(t) := \frac{1}{S_0} \mathbb{E} \left[ (S_t - S_0 e^{\kappa_t})^+ \right] = d_1 t^{\frac{1}{2}} + d_2 t^{\frac{3}{2} - Y} + d_{31} t + d_{32} t^{\frac{3}{2} - Y} + o(t) + o \left( t^{\frac{3}{2} - Y} \right), \quad t \to 0 ,
\]
where \( d_1 \) and \( d_2 \) are given as in (3.11). In particular, if \( Y \in (1, 3/2) \), the third-order term is \( d_{31} t \), if \( Y \in (3/2, 2) \), the third-order term is \( d_{32} t^{\frac{3}{2} - Y} \), and, finally, if \( Y = 3/2 \), the third-order term is \( (d_{31} + d_{32}) t \).

**Remark 3.5.** Interestingly, the third-order term depends on the parameters \( G \) and \( M \), which controls the relative intensities of the negative and positive jumps, respectively, but only if \( Y < 3/2 \).

Our final proposition gives the small-time asymptotic behavior for the ATM Black-Scholes implied volatility, denoted again by \( \hat{\sigma} \), corresponding to the option prices of Theorem 3.4 with \( \kappa_t \equiv 0 \) in (3.12). Unlike the pure-jump case, we can only derive the second-order asymptotic behavior using Theorem 3.4. In fact, the first-order term of the ATM call option price for the generalized CGMY model is the same as the one for the Black-Scholes model. The third-order term of \( \hat{\sigma} \) requires higher-order asymptotics of the ATM call option price. The proof is similar to that of [13, Corollary 4.3] and is therefore omitted.

**Proposition 3.6.** Let \( \kappa_t \equiv 0 \) in (3.12). Let \( d_2, d_{31} \) and \( d_{32} \) be respectively given by (3.11), (3.13) and (3.14), and let \( d_3 = d_{31} \mathbb{1}_{\{Y \leq 3/2\}} + d_{32} \mathbb{1}_{\{Y \geq 3/2\}} \). Then, under the exponential CGMY model (1.2) with an independent Brownian component, as \( t \to 0 \),
\[
\frac{1}{\sqrt{2\pi}} \hat{\sigma}(t) = \begin{cases} 
\sigma + d_2 t^{1-\tilde{Y}} + d_3 t^{\frac{3}{2} - Y} + o \left( t^{\frac{3}{2} - Y} \right), & \text{if } 1 < Y \leq \frac{3}{2}, \\
\sigma + d_2 t^{1-\tilde{Y}} + d_3 t^{2-Y} + o \left( t^{2-Y} \right), & \text{if } \frac{3}{2} < Y < 2.
\end{cases}
\]

### 4 Numerical Examples

This section is devoted to assess the performance of the previous approximations through a detailed numerical analysis. For simplicity, we assume \( S_0 = 1 \) and zero interest rate throughout this section.
In the sequel, we simply take

\[ \varphi_t^{BS,\Sigma}(v) := \exp \left( -\frac{\Sigma^2 t}{2} (v^2 + iv) \right), \]

and let \( C_{BS}(\kappa) \) denote the corresponding call option price at the log-moneyness \( \kappa = \ln K \). Also, let \( C(\kappa) \) denote the call option price at the log-moneyness \( \kappa = \ln K \), under the mixed CGMY model with an independent Brownian component. As illustrated in [13], Inverse Fourier Transform (IFT) methods are slower and less accurate than the Monte Carlo (MC) method, in computing close-to-the-money option prices with short maturities. For instance, consider the IFT method described in [8, Section 11.1.3], which is based on the formula

\[ z_t(\kappa) := C(\kappa) - C_{BS}(\kappa) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iv\kappa} \varphi_t(v-i) - \varphi_t^{BS,\Sigma}(v-i) \frac{dv}{iv(1+iv)}, \]

where \( \varphi_t \) is the characteristic function of the log-return, under the mixed CGMY model with an independent Brownian component, given as in (2.1). For a close-to-the-money regime, where \( \kappa \) is close to zero, the integrand in (4.1) approximately reduces to \( \zeta_t \), which is not easy to integrate numerically for small \( t \), since in that case \( \varphi_t \) and \( \varphi_t^{BS,\Sigma} \) are quite flat in a large domain of the integration variable \( v \) (see [13, Section 10] for numerical results using Simpson’s rule).

Next, we introduce our MC method to compute the close-to-the-money option prices under the CGMY model, which is based on the risk-neutral option price representation under the probability measure \( \tilde{P} \) (see also [13, Section 6.1]). Using (2.4)-(2.6), we have

\[
\mathbb{E}\left[ (e^{X_t} - e^{\kappa n})^+ \right] = \mathbb{E}^*\left[ e^{-X_t} (e^{X_t} - e^{\kappa n})^+ \right] = \tilde{\mathbb{E}}\left[ e^{-U_t} (1 - e^{\kappa n - U_t})^+ \right] = \tilde{\mathbb{E}}\left[ e^{-M^*U_t^{(p)} + G^*U_t^{(n)} - \eta t} (1 - e^{\kappa n - U_t^{(p)} - U_t^{(n)} - \xi t - \sigma W_t^*)}^+ \right],
\]

which can be easily computed by the MC method using the fact that, under \( \tilde{P} \), \( U_t^{(p)} \) and \( U_t^{(n)} \) are independent \( Y \)-stable random variables with scale, skewness and location parameters \( tC \cos(\pi Y/2)\Gamma(-Y), 1 \) and 0, respectively. We use the simulation method of [7] (see also [26]) to generate the stable random variables \( U_t^{(p)} \) and \( U_t^{(n)} \).

Our parameter settings are motivated by the studies in [3] and [24]. Concretely, in [3], a mixed exponential Lévy model with Lévy measure

\[ \nu(dx) = \left( \frac{C_+ e^{-Mx}}{x^{1+Y}} 1_{(x>0)} + \frac{C_- e^{Gx}}{|x|^{1+Y}} 1_{(x<0)} \right) dx, \]

was considered. The calibrated parameters were given as follows (see Table 5 therein):

\[
C_+ = 0.0069, \quad C_- = 0.0063, \quad G = 0.41, \quad M = 1.93, \quad Y = 1.5, \quad \sigma = 0,
\]

and

\[
C_+ = 0.0028, \quad C_- = 0.0025, \quad G = 0.41, \quad M = 1.93, \quad Y = 1.5, \quad \sigma = 0.1.
\]

In the sequel, we simply take \( C := (C_+ + C_-)/2 \). In [24], the CGMY model was considered, and the calibrated parameters were given as (see Table 6.3 therein):

\[
C = 0.0244, \quad G = 0.0765, \quad M = 7.5515, \quad Y = 1.2945, \quad \sigma = 0.
\]

We use 100,000 samples to simulate each of the MC-based option prices. The Figures 1-3 compare the first-, second- and third-order approximations, as given in Theorem 3.1 and Theorem 3.4, to the prices based on the MC method introduced above, for both the pure-jump CGMY model and the mixed CGMY model. In all cases, the third-order approximation is much more accurate than the first- and the second-order approximations, for a time \( t \) as large as one month.

Moreover, Table 1 summaries the elapsed time, in seconds, in simulating the MC-based prices as well as the first-, second- and third-order approximations in all cases. As expected, our asymptotic approximations are much more efficient than MC simulations, since all coefficients in our approximations are only made of simple algebraic computations of model parameters, except for \( d_{32} \) in the pure-jump case.
Figure 1: Comparisons of CGMY ATM and close-to-the-money call option prices with the first-, second- and third-order approximations. In both panels, \( C = 0.0244, G = 0.0765, M = 7.5515, Y = 1.2945, \sigma = 0 \). In the left panel, \( e_1 = e_2 = 0 \), while in the right panel, \( e_1 = 0.1 \) and \( e_2 = -0.1 \).

<table>
<thead>
<tr>
<th>Prices</th>
<th>Models</th>
<th>Figure 1 Left</th>
<th>Figure 1 Right</th>
<th>Figure 2 Left</th>
<th>Figure 2 Right</th>
<th>Figure 3 Left</th>
<th>Figure 3 Right</th>
</tr>
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<tr>
<td>MC-Based Prices</td>
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<td>183.48</td>
<td>182.83</td>
<td>182.81</td>
<td>150.33</td>
<td>150.48</td>
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<td>3.61 \times 10^{-4}</td>
<td>4.41 \times 10^{-4}</td>
<td>1.24 \times 10^{-4}</td>
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<tr>
<td>2nd-order Approx</td>
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<td>1.12 \times 10^{-3}</td>
<td>4.02 \times 10^{-4}</td>
<td>4.83 \times 10^{-3}</td>
<td>9.06 \times 10^{-4}</td>
<td>9.42 \times 10^{-4}</td>
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</tr>
<tr>
<td>3rd-order Approx</td>
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<td>0.98</td>
<td>1.90 \times 10^{-3}</td>
<td>1.94 \times 10^{-3}</td>
<td></td>
</tr>
</tbody>
</table>

Table 1: Comparisons of the elapsed time (in seconds) of MC-based ATM and close-to-the-money call option prices with the first-, second- and third-order approximations under both the pure-jump CGMY model and the mixed CGMY model.

The computation of \( d_{32} \), as given in (3.6), involves a double integral, which is numerically unstable if we first compute the tail probability as a function of \( w \) using the MC method, and then evaluate the integral with respect to \( w \). Instead, we will apply a two-dimensional MC method to evaluate the double integral term in \( \hat{d}_{32} \), which is denoted as \( \tilde{d}_{32} \). More precisely, let

\[
g(u, w) = w \left[ 1_{\{u \geq w\}} - \frac{CM^Y}{Yw^Y} - \frac{C(G + 1)^Y}{Yw^Y} \right], \quad u \in \mathbb{R}, \quad w \geq 0,
\]

and let \( V \) be an absolutely continuous random variable, supported on \([0, \infty)\) and with density \( f \) under \( \tilde{P} \), which is independent of \( U := Z_1^+ + U_1 \). Then,

\[
\tilde{d}_{32} = \tilde{E} \left( \frac{g(U, V)}{f(V)} \right).
\]

The choice of the random variable \( V \) will affect the efficiency and the stability of the MC method. Here we choose \( V \) to be a standard half-normal random variable, and simulate \( \tilde{d}_{32} \) using 1000\(^2\) pairs of samples \((U, V)\). As shown in Figure 2 as well as in the fourth and fifth column of Table 1, the third-order approximations are almost identical to the MC-based prices, for \( t \) as large as one month, while the corresponding elapsed time is negligible compared to that of the MC-based prices.
Proof of Theorem 3.1. where above

For simplicity, throughout all the proofs, we fix \( S_0 = 1 \).

A Proofs of the Main Results

For simplicity, throughout all the proofs, we fix \( S_0 = 1 \).

**Proof of Theorem 3.1.** Set \( \tilde{\gamma}_t \equiv t^{-1/Y} (\gamma - t^{-1} \kappa_t) \) and \( \vartheta := - CT(-Y) \left( M^Y + (G^*)^Y \right) \) and note that, in view of \( (2.3) \) and \( (2.5) \), \( d_2 = \vartheta + \eta + \tilde{\gamma}/2 - e_1/2 \). For future reference, it is also convenient to write \( \vartheta \) as

*\[ \vartheta = \frac{C}{Y} \left[ M^Y + (G^*)^Y \right] \int_0^\infty \frac{e^{-iYv} - 1}{t^{1-iY}} v^{-Y} dv, \]

which follows from the identity (see (14.18) in [23])

\[ -Y \Gamma(-Y) = \Gamma(1-Y) = \int_0^\infty (e^{-y} - 1) y^{-Y} dy = \int_0^\infty \frac{e^{-iYv} - 1}{t^{1-iY}} v^{-Y} dv. \]

Next, note the following decomposition for the option price derived from \( (2.2) \), \( (2.4) \), \( (2.6) \), \( (2.12) \), and the fact that \( (1 - e^{-x})^+ = 1 - e^{-x^+} \),

*\[ \Pi(t) = E \left[ e^{X_t} (1 - e^{-X_{t+}}) \right] = E^* \left[ \left( 1 - e^{-\tilde{X}_t} \right)^+ \right] = e^{-\eta t} E \left[ e^{-\tilde{U}_t} \left( 1 - e^{-\tilde{X}_t} \right) \right] = 1 - e^{-\eta t} E \left( e^{-\tilde{U}_t - \tilde{X}_t^+} \right), \]

where above \( \tilde{X}_t := X_t - \kappa_t, \ t \geq 0 \). Set

\[ \Delta_1(t) := t^{-\frac{1}{Y}} E \left[ 1 - e^{-\tilde{U}_t + \tilde{X}_t^+} - \left( \tilde{U}_t + \tilde{X}_t^+ \right) \right], \quad \Delta_2(t) := t^{-\frac{1}{Y}} \left( E \left( \tilde{X}_t^+ \right) - E \left( Z_t^+ \right) \right). \]

Then, recalling that \( E \left( \tilde{U}_t \right) = 0 \) and \( E \left( Z_t^+ \right) = t^{1/Y} E \left( Z_t^+ \right) \), we have the decomposition:

\[ A(t) := t^{\frac{1}{Y} - 1} \left[ t^{\frac{1}{Y}} \Pi(t) - E \left( Z_t^+ \right) \right] - d_2 \]

\[ = \left( t^{\frac{1}{Y} - 1} \Delta_1(t) - \vartheta \right) + \left( t^{\frac{1}{Y} - 1} \Delta_2(t) - \frac{\tilde{\gamma} - e_1}{2} \right) - \frac{e^{-\eta t} - 1 + \eta t}{t} E \left( e^{-\tilde{U}_t - \tilde{X}_t^+} \right) - \eta t^{\frac{1}{Y}} \Delta_1(t) - \eta E \left( \tilde{X}_t^+ \right) \]

\[ := A_1(t) + A_2(t) + A_3(t) - A_4(t) - A_5(t). \]

(A.3)
We will prove that $A_1(t) = O(t^{\frac{1}{\gamma}})$ (and so $t^{\frac{1}{\gamma}} \Delta_1(t) = O(t)$), and that $A_2 = O(t^{1-\frac{1}{\gamma}})$. These results, in turn, imply that $A_i(t) = O(t) = o(t^{1-\frac{1}{\gamma}})$, $i = 3, 4$, and $A_3(t) = O(t^{\frac{1}{\gamma}}) = o(t^{1-\frac{1}{\gamma}})$ since $\kappa t^{-1/Y} \to 0$. So, it remains to analyze the asymptotic behavior of $A_1(t)$ and $A_2(t)$. This is done in two steps:

**Step 1.** The identity

$$\bar{E}(1 - e^{-V} - V) = \int_0^{\infty} (e^{-y} - 1) \bar{P}(V \geq y) dy - \int_0^{\infty} (e^{y} - 1) \bar{P}(V \leq -y) dy,$$

together with the change of variables $v = t^{-1/Y} y$, lead to

$$A_1(t) = \left(\int_0^{\infty} e^{t^{\gamma} v} - \frac{1}{t^{1-\frac{1}{\gamma}}} \bar{P} \left(t^{\frac{1}{\gamma}} (\bar{X}^+_i + \bar{U}_i) \geq v\right) dv - \bar{v}\right) - \int_0^{\infty} e^{-t^{\gamma} v} - \frac{1}{t^{1-\frac{1}{\gamma}}} \bar{P} \left(t^{\frac{1}{\gamma}} (\bar{X}^+_i + \bar{U}_i) \leq -v\right) dv

:= B_1(t) - B_2(t). \tag{A.4}$$

For $B_2(t)$, using the decompositions (2.4)-(2.6) as well as the self-similarity of $\{(Z_t, \bar{U}_t)\}_{t \geq 0}$,

$$\lim_{t \to 0} t^{\frac{1}{\gamma}} B_2(t) = \lim_{t \to 0} \int_0^{\infty} e^{t^{\gamma} v} - \frac{1}{t^{1-\frac{1}{\gamma}}} \bar{P} \left((Z_1 + \bar{\gamma}_t)^+ + \bar{U}_1 \leq -v\right) dv = \int_0^{\infty} v \bar{P} \left(Z_1^+ + \bar{U}_1 \leq -v\right) dv, \tag{A.5}$$

where the second equality follows from the dominated convergence theorem, which applies in view of the following direct consequences of (2.12):

$$e^{t^{\gamma} v} - \frac{1}{t^{1-\frac{1}{\gamma}}} \bar{P} \left((Z_1 + \bar{\gamma}_t)^+ + \bar{U}_1 \leq -v\right) \leq e^{t^{\gamma} v} - \frac{1}{t^{1-\frac{1}{\gamma}}} \bar{P} \left(\bar{U}_1 \leq -v\right) \leq ve^{t^{\gamma} v} e^{-v} \bar{E} \left(e^{-\bar{U}_1}\right) \leq \eta v e^{(1-\frac{1}{\gamma}) v} \leq e^{\eta v} e^{-v/2}.$$

We now analyze the asymptotic behavior of $B_1(t)$, which is shown to be $O(t^{\frac{1}{\gamma}})$.

To this end, again use the decompositions (2.4)-(2.6) as well as the self-similarity of $\{(Z_t, \bar{U}_t)\}_{t \geq 0}$ to express $t^{1-\frac{1}{\gamma}} B_1(t)$ as:

$$t^{1-\frac{1}{\gamma}} B_1(t) = \int_0^{\infty} e^{-t^{\gamma} v} - \frac{1}{t^{1-\frac{1}{\gamma}}} \left(\bar{P} \left(Z_1 + \bar{\gamma}_t > 0, Z_1 + \bar{\gamma}_t + \bar{U}_1 \geq v\right) - \frac{CM^Y}{Y v^Y}\right) dv

+ \int_0^{\infty} e^{-t^{\gamma} v} - \frac{1}{t^{1-\frac{1}{\gamma}}} \left(\bar{P} \left(Z_1 + \bar{\gamma}_t \leq 0, \bar{U}_1 \geq v\right) - \frac{C(G^*)^Y}{Y v^Y}\right) dv

:= B_{11}(t) + B_{12}(t), \tag{A.6}$$

Figure 3: Comparisons of CGMY ATM and close-to-the-money call option prices with the first-, second- and third-order approximations. In both panels, $C = 0.00265$, $G = 0.4087$, $M = 1.932$, $Y = 1.5$, $\sigma = 0.1$. In the left panel, $e_1 = e_2 = 0$, while in the right panel, $e_1 = 0.1$ and $e_2 = -0.1$. 
where we have used (A.1). As suggested from the previous arguments involving \( B_2(t) \), the limit of each of the terms therein can be obtained by passing \( \lim_{t \to 0} \) into the various integrals to get
\[
\lim_{t \to 0} t^{1-\frac{\gamma}{2}} B_1(t) = - \int_0^\infty v \left( \frac{C}{Yv^\gamma} + \frac{C(G^* Y^*)}{Yv^\gamma} \right) dv.
\] (A.7)

For the sake of a more streamlined proof, we defer the justification of the latter operation to Appendix B. Combining (A.4), (A.5) and (A.7), we obtain that
\[
\lim_{t \to 0} t^{1-\frac{\gamma}{2}} A_1(t) = - \int_0^\infty v \left( \frac{C}{Yv^\gamma} + \frac{C(G^* Y^*)}{Yv^\gamma} \right) dv - \frac{1}{2} \mathbb{E} \left[ \frac{\left( Z_1^+ + U_1 \right)^2}{\left( Z_1^+ + U_1 \right)} \right] := d_{32}, \quad (A.8)
\]

where, for the second term above, we used the identity \( 2 \int_0^\infty v \mathbb{P}(V \leq -v)dv = \mathbb{E}(V^2 1_{\{V \leq 0\}}) \).

**Step 2.** Now, we analyze the behavior of \( A_2 = t^{1-\gamma/2} (Z_1 + \gamma_1 - \gamma_1)/2 \). By the self-similarity of \((Z_t)_{t \geq 0}\),
\[
\Delta_2(t) = \mathbb{E} \left[ (Z_1 + \gamma_1 - \gamma_1)^+ \right] = \int_0^\infty \mathbb{P} \left( Z_1 \geq u - \gamma_1 \right) - \mathbb{P} \left( Z_1 \leq u \right) du = \int_0^\infty \int_{u-\gamma_1}^u p_Z(w) dw du,
\]

where for simplicity we have written \( p_Z(u) \) for the density \( p_Z(1, u) \) of \( Z_1 \). From the symmetry of \( Z_1 \), \( (\gamma - e_1)/2 = (\gamma - e_1) \int_0^\infty p_Z(u) du \) and, thus, recalling that \( \gamma_1 := t^{-1/2} (\gamma - t^{-1} \kappa_1) \),
\[
A_2(t) = t^{1-\gamma_1} \int_0^\infty \left( \frac{1}{\gamma_1} \int_{u - \gamma_1}^u p_Z(w) dw - p_Z(u) \right) du + \frac{1}{2} (e_1 - t^{-1} \kappa_1) := A_{21}(t) + A_{22}(t).
\]
The identity
\[
p_Z(w) = p_Z(u) + (w - u) \int_0^1 p_Z'(u + \beta(u - w)) d\beta,
\]
followed by the change of variables \( v = \gamma_1^{-1} (w - u) \), gives
\[
A_{21}(t) = t^{1-\gamma_1} \int_0^\infty \left[ \int_{-1}^0 v \left( \int_0^1 p_Z'(u + \beta v) d\beta \right) dv \right] du.
\]
By Fubini’s theorem,
\[
A_{21}(t) = t^{1-\gamma_1} \int_{-1}^0 dv \left( \int_0^1 p_Z'(u + \beta v) d\beta \right) dv = -t^{1-\gamma_1} \int_{-1}^0 v \left( \int_0^1 p_Z'(\beta v t^{1/2}) d\beta \right) dv.
\]
It is now clear that
\[
\lim_{t \to 0} t^{1-\gamma/2} A_2(t) = \frac{(\gamma - e_1)^2 p_Z(0)}{2} - \frac{e_2}{2} := d_{31}, \quad (A.9)
\]
since, by assumption, \( \lim_{t \to 0} t^{1/2-1} (e_1 - t^{-1} \kappa_1) = -e_2 \). The expression for \( d_{31} \) given as in (3.5) is obtained from the power series representation of \( p_Z \) around \( z = 0 \) as given, for example, in (14.30) in [23]. Finally, combining (A.8) and (A.9) with (A.3) (together with the remarks thereafter), we obtain (3.2).

**Proof of Theorem 3.4.** Fix \( \gamma_1 := t^{1/2} (\gamma - t^{-1} \kappa_1) = (\gamma - e_1) t^{1/2} - e_2 t^{2-Y} + o(t^{1/2-Y}) \), and let
\[
\Delta_0(t) := \frac{1}{\sqrt{t}} \mathbb{E} \left[ (S_t - e^{\kappa_1})^+ \right] - d_1, \quad (A.10)
\]
with the constant \( d_1 \) given in (3.11). First, note the following easy representation
\[
\frac{1}{\sqrt{t}} \mathbb{E} \left[ (S_t - e^{\kappa_1})^+ \right] = \frac{1}{\sqrt{t}} \mathbb{E} \left[ e^{X_t} (1 - e^{-X_t + \kappa_1})^+ \right] = \frac{1}{\sqrt{t}} \mathbb{E}^* \left( 1 - e^{-X_t^+} \right) = \int_0^\infty e^{-\sqrt{t}v} \mathbb{P}^* \left( \frac{1}{\sqrt{t}} \tilde{X}_t \geq v \right) dv,
\]
where $\tilde{X}_t := X_t - \kappa_t$, $t \geq 0$. Next, recalling that $X_t = \tilde{\gamma} t + Z_t + \sigma W^*_t$ and using the self-similarity of $W^*$ and the change of variables $y = v - \tilde{\gamma} t$,

$$
\Delta_0(t) = \int_{-\tilde{\gamma} t}^{\infty} e^{-\sqrt{\gamma} y - \sqrt{\gamma} t} \mathbb{P}^*(\sigma W^*_t \geq y - t + \frac{1}{2} Z_t) \, dy - \int_{0}^{\infty} \mathbb{P}^*(\sigma W^*_0 \geq y) \, dy.
$$

Moreover, by changing the probability measure $\mathbb{P}^*$ to $\bar{\mathbb{P}}$, and using (2.4) as well as the self-similarity of $\{(Z_t, U_t)\}_{t \geq 0}$,

$$
\Delta_0(t) = \int_{-\tilde{\gamma} t}^{\infty} e^{-\sqrt{\gamma} y - \sqrt{\gamma} t} \bar{\mathbb{E}} \left( e^{-t \tilde{\gamma} \tilde{U}_1 - \sqrt{t} \gamma_1 - \sqrt{t} \gamma Z_1} \right) \, dy - \int_{0}^{\infty} \bar{\mathbb{E}} \left( e^{-t \tilde{\gamma} \tilde{U}_1 - \sqrt{t} \gamma_1 - \sqrt{t} \gamma Z_1} \right) \, dy
$$

$$
+ e^{-\sqrt{\gamma} t} \int_{-\tilde{\gamma} t}^{0} e^{-\sqrt{\gamma} y} \bar{\mathbb{E}} \left( e^{-t \tilde{\gamma} \tilde{U}_1 - \sqrt{t} \gamma_1 - \sqrt{t} \gamma Z_1} \right) \, dy + \int_{0}^{\infty} \left( e^{-\sqrt{\gamma} t - \sqrt{\gamma} y} - 1 \right) \bar{\mathbb{P}}(\sigma W^*_1 \geq y) \, dy
$$

$$
:= A_1(t) + A_2(t) + A_3(t). \quad (A.11)
$$

Since $\tilde{\gamma} t \to 0$, as $t \to 0$, it is clear that

$$
\lim_{t \to 0} t^{-\frac{1}{2}} A_3(t) = \lim_{t \to 0} \int_{0}^{\infty} \frac{e^{-\sqrt{\gamma} t - \sqrt{\gamma} y} - 1}{\sqrt{t}} \bar{\mathbb{P}}(\sigma W^*_1 \geq y) \, dy = - \int_{0}^{\infty} y \bar{\mathbb{P}}(\sigma W^*_1 \geq y) \, dy = - \frac{\sigma^2}{4}. \quad (A.12)
$$

For $A_2(t)$, we consider two cases. First, assume $Y < 3/2$. Then, by changing variables $u = t^{-1/2} y$, the integral appearing in the $A_2(t)$ term can be written as

$$
t^{\frac{1}{2}} \int_{-\tilde{\gamma} - e_{1 + R^t}}^{0} e^{-tu} \bar{\mathbb{E}} \left( e^{-t \tilde{\gamma} \tilde{U}_1 - \sqrt{t} \gamma_1 - \sqrt{t} \gamma Z_1} \right) \, du,
$$

where $\mathcal{R}_t := e t^{3/2 - Y} + o(t^{3/2 - Y})$. The above integral can further be decomposed as $t^{1/2} \int_{-\tilde{\gamma} - e_{1 + R^t}}^{0} + t^{1/2} \int_{-\tilde{\gamma} - e_{1 + R^t}}^{\gamma - e_{1}}$. By the dominated convergence theorem, the first integral is clearly equal to $t^{1/2} (\tilde{\gamma} - e_{1})/2 + o(t^{1/2})$. The second integral, by changing variables $w = (u + (\tilde{\gamma} - e_{1})) t^{-1/2} y$, becomes

$$
e^{-t(\tilde{\gamma} - e_2)} t^{2 - Y} \int_{e_{2} + o(1)}^{0} e^{-t^{\frac{1}{2} - Y} w} \bar{\mathbb{E}} \left( e^{-t \tilde{\gamma} \tilde{U}_1 - \sqrt{t} \gamma_1 - \sqrt{t} \gamma Z_1} \right) \, dw.
$$

Proceeding as before and further decomposing the last integral as $t^{2 - Y} \int_{e_{2} + o(1)}^{0} + t^{2 - Y} \int_{e_{2} + o(1)}^{e_{2}} + o(t^{2 - Y})$, it follows from the dominated convergence theorem that the first term is $- t^{2 - Y} e_{2}/2 + o(t^{2 - Y})$, while the second is $o(t^{2 - Y})$, since the integrand is uniformly bounded by 1. If $Y \geq 3/2$, we make instead the change of variables $u = t^{-1/2} y$ and follow a similar argument. In either case,

$$
A_2(t) = \frac{\tilde{\gamma} - e_1}{2} t^{\frac{1}{2}} - \frac{e_2}{2} t^{2 - Y} + o \left( t^{\frac{1}{2} \gamma(2 - Y)} \right), \quad t \to 0. \quad (A.13)
$$

It remains to study the asymptotic behavior of $A_1(t)$. To this end, we first decompose it as:

$$
A_1(t) = e^{-(\eta t + \sqrt{\gamma} t)} \bar{\mathbb{E}} \left( e^{-t \tilde{\gamma} \tilde{U}_1} \mathbb{1}_{\left\{ W^*_1 \geq 0, \sigma W^*_1 + t \tilde{\gamma} \tilde{U}_1 + \sqrt{t} \gamma Z_1 \geq 0 \right\}} \int_{\sigma W^*_1}^{\sigma W^*_1 + t \tilde{\gamma} \tilde{U}_1 + \sqrt{t} \gamma Z_1} e^{-\sqrt{\gamma} y} \, dy
$$

$$
- e^{-(\eta t + \sqrt{\gamma} t)} \bar{\mathbb{E}} \left( e^{-t \tilde{\gamma} \tilde{U}_1} \mathbb{1}_{\left\{ 0 \leq \sigma W^*_1 \leq -t \tilde{\gamma} \tilde{U}_1 - \sqrt{t} \gamma Z_1 \right\}} \int_{0}^{\sigma W^*_1} e^{-\sqrt{\gamma} y} \, dy
$$

$$
+ e^{-(\eta t + \sqrt{\gamma} t)} \bar{\mathbb{E}} \left( e^{-t \tilde{\gamma} \tilde{U}_1} \mathbb{1}_{\left\{ 0 \leq -\sigma W^*_1 \leq -t \tilde{\gamma} \tilde{U}_1 - \sqrt{t} \gamma Z_1 \right\}} \int_{0}^{\sigma W^*_1} e^{-\sqrt{\gamma} y} \, dy
$$

$$
:= B_1(t) - B_2(t) + B_3(t). \quad (A.14)
$$
Next, by Fubini’s theorem, the independence of $W^*_1$ and $(Z_1, \tilde{U}_1)$, and the symmetry of $Z_1$,

$$B_1(t) = e^{-(\sigma t + \sqrt{\tau} t)} \mathbb{E} \left[ e^{-\sqrt{\tau} \sigma W^*_1} \mathbb{1}_{\{W^*_1 > 0, \sigma W^*_1 + t^{\frac{1}{2}} \sigma W^*_1 \leq Z_1 \geq 0\}} \right]$$

$$= e^{-(\sigma t + \sqrt{\tau} t)} \mathbb{E} \left[ e^{-\sqrt{\tau} \sigma W^*_1} \mathbb{1}_{\{W^*_1 > 0, Z_1 \geq 0\}} \right]$$

$$- e^{-(\sigma t + \sqrt{\tau} t)} \mathbb{E} \left[ e^{-\sqrt{\tau} \sigma W^*_1} \mathbb{1}_{\{W^*_1 > 0, -t^{\frac{1}{2}} \sigma W^*_1 \leq Z_1 \leq 0\}} \right]$$

$$+ e^{-(\sigma t + \sqrt{\tau} t)} \mathbb{E} \left( Z_1 \mathbb{1}_{\{W^*_1 \geq 0, Z_1 \geq -t^{\frac{1}{2}} \sigma W^*_1 \}} e^{-\sqrt{\tau} \sigma W^*_1} \right) \cdot$$

In order to obtain the asymptotic behavior of $B_{11}(t)$, let

$$B_{11}^{(1)}(t) := \int_{-\infty}^{0} \left( e^{-t^{\frac{1}{2}} u - 1} \right) \mathbb{P} \left( Z_1 \geq 0, \tilde{U}_1 \leq u \leq \tilde{U}_1 + Z_1 \right) du, \quad B_{11}^{(2)}(t) := \int_{0}^{\infty} \left( e^{-t^{\frac{1}{2}} u - 1} \right) \mathbb{P} \left( Z_1 \geq 0, \tilde{U}_1 \leq u \leq \tilde{U}_1 + Z_1 \right) du.$$

For $B_{11}^{(1)}(t)$, arguments similar to those used to obtain (A.5) lead to

$$B_{11}^{(1)}(t) = t^{\frac{1}{2}} \int_{-\infty}^{0} \left( -u \right) \mathbb{P} \left( Z_1 \geq 0, \tilde{U}_1 \leq u \leq \tilde{U}_1 + Z_1 \right) du + o(t^{\frac{1}{2}}), \quad t \to 0. \quad (A.16)$$

For $B_{11}^{(2)}(t)$, we use arguments similar to those used to obtain (B.10). Concretely, let

$$B_{11}^{(21)}(t) := \int_{0}^{\infty} \left( e^{-t^{\frac{1}{2}} u - 1} \right) \mathbb{P} \left( Z_1 \geq 0, u \leq \tilde{U}_1 + Z_1 \right) du$$

$$= t^{\frac{1}{2}} \int_{0}^{\infty} \frac{e^{-t^{\frac{1}{2}} u - 1}}{t^{\frac{1}{2}}} \left[ \mathbb{P} \left( Z_1 \geq 0, u \leq \tilde{U}_1 + Z_1 \right) \right] - \frac{CM}{Y u^{Y}} du - t^{1 - \frac{1}{2}} \mathcal{C} \Gamma(-Y) M^Y,$$

where, in the second equality, the identity (A.2) was used. The integral on the right-hand side of the previous identity is precisely the first integral defined in (A.6), after setting $\gamma_t = 0$. Then, one uses arguments as those leading to (B.10) to conclude that

$$B_{11}^{(21)}(t) = -t^{1 - \frac{1}{2}} \mathcal{C} \Gamma(-Y) M^Y + O(t^{\frac{1}{2}}), \quad t \to 0.$$
which, together with (A.16), implies that the term $B_{11}(t)$ introduced in (A.15) is such that
\[ B_{11}(t) = -\frac{1}{2} t^{\frac{1}{2}}CT(-Y) \left( M^Y - (M^*)^Y \right) + O(t^{\frac{1}{2} - \frac{1}{2}}), \quad t \to 0. \] (A.17)

For $B_{12}(t)$, it turns out that
\[ B_{12}(t) = o(t^{\frac{1}{2}}), \quad t \to 0. \] (A.18)

For a more streamlined proof, we again defer the proof of (A.18) to Appendix B.

Finally, we deal with $B_{13}(t)$ and study the asymptotic behavior of
\[ \tilde{B}_{13}(t) := t^{\frac{\gamma}{2} - 1}B_{13}(t) - \frac{C\sigma^{-1}Y}{2(Y - 1)} \tilde{E} \left( |W_1|^{1-Y} \right). \] (A.19)

First, $\tilde{B}_{13}(t)$ is further decomposed as:
\[
\begin{align*}
\tilde{B}_{13}(t) &= t^{\frac{\gamma}{2} - \frac{1}{2}} \left[ e^{- (\eta t + \sqrt{\tau t})} \tilde{E} \left( e^{-\sqrt{\tau} \sigma W_1} \mathbf{1}_{\{W_1 \geq 0\}} \int_{t^{\frac{1}{2} - \frac{1}{2}} - \sqrt{\tau} \sigma W_1}^{\infty} z p_Z(1, z) \, dz \right) - \tilde{E} \left( \mathbf{1}_{\{W_1 \geq 0\}} \int_{t^{\frac{1}{2} - \frac{1}{2}} - \sqrt{\tau} \sigma W_1}^{\infty} Cz^{-Y} \, dz \right) \right] \\
&= t^{\frac{\gamma}{2} - \frac{1}{2}} \left( e^{- (\eta t + \sqrt{\tau} t)} - 1 \right) \tilde{E} \left( e^{-\sqrt{\tau} \sigma W_1} \mathbf{1}_{\{W_1 \geq 0\}} \int_{t^{\frac{1}{2} - \frac{1}{2}} - \sqrt{\tau} \sigma W_1}^{\infty} z p_Z(1, z) \, dz \right) \\
&\quad + t^{\frac{\gamma}{2} - \frac{1}{2}} \left[ e^{-\sqrt{\tau} \sigma W_1} \mathbf{1}_{\{W_1 \geq 0\}} \int_{t^{\frac{1}{2} - \frac{1}{2}} - \sqrt{\tau} \sigma W_1}^{\infty} z p_Z(1, z) \, dz \right] \\
&\quad + t^{\frac{\gamma}{2} - \frac{1}{2}} \left[ \mathbf{1}_{\{W_1 \geq 0\}} \int_{t^{\frac{1}{2} - \frac{1}{2}} - \sqrt{\tau} \sigma W_1}^{\infty} z (p_Z(1, z) - Cz^{-Y}) \, dz \right] \\
&:= B^{(1)}_{13}(t) + B^{(2)}_{13}(t) + B^{(3)}_{13}(t).
\end{align*}
\] (A.20)

As shown next,
\[ B^{(1)}_{13}(t) = O \left( t^{1\wedge(\frac{1}{2} - Y)} \right) = O(\sqrt{t}), \quad B^{(2)}_{13}(t) = O(\sqrt{t}), \quad t \to 0. \] (A.21)

Indeed, for $B^{(1)}_{13}(t)$, we first rewrite the expectation as
\[
\tilde{E} \left( e^{-\sqrt{\tau} \sigma W_1} \mathbf{1}_{\{W_1 \geq 0\}} \int_{t^{\frac{1}{2} - \frac{1}{2}} - \sqrt{\tau} \sigma W_1}^{\infty} z p_Z(1, z) \, dz \right) = \int_0^\infty \left( \int_{t^{\frac{1}{2} - \frac{1}{2}} - \sqrt{\tau} \sigma W_1}^{\infty} z p_Z(1, z) \, dz \right) e^{-\sqrt{\eta} w} e^{- \frac{w^2}{2\pi^2}} \, dw.
\] (A.22)

Next, by (2.10), there exists $H_1 > 0$ such that, for any $z \geq H_1$, $p_Z(1, z) \leq 2Cz^{-Y-1}$. Hence, for any $w > 0$,
\[
t^{\frac{\gamma}{2} - \frac{1}{2}} \int_{t^{\frac{1}{2} - \frac{1}{2}} - \sqrt{\tau} \sigma w}^{\infty} z p_Z(1, z) \, dz \leq t^{\frac{\gamma}{2} - \frac{1}{2}} \int_{t^{\frac{1}{2} - \frac{1}{2}} - \sqrt{\tau} \sigma w}^{\infty} 2Cu^{-Y}du + 1_{(t^{\frac{1}{2} - \frac{1}{2}} - \sqrt{\tau} \sigma w < H_1)} H_1 \tilde{P}(Z_1 \geq t^{\frac{1}{2} - \frac{1}{2}} - \sqrt{\tau} \sigma w) \leq \frac{2Cu^{-Y}}{Y - 1} + H_1^{-1}u^{-1},
\]
where to derive the second term in the last inequality we used that $\tilde{P}(Z_1 \geq t^{\frac{1}{2} - \frac{1}{2}} w) \leq H_1^{-1}(t^{\frac{1}{2} - \frac{1}{2}} w)^{-Y-1}$, when $t^{\frac{1}{2} - \frac{1}{2}} w < H_1$. Together with (A.22) and since $Y \in (1, 2)$, the first relation in (A.21) follows, since $\sqrt{\eta} t = (\gamma - e_1) t - e_2 t^{5/2 - Y} + o(t^{(5/2-Y)\nu})$. The second relation therein is obtained using similar arguments.

It remains to deal with $B^{(3)}_{13}(t)$, which can be rewritten as:
\[
B^{(3)}_{13}(t) = \frac{1}{2} t^{\frac{\gamma}{2} - \frac{1}{2}} \int_{\mathbb{R}} \left( \int_{t^{\frac{1}{2} - \frac{1}{2}} - \frac{\sqrt{\tau} \sigma w}{2\pi^2}}^{\infty} \frac{1}{\sqrt{2\pi^2}} e^{- \frac{w^2}{2\pi^2}} \, dw \right) \left| z \right| (p_Z(1, z) - C\left| z \right|^{-Y-1}) \, dz du,
\] (A.23)
by changing variables, $u = t^\frac{1}{2} - \frac{\dot{t}}{2} w/|z|$, and applying Fubini’s theorem in the second equality. For simplicity, we write $p_Z(z)$ instead of $p_Z(1, z)$ hereafter. Next, denoting the characteristic function of $Z_1$ by $\hat{p}_Z(x)$, we have

$$p_Z(z) = \mathcal{F} \left( \frac{1}{\sqrt{2\pi}} \hat{p}_Z \right) (z), \quad z^2 p_Z(z) = \mathcal{F} \left( \frac{-1}{\sqrt{2\pi}} \hat{p}_Z'' \right) (x), \quad (A.24)$$

where $\mathcal{F}(h)(z) := \int_{\mathbb{R}} e^{-izv} h(v) dv/\sqrt{2\pi}$ denotes the Fourier transformation of $h \in L_1(\mathbb{R})$. Also, regarding $|x|^{-2}$ as a tempered distribution, it is known that

$$|z|^{1-Y} = \mathcal{F} \left( K^{-1} |x|^{-2} \right) (z),$$

with $K := -2\sin(\pi(Y - 2)/2)\Gamma(Y - 1)/\sqrt{2\pi}$. In particular, by definition,

$$\int_{\mathbb{R}} |z|^{1-Y} \phi(z) dz = \int_{\mathbb{R}} K^{-1} |x|^{-2} \mathcal{F}(\phi)(x) dx, \quad (A.25)$$

for any Schwartz function $\phi$. Thus, combining (A.23)-(A.25), lead to

$$B_{13}^{(3)}(t) = \frac{1}{2} t^{\frac{1}{2} + \frac{1}{3}} \int_{0}^{1} \int_{\mathbb{R}} \mathcal{F} \left( \frac{t^\frac{1}{3} t^\frac{1}{2} e^{-i\frac{1}{2}+\frac{1}{2} \theta u^2 + 2\pi \theta}}{2\pi \sigma^2} \right) (x) \left( -\frac{1}{\sqrt{2\pi}} \hat{p}_Z''(x) - \frac{C}{K} |x|^{-2} \right) dx du
\begin{align*}
&= \frac{1}{2\sqrt{2\pi}} t^{\frac{1}{2} + \frac{1}{3}} \int_{0}^{1} \int_{\mathbb{R}} u^{-1} e^{-i\frac{1}{2} \theta u^2} \left( \frac{1}{\sqrt{2\pi}} \hat{p}_Z''(x) + \frac{C}{K} |x|^{-2} \right) dx du,
\end{align*}

Since $\hat{p}_Z(x) = e^{-c|x|^Y}$ with $c := 2C|\cos(\pi Y/2)|\Gamma(-Y)$,

$$\frac{1}{\sqrt{2\pi}} p''_Z(x) + \frac{C}{K} |x|^{-2} = -cY(Y - 1) e^{-c|x|^Y} |x|^{-2} + \frac{1}{\sqrt{2\pi}} (cY|x|^{-1})^2 e^{-c|x|^Y} + \frac{C}{K} |x|^{-2}
\begin{align*}
&= \frac{1}{\sqrt{2\pi}} (cY|x|^{-1})^2 e^{-c|x|^Y} + \frac{cY(Y - 1)}{\sqrt{2\pi}} |x|^{-2} \left( 1 - e^{-c|x|^Y} \right),
\end{align*}

where, in the last equality, we used $C/K = cY(Y - 1)/\sqrt{2\pi}$. Hence,

$$B_{13}^{(3)}(t) = \frac{-cY^2}{2\pi} t^{\frac{1}{2} + \frac{1}{3}} \int_{0}^{1} \int_{\mathbb{R}} u^{-1} e^{-i\frac{1}{2} \theta u^2} x^{2\gamma - 2} e^{-cx^\gamma} dx du
\begin{align*}
&+ \frac{-cY(Y - 1)}{2\pi} t^{\frac{1}{2} + \frac{1}{3}} \int_{0}^{1} \int_{\mathbb{R}} u^{-1} e^{-i\frac{1}{2} \theta u^2} x^{2\gamma - 2} \left( 1 - e^{-cx^\gamma} \right) dx du
\begin{align*}
&:= B_{13}^{(31)}(t) + B_{13}^{(32)}(t).
\end{align*}

(A.26)

For $B_{13}^{(31)}(t)$, changing variables $v = t^\frac{1}{2} - \frac{\dot{t}}{2} \sigma x/u$, gives,

$$B_{13}^{(31)}(t) = \frac{-cY^2}{2\pi} t^{\frac{1}{2} + \frac{1}{3}} \int_{0}^{1} \int_{\mathbb{R}} u^{-1} e^{-i\frac{1}{2} \theta u^2} \left( \frac{t^\frac{1}{3} + \frac{1}{2} \theta u^2}{\sigma} \right)^{2\gamma - 2} e^{-c \left( \frac{t^\frac{1}{3} + \frac{1}{2} \theta u^2}{\sigma} \right)^\gamma} t^{\frac{1}{3} - \frac{1}{2} \theta u^2} du dv
\begin{align*}
&= \frac{-cY^2}{2\pi \sigma^{2\gamma - 2}} t^{\frac{1}{2} + \frac{1}{3}} \int_{0}^{1} \left( \int_{\mathbb{R}} e^{-i\frac{1}{2} \theta u^2} \right) \left( e^{-c \gamma \left( \frac{t^\frac{1}{3} + \frac{1}{2} \theta u^2}{\sigma} \right)^\gamma} \right) t^{\frac{1}{3} - \frac{1}{2} \theta u^2} du.
\end{align*}

Hence, by the dominated convergence theorem,

$$\lim_{t \to 0} t^{\frac{1}{2} - 1} B_{13}^{(31)}(t) = \frac{-cY^2}{2\sqrt{2\pi}(2Y - 1)\sigma^{2\gamma - 1}} \mathbb{E} \left( |W_1|^2 \gamma - 2 \right) = \frac{-2cY^2 \cos^2 \left( \frac{\pi Y}{2} \right) \Gamma(2Y - 2)}{2\sqrt{2\pi}(2Y - 1)\sigma^{2\gamma - 1}} \mathbb{E} \left( |W_1|^2 \gamma - 2 \right). \quad (A.27)$$

Similarly, the asymptotic behavior of $B_{13}^{(32)}(t)$ is given by

$$\lim_{t \to 0} t^{\frac{1}{2} - 1} B_{13}^{(32)}(t) = \frac{-cY(Y - 1)}{2\sqrt{2\pi}(2Y - 1)\sigma^{2\gamma - 1}} \mathbb{E} \left( |W_1|^2 \gamma - 2 \right) = \frac{-2cY(Y - 1) \cos^2 \left( \frac{\pi Y}{2} \right) \Gamma(2Y - 2)}{2\sqrt{2\pi}(2Y - 1)\sigma^{2\gamma - 1}} \mathbb{E} \left( |W_1|^2 \gamma - 2 \right). \quad (A.28)$$
Combining (A.20), (A.21) and (A.26)-(A.28),
\[
\lim_{t \to 0} t^{\frac{Y}{2} - 1} \tilde{B}_{13}(t) = -\frac{2C^2Y \cos^2 \left( \frac{\pi Y}{2} \right) \Gamma^2(-Y)}{\sqrt{2\pi} \sigma^2 Y - 1} \mathbb{E} \left[ |W_1|^2 Y - 2 \right] := d_3'.
\] (A.29)

Combining (A.15), (A.17)-(A.19) and (A.29), and setting \(d_3 := C \Gamma(-Y) \left[ M^Y - (M^*)^Y \right] \), lead to
\[
B_1(t) = \frac{1}{2} C \Gamma(-Y) \left( M^Y - (M^*)^Y \right) t^{\frac{1}{2}} + t^{1 - \frac{Y}{2}} \left[ t^{1 - \frac{Y}{2}} d_{31} + o(t^{1 - \frac{Y}{2}}) \right] + \frac{C \sigma^{1 - Y}}{2(Y - 1)} \mathbb{E} \left[ |W_1|^{1 - Y} \right] + o(t^{1 - \frac{Y}{2}})
\] (A.30)

**Step 2.** Next, we tackle \(B_2(t)\) by decomposing it as:
\[
B_2(t) = e^{-\eta t + \sqrt{\pi} \eta t} \int_0^\infty \mathbb{E} \left[ \left( e^{- \tilde{t}^\frac{1}{2} \tilde{u}_1} - 1 \right) \mathbf{1}_{\{z_1 \leq -t^{\frac{1}{2}} + w, \tilde{u}_1 < 0\}} \right] \frac{1 - e^{-\sqrt{\pi} w} e^{-\frac{w^2}{2\pi \sigma^2}}}{\sqrt{t}} dw
\] (A.31)

We begin with proving that
\[
\lim_{t \to 0} t^{-\frac{1}{2}} B_2(t) = 0.
\] (A.32)

To this end, consider first
\[
B_{21}^{(1)}(t) := \int_0^\infty b_{21}^{(1)}(t; w) \frac{1 - e^{-\sqrt{\pi} w} e^{-\frac{w^2}{2\pi \sigma^2}}}{\sqrt{t}} dw,
\]
where
\[
b_{21}^{(1)}(t; w) := \mathbb{E} \left[ \left( e^{-t^{\frac{1}{2}} \tilde{u}_1} - 1 \right) \mathbf{1}_{\{z_1 \leq -t^{\frac{1}{2}} + w, \tilde{u}_1 < 0\}} \right].
\]
Note that, for any \(0 < t < 1\) and \(w > 0\), by (2.12),
\[
0 \leq t^{-\frac{1}{2}} b_{21}^{(1)}(t; w) = t^{-\frac{1}{2}} \mathbb{E} \left[ \mathbf{1}_{\{z_1 \leq -t^{\frac{1}{2}} + w, \tilde{u}_1 < 0\}} \int_{-\infty}^0 \mathbf{1}_{\{u \leq \tilde{u}_1 \leq u + t^{\frac{1}{2}}\}} e^{-t^{\frac{1}{2}} u} du \right] \leq t^{-\frac{1}{2}} \int_{-\infty}^0 e^{-u} \mathbb{E} \left[ \mathbf{1}_{\{U_1 \leq t^{\frac{1}{2}} + u\}} \right] du \leq \mathbb{E} \left[ e^{-\tilde{u}_1} \right] t^{-\frac{1}{2}} \int_{-\infty}^0 e^{-u} \mathbf{1}_{\{u \geq 0\}} du = e^{\eta t} t^{\frac{1}{2}} \frac{1}{1 - t^{\frac{1}{2}}^2}.
\]
Since \(Y \in (1, 2)\), the dominated convergence implies that \(B_{21}^{(1)}(t) = o(t^{1/2})\), as \(t \to 0\). Next, consider
\[
B_{21}^{(2)}(t) := \int_0^\infty b_{21}^{(2)}(t; w) \frac{1 - e^{-\sqrt{\pi} w} e^{-\frac{w^2}{2\pi \sigma^2}}}{\sqrt{t}} dw,
\]
where \(b_{21}^{(2)}(t; w)\) is defined and further decomposed as:
\[
b_{21}^{(2)}(t; w) := \mathbb{E} \left[ e^{-t^{\frac{1}{2}} \tilde{u}_1} - 1 \mathbf{1}_{\{z_1 \leq -t^{\frac{1}{2}} + w, \tilde{u}_1 \geq 0\}} \right] = \mathbb{E} \left[ e^{-t^{\frac{1}{2}} \tilde{u}_1} - 1 + t^{\frac{1}{2}} \tilde{U}_1 \mathbf{1}_{\{z_1 \leq -t^{\frac{1}{2}} + w, \tilde{u}_1 \geq 0\}} \right] - t^{\frac{1}{2}} \tilde{U}_1 \mathbb{E} \left[ \mathbf{1}_{\{z_1 \leq -t^{\frac{1}{2}} + w, \tilde{u}_1 \geq 0\}} \right].
\]
Note that (since $1 < Y < 2$), as $t \to 0$,

$$0 \leq t^{-\frac{1}{2}} \int_0^\infty \frac{\mathbb{E} \left( \tilde{U}_1 \mathbf{1}_{\{Z_1 \leq -t^{\frac{1}{2}} + w, \tilde{w}_1 \geq 0\}} \right) \frac{1 - e^{-\sqrt{7}w} e^{-\frac{w^2}{2\pi\sigma^2}}}{\sqrt{t}}}{\sqrt{2\pi\sigma^2}} \, dw \leq t^{\frac{1}{2}} \frac{\mathbb{E} \left| \tilde{U}_1 \right|}{\sqrt{2\pi\sigma^2}} \, dw \to 0. \quad (A.33)$$

Moreover, by (2.9-i) and the decomposition $\tilde{U}_i = M^* \tilde{U}_i^{(p)} - G^* \tilde{U}_i^{(n)}$, for any $t > 0$ and $w > 0$,

$$\mathbb{E} \left[ \left( e^{-t^2 \tilde{U}_1} - 1 + t^2 \tilde{U}_1 \right) \mathbf{1}_{\{Z_1 \leq -t^{\frac{1}{2}} + w, \tilde{w}_1 \geq 0\}} \right] = \mathbb{E} \left[ \int_0^{t^2 \tilde{U}_1} (1 - e^{-u}) \, du \right] \mathbf{1}_{\{Z_1 \leq -t^{\frac{1}{2}} + w, \tilde{w}_1 \geq 0\}} \leq \int_0^\infty (1 - e^{-u}) \bar{P} \left( \tilde{U}_1 \geq t^{-\frac{1}{2}} u \right) \, du \leq 2^{Y+1} K_1 \left( (M^*)^Y + (G^*)^Y \right) t \int_0^\infty (1 - e^{-u}) \, u^{-Y} \, du.$$

Hence, by the dominated convergence theorem,

$$0 \leq t^{-\frac{1}{2}} \int_0^\infty \mathbb{E} \left[ \left( e^{-t^2 \tilde{U}_1} - 1 + t^2 \tilde{U}_1 \right) \mathbf{1}_{\{Z_1 \leq -t^{\frac{1}{2}} + w, \tilde{w}_1 \geq 0\}} \right] \frac{1 - e^{-\sqrt{7}w} e^{-\frac{w^2}{2\pi\sigma^2}}}{\sqrt{t}} \, dw \leq 2^{Y+1} K_1 \left( (M^*)^Y + (G^*)^Y \right) \sqrt{t} \int_0^\infty (1 - e^{-u}) \, u^{-Y} \, du \int_0^\infty \frac{w \cdot \frac{e^{-\frac{w^2}{2\pi\sigma^2}}}{2\pi\sigma^2}}{\sqrt{2\pi\sigma^2}} \, dw \to 0, \quad t \to 0. \quad (A.34)$$

In view of (A.33) and (A.34), $B_{22}^{(2)} (t) = O(t^{1/2})$, as $t \to 0$, and (A.32) follows.

To finish, we analyze $B_{22} (t)$ defined via (A.31). To this end, let

$$\tilde{B}_{22} (t) := t^{-\frac{1}{2}} B_{22} (t) - \frac{C^1 Y}{2Y} \mathbb{E} \left[ |W_1^*|^{1-Y} \right] = t^{-\frac{1}{2}} \left[ e^{-\frac{(n+\sqrt{\gamma}n)(-z)}{2}} \right] \int_0^\infty \mathbb{E} \left[ \frac{1 - e^{-\sqrt{7}w} e^{-\frac{w^2}{2\pi\sigma^2}}}{\sqrt{t}} \right] \mathbf{1}_{\{0 \leq \sigma W_1^* \leq t^{\frac{1}{2}} - \frac{1}{2} z\}} p_z(1, z) \, dz \int_0^\infty \mathbb{E} \left[ (\sigma W_1^* \mathbf{1}_{\{0 \leq \sigma W_1^* \leq t^{\frac{1}{2}} - \frac{1}{2} z\}}) C z^{-Y-1} \right] dz\right]$$

$$= t^{-\frac{1}{2}} \left[ e^{-\frac{(n+\sqrt{\gamma}n)(-z)}{2}} \right] \int_0^\infty \mathbb{E} \left[ \frac{1 - e^{-\sqrt{7}w} e^{-\frac{w^2}{2\pi\sigma^2}}}{\sqrt{t}} \right] \mathbf{1}_{\{0 \leq \sigma W_1^* \leq t^{\frac{1}{2}} - \frac{1}{2} z\}} p_z(1, z) \, dz$$

$$+ t^{-\frac{1}{2}} \int_0^\infty \mathbb{E} \left[ \left( 1 - \sigma W_1^* \mathbf{1}_{\{0 \leq \sigma W_1^* \leq t^{\frac{1}{2}} - \frac{1}{2} z\}} \right) C z^{-Y-1} \right] dz$$

$$:= B_{22}^{(1)} (t) + B_{22}^{(2)} (t) + B_{22}^{(3)} (t), \quad (A.35)$$

where we used the symmetry of $Z_1$ in the second equality. As explained next,

$$B_{22}^{(1)} (t) = O \left( t^{1 \wedge \left( \frac{2}{2-Y} \right)} \right) = O(\sqrt{t}), \quad B_{22}^{(2)} (t) = O(\sqrt{t}), \quad t \to 0. \quad (A.36)$$

Indeed, for the first relation above, note that, by (2.11),

$$\int_0^\infty \mathbb{E} \left[ \frac{1 - e^{-\sqrt{7}w} e^{-\frac{w^2}{2\pi\sigma^2}}}{\sqrt{t}} \right] \mathbf{1}_{\{0 \leq \sigma W_1^* \leq t^{\frac{1}{2}} - \frac{1}{2} z\}} p_z(1, z) \, dz$$

$$\leq \mathbb{E} \left[ (\sigma W_1^* \mathbf{1}_{\{0 \leq \sigma W_1^* \leq t^{\frac{1}{2}} - \frac{1}{2} z\}}) C z^{-Y-1} \right] dz$$

while the second relationship in (A.36) follows in a similar fashion.
It remains to deal with \( B_{22}^{(3)}(t) \), which can be rewritten as:

\[
B_{22}^{(3)}(t) = \frac{1}{2} t^{\frac{Y-1}{2}} \int_0^1 \int_{-\infty}^\infty \left( \frac{t^{\frac{Y}{2}}}{\sqrt{2\pi\sigma^2}} e^{-\frac{u^2}{2\sigma^2}} \right) \left( \frac{t^{\frac{Y}{2}}}{\sqrt{2\pi\sigma^2}} e^{-\frac{u^2}{2\sigma^2}} \right) \left( p_Z(1, z) - C|z|^{-Y-1} \right) dz \]

by changing variables, \( u = t^{\frac{Y}{2}} w/|z| \), and applying Fubini’s theorem. Using the argument given after (A.23),

\[
B_{22}^{(3)}(t) = \frac{1}{2} t^{\frac{Y}{2} + \frac{Y-3}{2}} \int_0^1 \int_{-\infty}^\infty \left( \frac{t^{\frac{Y}{2}}}{\sqrt{2\pi\sigma^2}} e^{-\frac{u^2}{2\sigma^2}} \right) \left( \frac{t^{\frac{Y}{2}}}{\sqrt{2\pi\sigma^2}} e^{-\frac{u^2}{2\sigma^2}} \right) \left( -\frac{1}{\sqrt{2\pi}} \hat{\rho}_2^e(x) - \frac{C}{K}|x|^{-Y-2} \right) dx \ du
\]

\[
:= B_{22}^{(31)}(t) + B_{22}^{(32)}(t),
\]

with

\[
B_{22}^{(31)}(t):= \frac{-c^2 Y^2}{2\pi} t^{\frac{Y}{2} + \frac{Y-3}{2}} \int_0^1 \int_{-\infty}^\infty \left( \frac{t^{\frac{Y}{2}}}{\sqrt{2\pi\sigma^2}} e^{-\frac{u^2}{2\sigma^2}} \right) x^{2Y-2} e^{-cxY} dx \ du,
\]

\[
B_{22}^{(32)}(t):= \frac{-cY(Y-1)}{2\pi} t^{\frac{Y}{2} + \frac{Y-3}{2}} \int_0^1 \int_{-\infty}^\infty \left( \frac{t^{\frac{Y}{2}}}{\sqrt{2\pi\sigma^2}} e^{-\frac{u^2}{2\sigma^2}} \right) x^{Y-2} \left( 1 - e^{-cxY} \right) dx \ du.
\]

Changing variables to \( v = t^{\frac{Y}{2}} \sigma x/u \),

\[
B_{22}^{(31)}(t) = -\frac{c^2 Y^2}{2\pi \sigma^{2Y-1}} t^{1-\frac{Y}{2}} \int_0^1 \left[ \int_0^\infty e^{-\frac{v^2}{2\sigma^2}} v^{2Y-2} e^{-cxY} t^{\frac{Y}{2}} u^{Y} dv \right] u^{2Y-1} du,
\]

\[
B_{22}^{(32)}(t) = -\frac{cY(Y-1)}{2\pi \sigma^{2Y-1}} t^{1-\frac{Y}{2}} \int_0^1 \left[ \int_0^\infty e^{-\frac{v^2}{2\sigma^2}} v^{Y-2} \left( 1 - e^{-cxY} \right) u^{Y} dv \right] u^{Y-1} du.
\]

Hence, by the dominated convergence theorem,

\[
\lim_{t \to 0} t^{\frac{Y}{2} - 1} B_{22}^{(31)}(t) = -\frac{c^2 Y}{4\sqrt{2\pi\sigma^{2Y-1}}} \mathbb{E} \left( |W_1|^{2Y-2} \right) = -\frac{C^2 Y \cos^2 \left( \frac{\pi Y}{2} \right) \Gamma^2(-Y) \mathbb{E} \left( |W_1|^2 \right)}{\sqrt{2\pi\sigma^{2Y-1}}},
\]

\[
\lim_{t \to 0} t^{\frac{Y}{2} - 1} B_{22}^{(32)}(t) = -\frac{c^2 Y(Y-1)}{4\sqrt{2\pi\sigma^{2Y-1}}} \mathbb{E} \left( |W_1|^2 \right) = -\frac{C^2 (Y-1) \cos^2 \left( \frac{\pi Y}{2} \right) \Gamma^2(-Y) \mathbb{E} \left( |W_1|^2 \right)}{\sqrt{2\pi\sigma^{2Y-1}}}.
\]

Combining (A.35)-(A.39),

\[
\lim_{t \to 0} t^{\frac{Y}{2} - 1} B_{22}(t) = -\frac{C^2 (2Y-1) \cos^2 \left( \frac{\pi Y}{2} \right) \Gamma^2(-Y) \mathbb{E} \left( |W_1|^2 \right)}{\sqrt{2\pi\sigma^{2Y-1}}} := g_{32}.
\]

Thus, by combining (A.31), (A.32) and (A.40),

\[
B_2(t) = t^{1-\frac{Y}{2}} \left[ d_{32} t^{1-\frac{Y}{2}} + o \left( t^{1-\frac{Y}{2}} \right) + \frac{C \sigma^{1-Y}}{2Y} \mathbb{E} \left( |W_1|^{1-Y} \right) \right] + o(t^{1-\frac{Y}{2}}),
\]

\[
\frac{C \sigma^{1-Y}}{2Y} \mathbb{E} \left( |W_1|^{1-Y} \right) t^{1-\frac{Y}{2}} + d_{32} t^{2-Y} + o(t^{2-Y}) + o(t^{2-Y}), \quad t \to 0.
\]

(A.41)
Step 3. Next, we study the behavior of $B_3(t)$ by further decomposing it as:

$$B_3(t) = e^{-(\eta+\sqrt{\tau_t})} \int_0^\infty \mathbb{E} \left[ (e^{-t^{\frac{1}{2}} U_1} - 1) 1\{z_1 \ge t^{\frac{1}{2}} + w\} \right] \frac{1 - e^{\sqrt{w}}}{\sqrt{t}} \frac{e^{-\frac{w^2}{2\pi\sigma^2}}}{\sqrt{2\pi\sigma^2}} dw$$

$$+ e^{-(\eta+\sqrt{\tau_t})} \int_0^\infty \tilde{\mathbb{P}} \left( Z_1 \ge t^{\frac{1}{2}} + w \right) \frac{1 - e^{\sqrt{w}}}{\sqrt{t}} \frac{e^{-\frac{w^2}{2\pi\sigma^2}}}{\sqrt{2\pi\sigma^2}} dw$$

$$+ e^{-(\eta+\sqrt{\tau_t})} \int_0^\infty \mathbb{E} \left[ 1\{z_1 \ge t^{\frac{1}{2}} + w\} \right] \frac{e^{-t^{\frac{1}{2}} U_1} - e^{-t^{\frac{1}{2}} (U_1 + \tilde{U}_1)}}{t^{\frac{1}{2}} - \sqrt{2\pi\sigma^2}} dw$$

$$= e^{-(\eta+\sqrt{\tau_t})} \int_0^\infty \mathbb{E} \left( Z_1 1\{z_1 \ge t^{\frac{1}{2}} + w\} \right) e^{\sqrt{w}} \frac{e^{-\frac{w^2}{2\pi\sigma^2}}}{\sqrt{2\pi\sigma^2}} dw$$

$$:= B_{31}(t) + B_{32}(t) + B_{33}(t) + B_{34}(t).$$

(A.42)

First, note that the term $B_{32}(t)$ is similar to the term $B_{22}(t)$ in (A.31) and, thus, using arguments similar to those leading to (A.40) gives:

$$B_{32}(t) = -\frac{C \sigma^{1-Y}}{2Y} \mathbb{E} \left[ (W_1^*)^{1-Y} t^{1-\frac{Y}{2}} - d_{32}^* t^{2-Y} + o(t^{2-Y}), \ t \to 0. \right. \tag{A.43}$$

Next, the term $B_{34}(t)$ is similar to the term $B_{13}(t)$ introduced in (A.15) and, thus, using arguments similar to those leading to (A.29) gives:

$$B_{34}(t) = \frac{C \sigma^{1-Y}}{2(Y-1)} \mathbb{E} \left[ (W_1^*)^{1-Y} t^{1-\frac{Y}{2}} + d_{34}^* t^{2-Y} + o(t^{2-Y}) + o(t^{2-Y}), \ t \to 0. \right. \tag{A.44}$$

It remains to analyze $B_{31}(t)$ and $B_{33}(t)$. For $B_{31}(t)$, the expectation therein can be written as

$$\mathbb{E} \left[ (e^{-t^{\frac{1}{2}} M^* U_1^{(p)} - G^* U_2^{(n)}} - 1) 1\{-U_1^{(p)} - U_2^{(n)} \le -t^{\frac{1}{2}} + w\} \right] = \mathbb{E} \left[ (e^{-t^{\frac{1}{2}} G^* U_1^{(n)} - M^* U_2^{(n)}} - 1) 1\{U_1^{(p)} + U_2^{(n)} \le -t^{\frac{1}{2}} + w\} \right],$$

where $(U_1^{(p)}, U_2^{(n)}) := (-U_1^{(p)}, -U_2^{(n)}) \equiv (U_1^{(p)}, U_2^{(n)})$. Thus, $B_{31}(t)$ is the same as $B_{21}(t)$ defined in (A.31) but with the roles of the parameters $M^*$ and $G^*$ reversed. In other words, writing $B_{21}(t; M^*, G^*):=B_{21}(t)$ to emphasize the dependence on the parameters $M^*$ and $G^*$, we have that $B_{31}(t) = B_{21}(t; G^*, M^*)$. Therefore, in view of (A.32),

$$\lim_{t \to 0} t^{\frac{1}{2}} B_{31}(t) = 0. \tag{A.45}$$

To finish, we further decompose $B_{33}(t)$ as:

$$B_{33}(t) = e^{-(\eta+\sqrt{\tau_t})} \int_0^\infty \mathbb{E} \left[ 1\{Z_1 \ge t^{\frac{1}{2}} + w\} \right] \int_{0}^{Z_1 + \tilde{U}_1} \left( e^{-t^{\frac{1}{2}} u} - 1 \right) du \frac{e^{\sqrt{w}}}{\sqrt{2\pi\sigma^2}} \frac{e^{-\frac{w^2}{2\pi\sigma^2}}}{\sqrt{2\pi\sigma^2}} dw$$

$$= t^{\frac{1}{2}} e^{-(\eta+\sqrt{\tau_t})} \int_0^\infty \left[ \int_{-\infty}^{0} (e^{-x} - 1) P_t(w, x) dx \right] \frac{e^{\sqrt{w}}}{\sqrt{2\pi\sigma^2}} \frac{e^{-\frac{w^2}{2\pi\sigma^2}}}{\sqrt{2\pi\sigma^2}} dw$$

$$+ t^{\frac{1}{2}} e^{-(\eta+\sqrt{\tau_t})} \int_0^\infty \left[ \int_{0}^{\infty} (e^{-x} - 1) P_t(w, x) dx \right] \frac{e^{\sqrt{w}}}{\sqrt{2\pi\sigma^2}} \frac{e^{-\frac{w^2}{2\pi\sigma^2}}}{\sqrt{2\pi\sigma^2}} dw, \tag{A.46}$$

where

$$P_t(w, x) := \tilde{\mathbb{P}} \left( Z_1 \ge t^{\frac{1}{2}} + w, \tilde{U}_1 \le t^{\frac{1}{2}} x \le Z_1 + \tilde{U}_1 \right).$$

When $x < 0$, by (2.12), for any $t > 0$ and $w > 0$,

$$P_t(w, x) \le \tilde{\mathbb{P}} \left( \tilde{U}_1 \le t^{\frac{1}{2}} x \right) \le \mathbb{E} \left( e^{-\tilde{U}_1} \right) e^{t^{\frac{1}{2}} x} = e^\eta e^{t^{\frac{1}{2}} x}.$$
Hence, for $0 < t < 1$ and since $1 < Y < 2$,

$$0 \leq t^{-1}e^{-(nt+\sqrt{7}t)} \int_{0}^{\infty} \left[ \int_{-\infty}^{0} (e^{-x} - 1) \tilde{P} \left( Z_1 \geq t^{1/2} + \tilde{U}_1, \tilde{U}_1 \leq t^{-1/2} x \leq Z_1 + \tilde{U}_1 \right) dx \right] e^{\sqrt{7}t}w e^{-\frac{w^2}{2\pi\sigma^2}} dw$$

$$\leq t^{-1}e^{-(nt+\sqrt{7}t)} e^{n} \int_{-\infty}^{0} (e^{-x} - 1) e^{t^{-1/2}x} dx \int_{0}^{\infty} w e^{w} e^{-\frac{w^2}{2\pi\sigma^2}} dw$$

$$= \frac{t^{\frac{1}{2}} - 1}{1 - t^{1/2}} e^{-(nt+\sqrt{7}t)} e^{n} \int_{0}^{\infty} w e^{w} e^{-\frac{w^2}{2\pi\sigma^2}} dw \to 0, \quad t \to 0. \quad (A.47)$$

For the second integral in (A.46), we show below (see Appendix B) that

$$\lim_{t \to 0} \frac{1}{t} P_t(w, x) = \frac{C}{Y} \left( M_Y - (M^*)_Y \right) x^{-Y}. \quad (A.48)$$

Moreover, by arguments similar to those leading to (2.9-i), there exists a constant $\lambda \in (0, \infty)$, such that

$$t^{-1} \tilde{P} \left( t^{1/2} x \leq Z_1 + \tilde{U}_1 \right) \leq \lambda x^{-Y}, \quad \text{for any } x > 0,$$

and thus, by the dominated convergence theorem,

$$\lim_{t \to 0} t^{-1}e^{-(nt+\sqrt{7}t)} \int_{0}^{\infty} \left[ \int_{0}^{\infty} (e^{-x} - 1) \tilde{P} \left( Z_1 \geq t^{1/2} + \tilde{U}_1, \tilde{U}_1 \leq t^{-1/2} x \leq Z_1 + \tilde{U}_1 \right) dx \right] e^{\sqrt{7}t}w e^{-\frac{w^2}{2\pi\sigma^2}} dw$$

$$= -\frac{C\Gamma(-Y)}{2} \left[ M_Y - (M^*)_Y \right]. \quad (A.49)$$

The above limit together with (A.47) leads to:

$$\lim_{t \to 0} t^{-\frac{1}{2}} B_{33}(t) = -\frac{d_3^3}{2}, \quad (A.50)$$

where $d_3^3 = C\Gamma(-Y) \left( M_Y - (M^*)_Y \right)$. Hence, by (A.42)-(A.45) and (A.50),

$$B_3(t) = -\frac{d_3^3}{2} t^{1/2} + \frac{C\sigma^1-Y}{2Y(Y-1)} \tilde{E} \left( |W_1^{Y-1}| \right) t^{1/2} + (d_3^3 - d_3^3) t^{2-Y} + o(t^{1/2}) + o(t^{2-Y}), \quad t \to 0. \quad (A.51)$$

Finally, combining (A.11), (A.12), (A.14), (A.30), (A.41) and (A.51) established that,

$$\Delta_0(t) = \left( \frac{\tilde{\gamma} - \epsilon_1}{2} - \frac{\sigma^2}{4} - d_3^3 \right) t^{1/2} + \frac{C\sigma^1-Y}{2Y(Y-1)} \tilde{E} \left( |W_1^{Y-1}| \right) t^{1/2} + 2 \left( d_3^3 - d_3^3 - \frac{\epsilon^2}{2} \right) t^{2-Y} + o(t^{1/2}) + o(t^{2-Y}),$$

which yields (3.16), by noting that the coefficient of the first term above reduces to the expression $d_3^3$ in (3.13) and that $d_3^3 = 2(d_3^3 - d_3^3)$.

\[\square\]

### B Further Proofs

**Proof of Lemma 2.1.** From the leading term in the expansion (2.8), there exists $N > 0$ such that, for any $x > 0$,

$$\tilde{P} \left( \tilde{U}_1^{(p)} \geq x \right) = \tilde{P} \left( \tilde{U}_1^{(p)} \geq x \right) (1_{x \geq N}) + 1_{x < N} \leq 2\frac{C}{Y} x^{-Y} 1_{x \geq N} + \frac{N}{x^Y} 1_{x < N} \leq (2CY^{-1} + NY) x^{-Y},$$

and the first relationship in (2.9) follows by setting $K_1 = 2CY^{-1} + NY$. Similarly, from (2.8), there exists $N > 0$ such that, for any $x > 0$,

$$\left| \tilde{P} \left( \tilde{U}_1^{(p)} \geq x \right) - C \frac{x^{-Y}}{Y} \right| = \left| \tilde{P} \left( \tilde{U}_1^{(p)} \geq x \right) - C \frac{x^{-Y}}{Y} (1_{x \geq N}) + 1_{x < N} \right|$$

$$\leq \frac{C^2}{\pi} |\sin(2\pi Y)| \Gamma(2Y) \Gamma^2(-Y) x^{-2Y} 1_{x \geq N} + \left( \tilde{P} \left( \tilde{U}_1^{(p)} \geq x \right) + C \frac{x^{-Y}}{Y} \right) 1_{x < N}$$

$$\leq \left( \frac{C^2}{\pi} |\sin(2\pi Y)| \Gamma(2Y) \Gamma^2(-Y) + 2NY + CNY^{-1} \right) x^{-2Y}.$$
The second identity in (2.9) follows by setting $K_2 = C^2 |\sin(2\pi Y)| \Gamma(2Y) \Gamma^2(-Y)/\pi + N^{2Y} + CN^Y Y^{-1}$.

**Proof of (A.7).** We begin with $B_{11}(t)$. Using (2.4) and (2.6), lead to the decomposition

$$
\bar{P}\left(Z_1 + \gamma t > 0, Z_t + \gamma t + \bar{U}_1 \geq v\right) = \bar{P}\left(\tilde{U}_1^{(p)} + \bar{U}_1^{(n)} + \gamma t > 0, M\bar{U}_1^{(p)} - G\bar{U}_1^{(n)} + \gamma t \geq v\right)
$$

$$
= \bar{P}\left(\tilde{U}_1^{(p)} \geq \frac{v + GU_1^{(n)} - \gamma t}{M}, -\bar{U}_1^{(n)} < \frac{v + M^*\gamma t}{M + G}\right) + \bar{P}\left(\bar{U}_1^{(p)} + \gamma t \geq -\bar{U}_1^{(n)} \geq \frac{v + M^*\gamma t}{M + G}\right).
$$

Hence,

$$
B_{11}(t) := \int_0^\infty e^{-t^\frac{1}{r}v} - 1 \left(\bar{P}\left(\tilde{U}_1^{(p)} \geq \frac{v + GU_1^{(n)} - \gamma t}{M}, -\bar{U}_1^{(n)} < \frac{v + M^*\gamma t}{M + G}\right) - \frac{CM^Y}{Yv^r}\right) dv \tag{B.1}
$$

$$
+ \int_0^\infty e^{-t^\frac{1}{r}v} - 1 \left(\bar{P}\left(\bar{U}_1^{(p)} + \gamma t \geq -\bar{U}_1^{(n)} \geq \frac{v + M^*\gamma t}{M + G}\right)\right) dv. \tag{B.2}
$$

By (2.9-i), for any $v > 0$ and $t$ small enough (so that $G^*|\gamma t| < 1$ and $M^*|\gamma t| < 1$), the expression inside the integral in (B.2), which we denote by $b_1^{(2)}(t;v)$, is such that

$$
\left|b_1^{(2)}(t;v)\right| \leq \bar{P}\left(\tilde{U}_1^{(p)} \geq \frac{v + M^*\gamma t}{M + G} - \gamma t\right) \left(\bar{U}_1^{(n)} \leq \frac{v + M^*\gamma t}{M + G}\right) \leq v\mathbf{1}_{\{v \leq 1\}} + v\mathbf{1}_{\{v > 1\}} \min\left\{1, K_1^2(M + G)^{2Y} v^{-2Y}\right\},
$$

where $K_1 \in (0,\infty)$ is given as in (2.9-i). Hence, by the dominated convergence theorem,

$$
\lim_{t \to 0} \int_0^\infty b_1^{(2)}(t;v) dv = - \int_0^\infty v\bar{P}\left(\tilde{U}_1^{(p)} \geq -\bar{U}_1^{(n)} \geq \frac{v}{M + G}\right) dv. \tag{B.3}
$$

We now bound the expression inside the integral in (B.1), which we denote by $b_1^{(1)}(t;v)$. It suffices to consider $v > 1$, since $|b_1^{(1)}(t;v)| \leq v(1 + CY^{-1}M^2v^{-Y})$, which is integrable on $\{v \leq 1\}$. We also let $t$ be small enough, so that $|\gamma t| < 1$, $G^*|\gamma t| < 1$, and $M^*|\gamma t| < 1$. Then, for any $v > 1$,

$$
\left|b_1^{(1)}(t;v)\right| \leq v \int_{\mathbb{R}} p_U(1, y) \left|\bar{P}\left(\tilde{U}_1^{(p)} \geq \frac{v - Gy - \gamma t}{M}\right) - \frac{CM^Y}{Yv^r}\right| dy
$$

$$
\leq \int_{-\infty}^{\frac{v - Gy - \gamma t}{M}} p_U(1, y) \left|\bar{P}\left(\tilde{U}_1^{(p)} \geq \frac{v - Gy - \gamma t}{M}\right) - \frac{CM^Y}{Yv^r}\right| dy
$$

$$
+ \int_{\frac{v - Gy - \gamma t}{M}}^{\infty} p_U(1, y) \frac{CM^Y}{Yv^{Y+1}} \left|(v - Gy - \gamma t)^{-Y} - v^{-Y}\right| dv + \frac{CM^Y}{Yv^r - 1} \bar{P}\left(\bar{U}_1^{(p)} \geq \frac{v + M^*\gamma t}{M + G}\right)
$$

$$
:= D_1^{(1)}(v) + D_1^{(2)}(v) + D_1^{(3)}(v). \tag{B.4}
$$

Next, since

$$
\frac{v - Gy - \gamma t}{M} \geq \frac{v - G\frac{M^*\gamma t}{M + G} - \gamma t}{M} = \frac{v - G^*\gamma t}{M + G} > 0, \quad \text{for any } y \leq \frac{v + M^*\gamma t}{M + G}, \tag{B.5}
$$

the first integral in (B.4) can be bounded, using (2.9-ii), via:

$$
D_1^{(1)}(v) \leq K_2 v \int_{-\infty}^{\frac{v + M^*\gamma t}{M + G}} \frac{p_U(1, y)M^{2Y}}{(v - Gy - \gamma t)^{2Y}} dy \leq K_2(M + G)^{2Y} v^{1 - 2Y}, \quad \text{for any } v > 1, \tag{B.6}
$$

where $K_2 \in (0,\infty)$ is given as in (2.9-ii). Moreover, using the convexity and monotonicity of the function $f(x) = x^{-Y}$ on $(0,\infty)$ and (B.5), the second integral in (B.4) can be upper estimated as

$$
D_1^{(2)}(v) \leq CM^Y v \int_{-\infty}^{\frac{v + M^*\gamma t}{M + G}} p_U(1, y)v^{-Y - 1}|Gy + \gamma t| dy \leq CM^Y v^{-Y} \left(G\bar{E}\left|\bar{U}_1^{(p)}\right| + 1\right). \tag{B.7}
$$
Finally, by (2.9-i), the last term in (B.4) can be upper bounded via
\[ D_i^{(3)}(v) \leq K_1 CM^Y Y^{-1} v^{1-2Y}, \quad \text{for any } v > 1. \] (B.8)

Combining (B.4) and (B.6)-(B.8), and by the dominated convergence theorem,
\[ \lim_{t \to 0} \int_0^\infty b_{11}^{(t)}(t; v) \, dv = -\int_0^\infty v \left[ \tilde{P} \left( \tilde{U}^{(n)} \geq \frac{v + G \tilde{U}^{(n)}}{M}, -\tilde{U}^{(n)} \leq \frac{v}{M + G} \right) - CM^Y \right] dv. \] (B.9)

Putting together (B.3) and (B.9), we obtain
\[ \lim_{t \to 0} B_{11}(t) = -\int_0^\infty v \tilde{P} \left( \tilde{U}^{(p)} \geq -\tilde{U}^{(n)} \geq \frac{v}{M + G} \right) dv - \int_0^\infty v \left[ \tilde{P} \left( \tilde{U}^{(p)} \geq \frac{v + G \tilde{U}^{(n)}}{M}, -\tilde{U}^{(n)} \leq \frac{v}{M + G} \right) - CM^Y \right] dv \]
\[ = -\int_0^\infty v \left[ \tilde{P} \left( Z_1 > 0, \tilde{U}_1 \geq v \right) - \frac{CM^Y}{Yv^Y} \right] dv. \] (B.10)

Applying the same arguments to the decomposition
\[ \tilde{P} \left( Z_1 + \check{\gamma}_t \leq 0, \tilde{U}_1 \geq v \right) = \tilde{P} \left( -\tilde{U}^{(n)} - \check{\gamma}_t \geq \tilde{U}^{(p)} \geq \frac{v + G^* \check{\gamma}_t}{M + G} \right) + \tilde{P} \left( -\tilde{U}^{(n)} \geq \frac{v - M* \tilde{U}^{(p)}}{G^*}, \tilde{U}^{(p)} < \frac{v - G^* \check{\gamma}_t}{M + G} \right), \]

it can be shown that
\[ \lim_{t \to 0} B_{12}(t) = -\int_0^\infty v \tilde{P} \left( \tilde{U}^{(n)} \geq \tilde{U}^{(p)} \geq \frac{v}{M + G} \right) dv - \int_0^\infty v \left[ \tilde{P} \left( \tilde{U}^{(n)} \geq \frac{v - M^* \tilde{U}^{(p)}}{G^*}, \tilde{U}^{(p)} \leq \frac{v}{M + G} \right) - \frac{C(G^*)^Y}{Yv^Y} \right] dv \]
\[ = -\int_0^\infty v \left[ \tilde{P} \left( Z_1 < 0, \tilde{U}_1 \geq v \right) - \frac{C(G^*)^Y}{Yv^Y} \right] dv. \] (B.11)

Combining (B.10) and (B.11), we obtain
\[ B_{12}(t) = -\int_0^\infty v \tilde{P} \left( \tilde{U}^{(n)} \geq \tilde{U}^{(p)} \geq \frac{v}{M + G} \right) dv - \int_0^\infty v \left[ \tilde{P} \left( \tilde{U}^{(n)} \geq \frac{v - M^* \tilde{U}^{(p)}}{G^*}, \tilde{U}^{(p)} \leq \frac{v}{M + G} \right) - \frac{C(G^*)^Y}{Yv^Y} \right] dv. \]

Proof of (A.18). First, change of variables, \( x = t^{\frac{1}{2}} u \), in the integral of the term \( B_{12}(t) \) defined in (A.15), so that
\[ B_{12}(t) = t^{-\frac{1}{2}} e^{-(\sqrt{7}t + \sqrt{7}w)} \int_0^\infty \left[ \int_R (e^{-x} - 1) \tilde{P} \left( -t^{\frac{1}{2}} \xi w \leq Z_1 \leq 0, \tilde{U}_1 + Z_1 \leq t^{\frac{1}{2}} x \leq \tilde{U}_1 \right) dx \right] e^{-\sqrt{7}w} e^{-w^2 / 2\pi \sigma^2} dw. \]

We next prove that \( B_{12}(t) = o(t^{1/2}) \) as \( t \to 0 \). To this end, let
\[ B_{12}^{(1)}(t) = \int_0^\infty \left[ \int_0^\infty (1 - e^{-x}) \tilde{P} \left( -t^{\frac{1}{2}} \xi w \leq Z_1 \leq 0, \tilde{U}_1 + Z_1 \leq t^{\frac{1}{2}} x \leq \tilde{U}_1 \right) dx \right] e^{-\sqrt{7}w} e^{-w^2 / 2\pi \sigma^2} dw, \]
\[ B_{12}^{(2)}(t) = \int_0^\infty \left[ \int_0^\infty (e^{-x} - 1) \tilde{P} \left( -t^{\frac{1}{2}} \xi w \leq Z_1 \leq 0, \tilde{U}_1 + Z_1 \leq t^{\frac{1}{2}} x \leq \tilde{U}_1 \right) dx \right] e^{-\sqrt{7}w} e^{-w^2 / 2\pi \sigma^2} dw. \]

For any \( t > 0, w > 0 \) and \( x > 0 \), by (2.9-i), \( P_t(x, w) := \tilde{P} \left( -t^{\frac{1}{2}} \xi w \leq Z_1 \leq 0, \tilde{U}_1 + Z_1 \leq t^{\frac{1}{2}} x \leq \tilde{U}_1 \right) \) is such that
\[ \frac{1}{t} P_t(x, w) = \tilde{P} \left( -t^{\frac{1}{2}} \xi w \leq \tilde{U}_1^{(p)} + \tilde{U}_1^{(n)} \leq 0, (M^* + 1) \tilde{U}_1^{(n)} - (G^* - 1) \tilde{U}_1^{(n)} \leq t^{\frac{1}{2}} x \leq M^* \tilde{U}_1^{(p)} - G^* \tilde{U}_1^{(n)} \right) \]
\[ \leq \frac{1}{t} \tilde{P} \left( -t^{\frac{1}{2}} x + (M* + 1) t^{\frac{1}{2}} - \tilde{w} \leq -\tilde{U}_1^{(n)} \leq \frac{t^{\frac{1}{2}} x + (M^* + 1) t^{\frac{1}{2}} - \tilde{w}}{M^* + G^*} \leq M^* \right) \]
\[ \leq \frac{1}{t} \tilde{P} \left( -t^{\frac{1}{2}} x + (M* + 1) t^{\frac{1}{2}} - \tilde{w} \leq -\tilde{U}_1^{(n)} \leq M^* \right) \tilde{P} \left( \frac{t^{\frac{1}{2}} M^* x - G^* (M^* + 1) t^{\frac{1}{2}} - \tilde{w}}{M^* (M^* + G^*)} \leq \tilde{U}_1^{(p)} \right) \]
\[ \leq K_1 (M^* + G^*) Y x^{-Y} \tilde{P} \left( \frac{t^{\frac{1}{2}} M^* x - G^* (M^* + 1) t^{\frac{1}{2}} - \tilde{w}}{M^* (M^* + G^*)} \leq \tilde{U}_1^{(p)} \right) \rightarrow 0, \quad t \to 0, \]
while for $t > 0$, $w > 0$ and $x < 0$,
\[
\frac{1}{t} P_t(x, w) \leq \frac{2}{t} \mathbb{P} \left( \bar{U}_1^{(p)} \leq \frac{t^{-\frac{1}{2}} x}{2(M^* + G^*)} \right) \leq \frac{2}{t} \mathbb{E} \left( e^{-\mathcal{G}_i^{(p)}} \right) \exp \left\{ \frac{t^{-\frac{1}{2}} x}{2(M^* + G^*)} \right\} \to 0, \quad t \to 0.
\]

it follows from the dominated convergence that $B_{12}^{(1)}(t) = o(t)$ and $B_{12}^{(2)}(t) = o(t)$, which in turn implies that
\[
B_{12}(t) = t^{-\frac{1}{2}} e^{-((p+\sqrt{t})t)} \left( B_{12}^{(2)}(t) - B_{12}^{(1)}(t) \right) = o(t^{\frac{1}{2}}), \quad t \to 0.
\]

(B.12)

The proof is now complete.

\[\square\]

**Proof of (A.48).** First, for any $t > 0$, $x > 0$ and $w > 0$,
\[
\frac{1}{t} P_t(w, x) = \frac{1}{t} \mathbb{P} \left( \bar{U}_1^{(p)} + \bar{U}_1^{(n)} \geq t^\frac{1}{2} \frac{-x}{w}, M^* \bar{U}_1^{(p)} - G^* \bar{U}_1^{(n)} \leq t^\frac{1}{2} x \leq M \bar{U}_1^{(p)} - G \bar{U}_1^{(n)} \right)
\]
\[
= \frac{1}{t} \int_{\mathbb{R}} \mathbb{P} \left( \bar{U}_1^{(p)} \geq t^\frac{1}{2} \frac{x}{w} + u, \frac{t^\frac{1}{2} x - G u}{M} \leq \bar{U}_1^{(p)} \leq \frac{t^\frac{1}{2} x - G^* u}{M^*} \right) p_U(1, u)\, du.
\]

Note that
\[
\frac{t^\frac{1}{2} x - G u}{M} \leq \frac{t^\frac{1}{2} x - G^* u}{M^*} \iff u \leq \frac{t^\frac{1}{2} x - G^* u}{M^*}, \quad \frac{t^\frac{1}{2} x + w}{M + G} \leq \frac{t^\frac{1}{2} x + w}{M + G} \iff u \leq \frac{t^\frac{1}{2} x}{M + G},
\]
\[
\therefore \quad t^\frac{1}{2} \frac{x}{w} + u \leq \frac{t^\frac{1}{2} x - G u}{M} \iff u \leq \frac{t^\frac{1}{2} x - M^* t^\frac{1}{2} \frac{x}{w}}{M + G}.
\]

Hence,
\[
\frac{1}{t} P_t(w, x) = \frac{1}{t} \int_{-\infty}^{\frac{t^\frac{1}{2} x - M^* t^\frac{1}{2} \frac{x}{w}}{M + G}} \mathbb{P} \left( \frac{t^\frac{1}{2} x - G u}{M} \leq \bar{U}_1^{(p)} \leq \frac{t^\frac{1}{2} x - G^* u}{M^*} \right) p_U(1, u)\, du
\]
\[
+ \frac{1}{t} \int_{\frac{t^\frac{1}{2} x - M^* t^\frac{1}{2} \frac{x}{w}}{M + G}}^{\frac{t^\frac{1}{2} x + w}{M + G}} \mathbb{P} \left( \frac{t^\frac{1}{2} x + w}{M + G} \leq \bar{U}_1^{(p)} \leq \frac{t^\frac{1}{2} x + w}{M^*} \right) p_U(1, u)\, du
\]
\[
:= I_1(t; w, x) + I_2(t; w, x).
\]

(B.13)

For the first integral in (B.13), note that for any $t > 0$, $x > 0$ and $w > 0$,
\[
u \leq \frac{t^\frac{1}{2} x - M^* t^\frac{1}{2} \frac{x}{w}}{M + G} < \frac{t^\frac{1}{2} x}{M + G} < \frac{t^\frac{1}{2} x - M w \sqrt{t}}{M + G} \quad \Rightarrow \quad t^\frac{1}{2} x - G u > 0, \quad \frac{x - M w \sqrt{t}}{M + G} > 0 \quad \Rightarrow \quad t < \frac{x^2}{M^2 w^2}.
\]

Hence, by (2.8) and the dominated convergence theorem, for any $x > 0$, $w > 0$ and $u \leq t^\frac{1}{2} (x - M w)/(M + G)$,
\[
\lim_{t \to 0} I_1(t; w, x) = \int_{\mathbb{R}} p_U(1, u) \left[ \lim_{t \to 0} \frac{1}{t} \mathbb{P} \left( \frac{t^\frac{1}{2} x - G u}{M} \leq \bar{U}_1^{(p)} \leq \frac{t^\frac{1}{2} x - G^* u}{M^*} \right) 1_{\left\{ u \leq \frac{t^\frac{1}{2} x - G u}{M} \right\}} \right] du
\]
\[
= \int_{\mathbb{R}} p_U(1, u) \left[ \lim_{t \to 0} \frac{1}{t} \mathbb{P} \left( \bar{U}_1^{(p)} \geq \frac{t^\frac{1}{2} x - G u}{M} \right) \right] du - \int_{\mathbb{R}} p_U(1, u) \left[ \lim_{t \to 0} \frac{1}{t} \mathbb{P} \left( \bar{U}_1^{(p)} \geq \frac{t^\frac{1}{2} x - G^* u}{M^*} \right) \right] du
\]
\[
= C \left( M^* Y - (M^*)^Y \right) x^{-Y}. \tag{B.14}
\]

For the second integral in (B.13), since for any $x > 0$ and $w > 0$, $t^\frac{1}{2} x - M^* t^\frac{1}{2} \frac{x}{w} > 0$ is equivalent to $t < w^2/(M^2 w^2)$,
\[
0 \leq \frac{1}{t} \int_{t^\frac{1}{2} \frac{x - M^* \frac{x}{w}}{w}}^{t^\frac{1}{2} \frac{x - M^* \frac{x}{w}}{w}} \mathbb{P} \left( \frac{t^\frac{1}{2} x + w + u}{M} \leq \bar{U}_1^{(p)} \leq \frac{t^\frac{1}{2} x - G^* u}{M^*} \right) p_U(1, u)\, du
\]
\[
\leq \frac{1}{t} \mathbb{P} \left( \bar{U}_1^{(p)} \geq t^\frac{1}{2} \frac{x}{w} \right) \mathbb{P} \left( -\bar{U}_1^{(n)} \geq \frac{t^\frac{1}{2} x - M^* t^\frac{1}{2} \frac{x}{w}}{M + G} \right) \to 0, \quad t \to 0, \tag{B.15}
\]

which completes the proof.

\[\square\]
References


