Abstract

We consider a univariate semimartingale model for (the logarithm of) an asset price, containing jumps having possibly infinite activity (IA). The nonparametric threshold estimator  \( \hat{IV}_n \) of the integrated variance 
\[ IV := \int_0^T \sigma_s^2 ds \]
proposed in [17] is constructed using observations on a discrete time grid, and precisely it sums up the squared increments of the process when they are below a threshold, a deterministic function of the observation step and possibly of the coefficients of \( X \). All the threshold functions satisfying given conditions allow asymptotically consistent estimates of \( IV \), however the finite sample properties of \( \hat{IV}_n \) can depend on the specific choice of the threshold. We aim here at optimally selecting the threshold by minimizing either the estimation mean square error (MSE) or the conditional mean square error (cMSE). The last criterion allows to reach a threshold which is optimal not in mean but for the specific volatility and jumps paths at hand.

A parsimonious characterization of the optimum is established, which turns out to be asymptotically proportional to the Lévy’s modulus of continuity of the underlying Brownian motion. Moreover, minimizing the cMSE enables us to propose a novel implementation scheme for approximating the optimal threshold. Monte Carlo simulations illustrate the superior performance of the proposed method.

Keywords: Threshold estimator, integrated variance, Lévy jumps, mean square error, conditional mean square error, modulus of continuity of the Brownian motion paths, numerical scheme

JEL classification codes: C6, C13

1 Introduction

The importance of including jump components in assets prices models has been extensively highlighted. For instance Huang and Tauchen (in [12]) documented empirically that jumps account for 7% of the S&P500 market price variance, and many different tests for the presence of jumps in asset prices have been proposed and applied in the literature (see [18], Sec. 17.3, for a review of the most used tests). From an economic point of view, jumps may reflect, for instance, reactions of the market to important announcements or events. Thus semimartingale models with jumps are broadly used in a variety of financial applications, for example for derivative pricing, and also infinite activity jump components have been considered (see e.g. [8], ch.15).

Separately identifying the contribution of the Brownian part (through the Integrated Variance IV) and the one of the jumps to the asset price variations when we can observe prices discretely is crucial in many respects, for instance, for model assessing and for improving volatility forecasting: e.g. in [5] the proposed test for the presence of jumps is obtained after having filtered out the jump component; in [1], the separation allows to construct two tests for recognizing whether the jumps have finite or infinite variation; in [2] it is shown that including a separate factor accounting for the jumps in an econometric model for the realized variance substantially improves the out

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of sample volatility forecasts. The correct identification of a model has a significant impact on option pricing and on risk management and thus on assets allocation: for instance Carr and Wu (in [7]) show that the asymptotic behavior of the price of an option as the time-to-maturity approaches zero is substantially different depending on whether the model for the underlying contains jumps or not, and whether the jumps have finite or infinite variation; Liu, Longstaff, and Pan (in [16]) find that incorporating jumps events in the model dramatically affects the optimal investment strategy.

With discrete (non-noisy) observations, non parametrically disentangling the jumps from integrated variance (IV) has mainly been done by using Multipower Variations (MPVs) and Truncated (or Threshold) Realized Variance (TRV) (see [18], Sec. 17.2, for a review of also other methods). MPV relies on the observation that, when the jumps have finite activity, the probability of having jumps among subsequent sampling intervals is very small, however with infinite activity jumps, this probability is much larger. Hence, MPV may not work well in the general case. In contrast, TRV has been shown to be consistent also in the presence of any infinite activity jumps component ([17]). Further, it is efficient as soon as the jumps have finite variation.

However the choice of the truncation level (threshold) has an impact on the estimation performance of IV on finite samples. The estimation error is large when either the threshold is too small or when it is too large. In the first case too many increments are discarded, included the increments bearing relevant information about the Brownian part, and TRV underestimates IV. In the second case too many increments are kept within TRV, included many increments containing jumps, leading to an overestimation of IV. Many different data driven choices of the threshold have been proposed in the literature, for instance Ait-Sahalia and Jacod [1] (Sec. 4 therein) chose a truncation level of the form $\alpha h^{0.2}$, where $h$ is the observation step and $\alpha$ is a multiplier of the standard deviations of the continuous martingale part of the process (other choices are described in [18], p.418). However it is important to control for the estimation error for a given time resolution $h$, and here we look for an endogenous, theoretically supported, optimal choice.

We consider the model

$$dX_t = \sigma_t dW_t + dJ_t, \quad (1)$$

where $W$ is a standard Brownian motion, $\sigma$ is a càdlàg process, and $J$ is a pure jump semimartingale (SM) process. We assume that we have at our disposal a record $\{x_0, X_1, \ldots, X_{n}\}$ of discrete observations of $X$ spanned on the fixed time interval $[0, T]$. We also define $\Delta_i Z$, or $\Delta^2_i Z$, the increment $Z_{t_i} - Z_{t_{i-1}}$ for any process $Z$, and a threshold function $r(\sigma, h)$ any deterministic non-negative function of the observation step $h$, and possibly of a summary measure $\sigma$ of the realized volatility path of $(\sigma_t)_{t \geq 0}$, such that for any value $\sigma \in \mathbb{R}$ the following conditions are satisfied

$$\lim_{h \to 0} r(\sigma, h) = 0, \quad \lim_{h \to 0} \frac{r(\sigma, h)}{h \log \frac{1}{h}} = +\infty.$$

We know that then TRV, given by

$$\hat{IV}_n := \sum_{i=1}^{n} (\Delta_i X)^2 I_{(\Delta_i X)^2 \leq r(\sigma_{t_{i-1}}, h_i))}, \quad (2)$$

where $h_i := t_i - t_{i-1}$, is a consistent estimator of $IV := \int_0^T \sigma_s^2 ds$, as $\sup_i h_i \to 0$, as soon as $(\sigma_t)_{t \geq 0}$ is a.s. bounded away from zero on $[0, T]$. In the case where the jump process $J$ has finite variation (FV) and the observations are evenly spaced, the estimator is also asymptotically Gaussian and efficient.

For the choice of the threshold (TH) in finite samples, we consider the following two optimality criteria: minimization of MSE, the expected quadratic error in the estimation of IV; and minimization of cMSE, the expected quadratic error conditional on the realized paths of the jump process $J$ and of the volatility process $(\sigma_t)_{t \geq 0}$. Even though, as mentioned above, many different TH selection procedures have been proposed, the literature for optimal TH selection is rather scarce. In [10] the TH that minimizes the expected number of jump misclassifications is considered for a class of additive processes with finite activity (FA) jumps and absolutely continuous characteristics. Even though it is shown therein that the proposed criterion is asymptotically equivalent to the minimization of the
MSE in the case of Lévy processes with FA jumps, the latter optimality criterion was not directly analyzed in [10]. Here we go beyond and not only investigate the MSE criterion in the presence of FA jumps but also consider infinite activity jumps and further introduce the novel cMSE criterion. The last criterion allows to reach a threshold which is optimal not in mean but for the specific volatility and jumps paths at hand, so it is particularly appealing in the cases of non-stationary processes, for which, even if the MSE was feasible, the deviation of each realization from the unconditional mean value could be quite large, yielding a poor performance of the unconditional criterion. Moreover, minimizing the cMSE is important from a practical point of view, as will be seen in Section 5, where we propose a new TH selection method in the presence of FA jump processes.

Assuming evenly spaced observations, it turns out that for any semimartingale X, for which the volatility and the jump processes are independent of the underlying Brownian motion, the two quantities MSE and cMSE are explicit functions of the TH and under each criterion an optimal TH exists, and is a solution of an explicitly given equation, the equation being different under the two criteria. Under certain specific assumptions we also show uniqueness of the optimal TH: for Lévy processes X, under the first criterion; for constant volatility processes with general FA jumps, under the second criterion.

The equation characterizing the optimal threshold depends on the observations’ time step h and so does its solution. The optimal TH has to tend to 0 as h tends to zero and, under each criterion, an asymptotic expansion with respect to h is possible for some terms within the equation, which in turn implies an asymptotic expansion of the optimal TH. Under the MSE criterion, when X is Lévy and J has either finite activity jumps or the activity is infinite but J is symmetric strictly stable, the leading term of the expansion is explicit in h, and in both cases is proportional to the modulus of continuity of the Brownian motion paths and to the spot volatility of X, the proportionality constant being $\sqrt{2 - Y}$, where Y is the jump activity index of X. Thus the higher the jump activity is, the lower the optimal threshold has to be if we want to discard the higher noise represented by the jumps and to catch information about IV. The leading term of the optimal TH does not satisfy the classical assumptions under which the truncation method has been shown in [17] to consistently estimate IV, however, at least in the finite activity jumps case, we show herein that the threshold estimator of IV constructed with the optimal TH is still consistent.

The assumptions needed for the asymptotic characterization for the cMSE criterion are less restrictive, and also allow for a drift. We find that, for constant $\sigma$ and general FA jumps, the leading term of the optimal TH still has to be proportional to the modulus of continuity of the Brownian motion paths and to $\sigma$. One of the main motivations for considering the cMSE arises from a novel application of this to tuneup the threshold parameter. The idea consists in iteratively updating the optimal TH and estimates of the increments of the continuous and jump components $X_t^c = \int_0^t \sigma_s dW_s$ and $\{J_t\}_{t \geq 0}$ of X. We illustrate this method on simulated data. Minimization of cMSE in the presence of infinite activity jumps in X is a further topic of ongoing research.

The constant volatility assumption of some of our results is obviously restrictive. It is possible to allow for stochastic volatility and leverage but, since the proofs are still ongoing, we only discuss here some ideas and present some simulations experiments that show that also in such contexts our methods outperform other popular estimators appearing in the literature.

An outline of the paper is as follows. Section 2 deals with the MSE: the existence of an optimal threshold $\varepsilon^*(h)$ is established for a SM X having volatility and jumps independent on the underlying Brownian motion W: for a Lévy process X, uniqueness is also established (Subsection 2.1) and the asymptotic expansion for the optimal TH is found in Section 2.3, in both the cases of a finite jump activity Lévy X and of an infinite activity symmetric strictly stable X. In Section 3, for any finite jump activity SM X, consistency of $\hat{IV}_n$ is verified even when the threshold function consists of the leading term of the optimal threshold, which does not satisfy the classical hypothesis. Section 4 deals with the cMSE in the case where X is a SM with constant volatility and FA jumps: existence of an optimal TH $\bar{\varepsilon}(h)$ is established, its asymptotic expansion is found, then uniqueness is obtained. In Section 5 the results of Section 4 are used to construct a new method for iteratively determine the optimal threshold value in finite samples, and a reliability check is executed on simulated data. Section 6 presents a Monte Carlo
study that shows the superior performance of the new methods over other methods available in the literature under stochastic volatility and leverage. Section 7 concludes and Section 8 contains the proofs of the presented results.

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2 MEAN SQUARE ERROR

We compute and optimize the mean square error (MSE) of \( \hat{V}_n \) passing through the conditional expectation with respect to the paths of \( \sigma \) and \( J \):

\[
MSE := E[(\hat{V}_n - IV)^2] = E \left[ E \left[ (\hat{V}_n - IV)^2 | \sigma, J \right] \right].
\]

Conditioning on \( \sigma \), as well as assuming no drift in \( X \), is standard in papers where MSE-optimality is looked for, in the absence of jumps (see e.g. [4]). We also assume evenly spaced observation over a fixed time horizon \([0, T]\), so that \( t_i = t_{i,n} = ih_n \), for any \( i = 1 \ldots n \), with \( h = h_n = T/n \). Denoted by \( \varepsilon \) the square root \( \sqrt{r(\sigma, h)} \) of a given threshold function, in this work we focus on the performance of the threshold estimator:

\[
\hat{V}_n(\varepsilon) := \sum_{i=1}^{n} (\Delta_i X)^2 I_{\{\Delta_i X \leq \varepsilon\}}.
\]

We indicate the corresponding MSE by \( MSE(\varepsilon) \). Note that for \( \varepsilon \equiv 0 \) we have \( \hat{V}_n = 0 \), so \( MSE(\varepsilon) = E[IV^2] \); as \( \varepsilon \) increases some squared increments \( (\Delta_i X)^2 \) are included within \( \hat{V}_n \), so \( \hat{V}_n \) becomes closer to \( IV \) and \( MSE(\varepsilon) \) decreases. However, if \( J \neq 0 \), for \( \varepsilon \to +\infty \) the quantity \( MSE(\varepsilon) \) increases again, since \( \hat{V}_n \) includes all the squared increments \( (\Delta_i X)^2 \) and thus \( \hat{V}_n \) estimates the global quadratic variation \( IV + \sum_{s \leq T} \Delta X_s^2 \) of \( X \) at time \( T \), and \( MSE(\varepsilon) \) becomes close to \( E[(\sum_{s \leq T} \Delta X_s^2)^2] \). We look for a threshold \( \varepsilon^* \) giving

\[
MSE(\varepsilon^*) = \min_{\varepsilon \in [0, \infty]} MSE(\varepsilon).
\]

In this section we analyze the first derivative \( MSE'(\varepsilon) \) and we find that an optimal threshold exists, in the general framework where \( X \) is a semimartingale satisfying \( A1 \) below, and we furnish an equation to which \( \varepsilon^* \) is a solution, while in Section 2.1, we find that \( \varepsilon^* \) is even unique. The equation has no explicit solution, but \( \varepsilon^* \) is a function of \( h \) and we can explicitly characterize the first order term of its asymptotic expansion in \( h \), for \( h \to 0 \). Clearly we can always find an approximation of the optimal threshold with arbitrary precision making use of numerical methods.

Let us denote

\[
\Delta_i X_s := \Delta_i X I_{\{\Delta_i X \leq \varepsilon^2\}}, \quad s_i^2 := \int_{t_{i-1}}^{t_i} \sigma_s^2 ds, \quad m_i := \Delta_i J.
\]

We assume the following

\( A1 \). A.s. \( \sigma_s^2 > 0 \) for all \( s \); \( J \neq 0 \); and \( \sigma, J \) are independent on \( W \).

The independence condition is needed to guarantee that \( W \) remains a Brownian motion conditionally to \( \sigma \) and \( J \). We analyze the leverage case in our simulation study of Sec. 6. With the next theorem we compute the first derivative \( MSE' \) of the mean square error. The proof is deferred to the Appendix.
Corollary 1. Under the same assumptions of Theorem 1 an optimal threshold exists and is solution of the equation

\[ G(\varepsilon) := \sum_{i=1}^{n} E\left[a_i(\varepsilon)\left(\varepsilon^2 + 2 \sum_{j=1}^{n} b_j(\varepsilon) - 2IV\right)\right], \quad (4) \]

with \(a_i(\varepsilon)\) and \(b_i(\varepsilon)\) defined as

\[ a_i(\varepsilon) := e^{-\frac{(\varepsilon-m_i)^2}{2\sigma_i^2}} + e^{-\frac{(\varepsilon+m_i)^2}{2\sigma_i^2}}, \]

\[ b_i(\varepsilon) := E[(\Delta_i X_*)^2 | \sigma, J] = -\left(e^{-\frac{(\varepsilon-m_i)^2}{2\sigma_i^2}} (\varepsilon + m_i) + e^{-\frac{(\varepsilon+m_i)^2}{2\sigma_i^2}} (\varepsilon - m_i)\right) \frac{\sigma_i}{\sqrt{2\pi}} + \frac{m_i^2 + \sigma_i^2}{\sqrt{2\pi}} \int_{\frac{m_i - \varepsilon}{\sigma_i}}^{\infty} e^{-\frac{x^2}{2}} dx. \]

It clearly follows that \(MSE'(\varepsilon) > 0\) if and only if \(G(\varepsilon) > 0\) and, thus, to our aim of finding an optimal threshold, it suffices to study the sign of \(G(\varepsilon)\) as \(\varepsilon\) varies.

**Notation.** For brevity we sometimes omit to precise the dependence on \(\varepsilon\) of \(a_i(\varepsilon)\) and \(b_i(\varepsilon)\).

For a function \(f(\varepsilon)\) we sometimes use \(f(+)\) for \(\lim_{x \to +\infty} f(\varepsilon)\).

For two functions \(f(x), g(x)\) of a non-negative variable \(x\) which tends to 0 (respectively to \(+\infty\)), by \(f \ll g\), we mean that \(f = o(g)\) as \(x \to 0\) (respectively \(x \to +\infty\)), by \(f \asymp g\) we mean that both \(f = O(g)\) as \(x \to 0\) (respectively \(x \to +\infty\)), while by \(f \sim g\) we mean that \(f(x)/g(x) \to 1\) as \(x \to 0\) (respectively \(x \to +\infty\)).

We denote \(\phi(x) = \frac{e^{-\frac{x^2}{2\sigma^2}}}{\sqrt{2\pi}}\), \(\Phi(x) = \int_{x}^{+\infty} \phi(s)ds\).

\(h.o.t\) means higher order terms.

**Remark 1.** Under A1 and the finiteness of the expectation of the terms below, for fixed \(h\) and \(\varepsilon > 0\), we have that \(MSE'(\varepsilon) = \varepsilon^2 G(\varepsilon)\), where

\[ G(\varepsilon) := \sum_{i=1}^{n} E\left[a_i(\varepsilon)\left(\varepsilon^2 + 2 \sum_{j=1}^{n} b_j(\varepsilon) - 2IV\right)\right], \]

and

\[ G(\varepsilon) = E[(\Delta_i X_* )^2 | \sigma, J] = -\left(e^{-\frac{(\varepsilon-m_i)^2}{2\sigma_i^2}} (\varepsilon + m_i) + e^{-\frac{(\varepsilon+m_i)^2}{2\sigma_i^2}} (\varepsilon - m_i)\right) \frac{\sigma_i}{\sqrt{2\pi}} + \frac{m_i^2 + \sigma_i^2}{\sqrt{2\pi}} \int_{\frac{m_i - \varepsilon}{\sigma_i}}^{\infty} e^{-\frac{x^2}{2}} dx. \]

It clearly follows that \(MSE'(\varepsilon) > 0\) if and only if \(G(\varepsilon) > 0\) and, thus, to our aim of finding an optimal threshold, it suffices to study the sign of \(G(\varepsilon)\) as \(\varepsilon\) varies.

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We denote \(\phi(x) = \frac{e^{-\frac{x^2}{2\sigma^2}}}{\sqrt{2\pi}}\), \(\Phi(x) = \int_{x}^{+\infty} \phi(s)ds\).

\(h.o.t\) means higher order terms.

**Remark 1.** Under A1 and the finiteness of the expectation of the terms in MSE we have

\[ MSE(0) = E[IV^2] > 0 \quad \text{and, for small } h, \lim_{\varepsilon \to +\infty} MSE(\varepsilon) > 0. \]

The next Corollary states the existence of an optimal threshold (see the proof in the Appendix).

**Corollary 1.** Under the same assumptions of Theorem 1 an optimal threshold exists and is solution of the equation \(G(\varepsilon) = 0\).

To find an optimal threshold \(\varepsilon^*\) to estimate \(\sigma\) we need to find the zeroes of \(G\), which in turn depends on \(\sigma\). Also, \(G\) depends on the jump process increments \(m = (m_1, \ldots, m_n)\), which we don’t know. An analogous problem arises when dealing with the minimization of the conditional MSE introduced in Section 4, where the optimal threshold \(\bar{\varepsilon}\) has to satisfy the equation \(F(\bar{\varepsilon}) = 0\), with \(F(\varepsilon) := a_i(\varepsilon)\varepsilon^2 + 2 \sum_{j \neq i} b_j(\varepsilon) - 2IV\). However, when we apply our theory to the case of constant \(\sigma\) and finite activity jumps, as precisely explained in Section 5, we can proceed by estimating \(\sigma, m\) and \(\bar{\varepsilon}\) iteratively. Another method yet to implement \(\varepsilon^*\) is to study the infill asymptotic behavior of \(\varepsilon^*\) in a stationary or deterministic state of \(\sigma\). In some situations, the leading order terms of \(\varepsilon^*\) will only depend on a few summary measures of the stationary distribution or path of \(\sigma\), which could be estimated separately or jointly with \(IV\).

**Remark 2.** In principle \(MSE(\varepsilon)\) could even have many points \(\varepsilon\) where the absolute minimum value \(MSE\) of MSE on \([0, +\infty)\) is reached; also, MSE could have an infinite number of local not absolute minima.

To determine the number of solutions to \(G(\varepsilon) = 0\), we need to study the sign of \(G'(\varepsilon)\) (corresponding to the convexity properties of \(MSE(\varepsilon)\)), but this is not easy. Define

\[ g_i(\varepsilon) := \varepsilon^2 + 2 \sum_{j \neq i} b_j - 2IV, \]

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so that
\[ G(\varepsilon) = \sum_{i} E[a_i(\varepsilon)g_i(\varepsilon)]. \]

We can easily study the functions \( g_i \), since we know that \( g_i(0) = -2IV < 0 \), \( \lim_{\varepsilon \to +\infty} g_i(\varepsilon) = +\infty \) and \( g'_i(\varepsilon) = 2\varepsilon(1 + \varepsilon \sum_{j \neq i} a_j) > 0 \) for all \( \varepsilon > 0 \). However within the joint function \( G(\varepsilon) \) the presence of the terms \( a_i(\varepsilon) \) makes it difficult even to know whether \((a_i g_i)'\) is positive.

### 2.1 When \( X \) is Lévy

Let us assume

**A2.** \( X \) is a Lévy process.

We now have that \( \sigma > 0 \) is constant and \( \Delta_n X \) are i.i.d., so the equation characterizing \( MSE'(\varepsilon) = 0 \) is much simpler to analyze. Indeed, from (4), since within \( a_i \sum_{j \neq i} b_j \), the term \( m_i \) of \( a_i \) is independent on the terms \( m_j \) of \( b_j \), we have

\[
MSE'(\varepsilon) = \varepsilon^2 G(\varepsilon) + \varepsilon^2 nE[a_1(\varepsilon)]\left(\varepsilon^2 + 2(n-1)E[b_1(\varepsilon)] - 2IV\right) + o(\varepsilon).
\]

The next result establishes uniqueness of the optimal threshold under **A2**. The proof is in the Appendix.

**Theorem 2.** If \( X \) is Lévy, equation

\[
\varepsilon^2 + 2(n-1)E[b_1(\varepsilon)] - 2IV = 0
\]

has a unique solution \( \varepsilon^* \) and, thus, there exists a unique optimal threshold, which is \( \varepsilon^* \).

The equation in (5) has no explicit solution, however we can give some important indications to approximate \( \varepsilon^* \).

### 2.2 Asymptotic behavior of \( \mathbb{E}(b_i(\varepsilon)) \)

For the rest of Section 2, in order to emphasize the dependence of \( \varepsilon^* \) on \( h \), we write \( \varepsilon := \varepsilon(h) = \varepsilon_h \). We still are under **A2**, so recall that

\[
\mathbb{E}[b_i(\varepsilon)] = \mathbb{E}\left[|\sigma \Delta_h^i W + \Delta_h^i J| \mathbb{1}_{\{|\sigma \Delta_h^i W + \Delta_h^i J| \leq 1\}}\right],
\]

is constant in \( i \). Note that \( \mathbb{E}[b_i(\varepsilon)] \) is finite for any Lévy process \( J \), regardless of whether \( J \) has bounded first moment or not. We consider two cases: the case where \( J \) is a finite jump activity process and the one where it is a symmetric strictly stable process. The asymptotic characterization of \( \mathbb{E}[b_i(\varepsilon)] \) will be used in Subsection 2.3 to deduce the asymptotic behavior in \( h \) of the optimal threshold \( \varepsilon^* \).

We anticipate that in Subsection 2.3 we will also see that an optimal threshold \( \varepsilon^* \) has to tend to 0 as \( h \to 0 \) and in such a way that \( \frac{\varepsilon^*}{\sqrt{h}} \to +\infty \).

#### 2.2.1 Finite Jump Activity Lévy process

The asymptotic characterization of \( \mathbb{E}[b_i(\varepsilon)] \) in the case where \( J \) has finite activity jumps is given in the following Theorem. Its proof is in the Appendix.

**Theorem 3.** Let \( X \) be a finite jump activity Lévy process with jump size density \( f \) and with jump intensity \( \lambda \). Suppose also that the restrictions of \( f \) on \((0, \infty)\) and \((-\infty, 0)\) admit \( C_1 \) extensions on \([0, \infty)\) and \((\infty, 0] \)

respectively. Then, for any \( \varepsilon = \varepsilon(h) \) such that \( \varepsilon \to 0 \) and \( \varepsilon \gg \sqrt{h} \), as \( h \to 0 \), we have

\[
\mathbb{E}[b_1(\varepsilon)] = \sigma^2 h - \frac{2}{\sqrt{2\pi}} \sigma \varepsilon \sqrt{he^{-\frac{\varepsilon^2}{2\pi h}}} + \lambda h \varepsilon^3 C(f) + O(h^2) + o\left(\varepsilon \sqrt{he^{-\frac{\varepsilon^2}{2\pi h}}} + \varepsilon^3 \right),
\]

where above \( C(f) := f(0^+) + f(0^-) \).
2.2.2 Strictly stable symmetric Lévy Jump process

Let us start by noting that

\[ \mathbb{E}[b_1(\varepsilon)] = \mathbb{E}\left[(\sigma W_h + J_h)^2 \mathbbm{1}_{|\sigma W_h + J_h| \leq \varepsilon}\right] \]

\[ = \sigma^2 \mathbb{E}\left[W_h^2 \mathbbm{1}_{|\sigma W_h + J_h| \leq \varepsilon}\right] + 2\sigma \mathbb{E}\left[W_h J_h \mathbbm{1}_{|\sigma W_h + J_h| \leq \varepsilon}\right] + \mathbb{E}\left[J_h^2 \mathbbm{1}_{|\sigma W_h + J_h| \leq \varepsilon}\right] \]

\[ =: C_h(\varepsilon) + D_h(\varepsilon) + E_h(\varepsilon). \]

The first term above can be written as

\[ C_h(\varepsilon) = \sigma^2 h - \sigma^2 \mathbb{E}\left[W_h^2 \mathbbm{1}_{|\sigma W_h + J_h| > \varepsilon}\right] = \sigma^2 h - \sigma^2 h \left(C_h^+(\varepsilon) + C_h^-(\varepsilon)\right), \]

where

\[ C_h^+(\varepsilon) = \mathbb{E}\left[W_h^2 \mathbbm{1}_{\{W_h > \sigma^{-1} h^{-1/2} J_h \geq \sigma^{-1} h^{-1/2}\}}\right], \quad C_h^-(\varepsilon) = \mathbb{E}\left[W_h^2 \mathbbm{1}_{\{W_h > \sigma^{-1} h^{-1/2} J_h < \sigma^{-1} h^{-1/2}\}}\right]. \]

By conditioning on \( J \) and using the fact that \( \mathbb{E}[W_h^2 \mathbbm{1}_{\{W_h > \varepsilon\}}] = x\phi(x) + \bar{\Phi}(x) \), for all \( x \in \mathbb{R} \), we have

\[ C_h^+(\varepsilon) = \mathbb{E}\left[\left(\frac{\varepsilon}{\sigma \sqrt{h}} \mp \frac{J_h}{\sigma \sqrt{h}}\right)\phi\left(\frac{\varepsilon}{\sigma \sqrt{h}} \mp \frac{J_h}{\sigma \sqrt{h}}\right) \mp \bar{\Phi}\left(\frac{\varepsilon}{\sigma \sqrt{h}} \mp \frac{J_h}{\sigma \sqrt{h}}\right)\right]. \]

The following Lemmas state the asymptotic behavior of the above quantities under the assumption that \( \varepsilon \gg \sqrt{h} \). Their proofs are in the Appendix.

**Lemma 1.** Suppose that \( \{J_t\}_{t \geq 0} \) is a symmetric \( Y \)-stable process with \( Y \in (0, 2) \). Then, there exist constants \( K_1 < 0 \) and \( K_2 \) such that:

\[ \mathbb{E}\left[\phi\left(\frac{\varepsilon}{\sigma \sqrt{h}} \mp \frac{J_h}{\sigma \sqrt{h}}\right)\right] = \frac{1}{\sqrt{2\pi}} e^{-\frac{\varepsilon^2}{2\sigma^2 h}} - K_1 \varepsilon^{-1 - Y} h^{\frac{3}{2}} + \text{h.o.t.} \quad (6) \]

\[ \mathbb{E}\left[J_h \phi\left(\frac{\varepsilon}{\sqrt{h}} \mp \frac{J_h}{\sqrt{h}}\right)\right] = K_2 \varepsilon^{1 - Y} + \text{h.o.t.}. \quad (7) \]

**Lemma 2.** Suppose that \( \{J_t\}_{t \geq 0} \) is a symmetric strictly stable process with Lévy measure \( C|x|^{-Y-1}dx \). Then, the following asymptotics hold:

\[ \mathbb{E}\left[\bar{\Phi}\left(\frac{\varepsilon}{\sigma \sqrt{h}} \mp \frac{J_h}{\sigma \sqrt{h}}\right)\right] = \frac{C}{Y} h^{-\frac{2}{Y}} + O\left(\varepsilon^{-2 Y h^2}\right) + O\left(\mathbb{E}\left[\phi\left(\frac{\varepsilon}{\sigma \sqrt{h}} \mp \frac{J_h}{\sigma \sqrt{h}}\right)\right]\right), \quad (8) \]

\[ \mathbb{E}\left[J_h^2 \mathbbm{1}_{|\sigma W_h + J_h| \leq \varepsilon}\right] = \frac{2C}{2 - Y} h^{\frac{2}{Y} - 2 Y + O\left(h^{4 Y - 2} + O\left(h^{4 Y - 2}\right) + O\left(h^{4 Y - 2}\right). \quad (9) \]

As a consequence, the following Theorem states explicitly the asymptotic behavior of \( \mathbb{E}[b_1(\varepsilon)] \). It’s proof is in the Appendix.

**Theorem 4.** Let \( X_t = \sigma W_t + J_t \), where \( W \) is a Wiener process and \( J \) is a symmetric strictly stable Lévy process with Lévy measure \( C|x|^{-Y-1}dx \). Then, for any \( \varepsilon = \varepsilon(h) \) such that \( \varepsilon \to 0 \) and \( \varepsilon \gg \sqrt{h} \), as \( h \to 0 \), we have

\[ \mathbb{E}[b_1(\varepsilon)] = \sigma^2 h - \frac{2\sigma}{\sqrt{2\pi}} \sqrt{h} e^{-\frac{\varepsilon^2}{2\sigma^2 h}} + \frac{2C}{2 - Y} h^{\frac{2}{Y} - 2 Y} + \text{h.o.t.}. \]

2.3 Asymptotic behavior of \( \varepsilon^* \)

We now assume

\[ \text{A3. The support of any jump size } \Delta J_t \text{ is } \mathbb{R}. \]

We firstly see that an optimal threshold \( \varepsilon^* = \varepsilon^*(h) \) has to tend to 0 as \( h \to 0 \) and in such a way that \( \frac{\varepsilon^*}{\sqrt{h}} \to +\infty \). Then we will show the asymptotic behavior of \( \varepsilon^* \) in more detail.
Remark 3. Note that under A3, if \( \varepsilon^*(h) \) minimizes MSE, then necessarily \( \varepsilon^*(h) \to 0 \) as \( h \to 0 \). Indeed, if \( \liminf_{h \to 0} \varepsilon^*(h) = c > 0 \), then on a sequence \( \varepsilon^*(h) \) converging to \( c \) we would have \( \hat{IV}_n - IV \to \sum_{t \leq T} \Delta J^2 \varepsilon I_j^j \leq c \) in probability, rather than \( \hat{IV}_n - IV \to 0 \); since \( P \{ \sum_{t \leq T} \Delta J^2 \varepsilon I_j^j \leq c \} > 0 \), the MSE could not be minimized.

**Lemma 3.** Suppose \( X_t = \sigma W_t + J_t \), where \( W \) is a Brownian motion and \( J \) is a pure-jump Lévy process of bounded variation or, more generally, such that, for some \( Y \in (0, 2) \), \( h^{-1/2} J^2 \varepsilon \to J \), for a real-valued random variable \( J \). Then, \( \varepsilon^*_n / \sqrt{\bar{n}} \to \infty \), as \( n \to \infty \).

**Remark.** If \( J \) has FA jumps, drift \( d \) and \( J_t = dt + \sum_{k=1}^{N_t} \gamma_k \), then we have \( h^{-1} J_h \varepsilon \to d \) and, thus, the assumption in Lemma 3 is satisfied with \( Y = 1 \). If \( J \) is a Lévy process with Blumenthal-Getoor index \( Y \), then \( Y \in (0, 2) \) and for any \( \eta \in (Y, 2) \) we have \( h^{-1/2} J^2 \varepsilon \to 0 \), and again the assumption is satisfied.

We are now ready to show more precisely the asymptotic behavior of \( \varepsilon^* \). Proposition 1 covers the FA jumps case, while Proposition 2 tackles the case of symmetric strictly stable jumps. Their proofs are deferred to the Appendix.

**Proposition 1.** Let \( J \) have FA jumps and satisfy the assumptions of Theorem 3, let \( \varepsilon^* = \varepsilon^*(h) \) be the optimal threshold. Then,

\[
\varepsilon^* \sim \sqrt{2\sigma^2 h \ln \frac{1}{h}}, \quad \text{as} \quad h \to 0.
\]

**Proposition 2.** Under the conditions of Theorem 4, the optimal threshold \( \varepsilon^* = \varepsilon^*(h) \) is such that

\[
\varepsilon^* \sim \sqrt{(2 - Y)\sigma^2 h \ln \frac{1}{h}}, \quad \text{as} \quad h \to 0.
\]

As explained in the introduction, the proportionality constant \( \sqrt{2 - Y} \) of the previous result says that the higher the jump activity is, the lower the optimal threshold has to be if we want to discard the higher noise represented by the jumps and to catch information about \( IV \).

### 3 CONSISTENCY WHEN \( \varepsilon_h = \sqrt{2Mh \log \frac{1}{h}} \)

Under the framework described in [17], in the case of equally spaced observations, the threshold criterion allows convergence of

\[
\hat{IV}_n := \sum_{i=1}^{n} (\Delta_i X)^2 I_{(\Delta_i X)^2 \leq r(\sigma_{t_i - 1}, h)}
\]

to

\[
IV_T := \int_0^T \sigma^2 ds
\]

when, for all \( i = 1, \ldots, n \), we have \( r(\sigma_{t_i - 1}, h) = r(h) \) and \( r(h) \) is a deterministic function of \( h \) s.t. \( r(h) \to 0 \), \( \frac{r(h)}{\text{log } \pi} \to \infty \), as \( h \to 0 \). Here we show that, under finite activity jumps, the same estimator is also consistent in the case where on any \( [t_i - 1, t_i] \) we consider a different truncation level \( r_i(\sigma, h) = 2M_i h \log \frac{1}{h} \), with suitably chosen random variables \( M_i \). Concretely, assume the following

**A4.** Let

\[
dX_t = a_t dt + \sigma_t dW_t + dJ_t,
\]

where \( J_t = \sum_{i=1}^{N(t)} \gamma_i \) for a non-explosive counting process \( N \) and real-valued random variables \( \gamma_j, a, \sigma \) are càdlàg and a.s. \( \sigma^2 := \inf_{s \in [0, T]} \sigma_s^2 > 0 \).

Recall that a.s. the paths of \( a \) and of \( \sigma \) are bounded on \([0, T]\). Define \( \sigma^2 := \sup_{s \in [0, T]} \sigma_s^2 \), then, the following Proposition and Corollary hold true. Their proofs are in the Appendix.

**Proposition 3.** Under A4, if we choose \( r_i(h) = 2 M_i h \log \frac{1}{h} \), with any \( M_i(\omega) \) such that \( M_i(\omega) \in [\inf_{s \in [t_i - 1, t_i]} \sigma_s^2(\omega), \bar{\sigma}] \), we have:

\[
\text{a.s.} \forall \eta > 0, \text{ for sufficiently small } h: \forall i = 1, \ldots, n, \quad I_{(\Delta_i X)^2 \leq (1+\eta)r_i(h)} \sim IV_T.
\]

**Corollary 2.** For all \( \eta > 0 \), we have \( \sum_{i=1}^{n} (\Delta_i X)^2 I_{(\Delta_i X)^2 \leq (1+\eta)r_i(h)} \to IV_T \), as \( h \to 0 \).
4 CONDITIONAL MEAN SQUARE ERROR: FA jumps case

We now put ourselves under A1. The quantity of our interest here, $cMSE(\varepsilon) \doteq E[(\hat{J} - J)^2]\sigma, J$, is such that $\forall \omega, cMSE(0) = IV^2$ and $cMSE(+\infty) > 0$, because $IV \rightarrow +\infty$ for variable. Further, from the proof of Theorem 1, we have

$$cMSE'(\varepsilon) = \varepsilon^2 F(\varepsilon),$$

with $F(\varepsilon) \doteq \sum_{i=1}^n a_i g_i, \quad g_i = \varepsilon^2 + 2 \sum_{j \neq i} b_j - 2IV.$ \hspace{1cm} (11)

We analyze the sign of $F(\varepsilon)$: for $n, h$ fixed, $\sigma_i^2$ and $m_i$ also are fixed, and we have $F(0) = -2IV \sum_{i=1}^n a_i < 0$, since $b_j(0) = 0$. Further we have $F(+\infty) = 0^+$: to see this, first note that, from the expression of $b_i(\varepsilon), b_i(+\infty) = m_i^2 + \sigma_i^2$, then $g_i(\varepsilon) \sim \varepsilon^2 + 2 \sum_{j \neq i} m_j^2 - 2 \sigma_i^2 \sim \varepsilon^2$, as $\varepsilon \rightarrow +\infty$. Moreover, each $a_i \sim 2(2\pi)^{-1/2}\sigma_i^{-1} \exp\left(-\frac{\varepsilon^2}{2\sigma_i^2}\right)$, thus, for sufficiently large $\varepsilon$, $F = \sum_{i=1}^n a_i g_i$ is a finite sum of $n$ positive terms $a_i g_i \leq K(2\pi)^{-1/2}\sigma_i^{-1} \varepsilon^2 \exp\left(-\frac{\varepsilon^2}{2\sigma_i^2}\right)$ for some constant $K$ and fixed $\sigma_i$, so $F(\varepsilon) \rightarrow 0^+$, as $\varepsilon \rightarrow +\infty$. Since $F$ is continuous, it follows that an optimal threshold exists and solves $F(\varepsilon) = 0$.

We now assume also A3.

**Remark 4.** Under A3, as in Remark 3, if $\tilde{\varepsilon} = \tilde{\varepsilon}(h)$ minimizes $cMSE$, then it has to be true that $\tilde{\varepsilon} \rightarrow 0$, as $h \rightarrow 0$.

In Proposition 4 below we again also find that under the following A4' then necessarily $\frac{\tilde{\varepsilon}(h)}{\sqrt{n}} \rightarrow +\infty$.

**A4’.** We assume A4 with $a \equiv 0$, constant $\sigma > 0$ and $nh = 1$.

Under FA jumps, when considering $h \rightarrow 0$, we assume to have a sufficiently small $h$ so that a.s. the number of jumps occurring during $[t_{i-1}, t_i]$ is at most 1; note that for any $t$ we have $m_i \rightarrow \Delta J_i$, when selecting $i = i(t)$ such that $t_{i-1} < t \leq t_i$. Thus, when considering a jump time $t$, we assume that $h$ is sufficiently small so that the sign of $m_{i(t(i)}}$ is the same as the one of $\Delta J_i$, in particular if $\Delta J_i \neq 0$ then the increments $m_i$ approaching it are non-zero.

4.1 Asymptotic behavior of $b_i(\varepsilon), a_i(\varepsilon),$ and $F$

The following result ensures that, as previously announced, an optimal threshold has to tend to 0, as $h \rightarrow 0$, but at a slower rate than $\sqrt{n}$. Its proof is in the Appendix.

**Proposition 4.** Under A1, A3, A4’, if $\tilde{\varepsilon} = \tilde{\varepsilon}(h)$ solves $F(\varepsilon) = 0$ and $\tilde{\varepsilon} = \tilde{\varepsilon}(h) \rightarrow 0$, then $\frac{\tilde{\varepsilon}(h)}{\sqrt{n}} \rightarrow +\infty$.

We now pass to consider the asymptotic behavior of $F(\varepsilon)$ for sequences $\varepsilon = \varepsilon(h) = \varepsilon_h$ satisfying the conditions of Proposition 4.

**Proposition 5.** Under A4’, if $\varepsilon_h \rightarrow 0$ as $h \rightarrow 0$ in such a way that $\frac{\varepsilon(h)}{\sqrt{n}} \rightarrow +\infty$ then $F(\varepsilon_h) = F_0(\varepsilon_h) + \text{h.o.t.}$, where

$$F_0(\varepsilon_h) \doteq \frac{2\varepsilon_h e^{-\frac{\varepsilon_h^2}{2\sigma^2}}}{\sqrt{\hat{h}}} \left(\varepsilon_h - \frac{\varepsilon_h^2}{\sqrt{\hat{h}}} \frac{4\sigma}{\sqrt{2\pi}} \frac{1}{\sigma \sqrt{2\pi}}\right).$$

With the notation $v_h := \frac{\varepsilon_h}{\sqrt{\hat{h}}}$ and $s_h := \frac{1}{\sqrt{2\pi}} e^{-\frac{s_h^2}{2}}$, we can write $F_0(\varepsilon(h)) = \frac{2}{\sigma} e_{\frac{v_h}{\sigma}} \left(v_h - 4s_h \right)$. Note that $v_h \ll n$, but $s_h \rightarrow 0$, so which is the leading term between $v_h$ and $ns_h$ depends on the choice of $v_h$. We also remark that a solution $\tilde{\varepsilon}$ of $F = 0$ not necessarily is such that $F_0(\tilde{\varepsilon}) = 0$, however if a sequence $\varepsilon_h$ is such that $F_0(\varepsilon_h) \rightarrow 0$ then the whole $F(\varepsilon_h) \rightarrow 0$, so it has to be true that $\varepsilon_h$ is close (in a way that will become explicit later) to one of the solutions $\tilde{\varepsilon}$ of $F = 0$.

**Remark 5.** The asymptotic behavior of $F(\varepsilon)$ stated in Proposition 5 also holds under the presence of a nonzero drift process $\{a_t\}_{t \geq 0}$ that has almost surely locally bounded paths (recall that any càdlàg process $a$ satisfies such a requirement) and that is independent on $W$. This is shown in the Appendix.
4.2 Asymptotic behavior of \( \bar{\varepsilon} \)

We show here that any cMSE optimal threshold \( \bar{\varepsilon} \) has the same asymptotic behavior as the MSE optimal threshold \( \varepsilon^* \). The proof of the following result is given in the Appendix.

Corollary 3. Under A1, A3, A4' we have that

\[
\bar{\varepsilon} \sim \sqrt{2\sigma^2 h \ln \frac{1}{h}}, \quad \text{as} \quad h \to 0.
\]

The previous result suggests an approximation for the optimal \( \bar{\varepsilon} := \bar{\varepsilon}_h \) of the form \( \varepsilon_h = \sigma w_h \sqrt{2h} \), with \( w_h = \sqrt{\ln(1/h)} \). It is natural to wonder about other choices for \( w_h \). Intuitively, we should aim at making \( F(\varepsilon_h) \) to converge to 0 as quickly as possible: in view of (49) within the proof of Corollary 3, the only possible way is rendering \( v_h \) and \( ns_h \) within \( F_0(\varepsilon) \) of the same order, so we choose \( w_h \) such that

\[
\begin{align*}
1) \ w_h & \to +\infty, \\
2) \ w_h \sqrt{h} & \to 0, \\
3) \ \frac{\varepsilon - \bar{\varepsilon}_h}{w_h} & \to \frac{\sqrt{\varepsilon_0}}{2},
\end{align*}
\]

as \( h \to 0 \). For example a function of type \( w_h = \sqrt{\ln \frac{1}{h} - \frac{1}{2} \ln \ln \frac{1}{h} - \ln y_h} \), with any continuous function \( y_h \) tending to \( \sqrt{\varepsilon_0} \) as \( h \to 0 \), satisfies the three above conditions. However the quickest convergence speed of \( F \) to 0 would be reached by choosing a function \( w_h \), which satisfies the following three more restrictive conditions, as \( h \to 0 \),

\[
\begin{align*}
1) \ w_h & \to +\infty \\
2) \ w_h \sqrt{h} & \to 0 \\
3') \ \frac{\varepsilon - \bar{\varepsilon}_h}{w_h} & \equiv \frac{\sqrt{\varepsilon_0}}{2},
\end{align*}
\]

where condition 3') means that \( F_0(\varepsilon_h) \equiv 0 \). In fact such a \( w_h \) exists, since the following holds true.

Theorem 5. There exists a unique deterministic function \( w_h : (0, 1] \to (0, +\infty) \) such that the three conditions 1), 2) and 3') above are satisfied. Such a \( w_h \) turns out to be differentiable and to satisfy also the ODE \( w_h' = \frac{w_h h}{1 + 2w_h} \), which entails that \( w_h \leq w_1 + \frac{1}{2\sqrt{2}} \log \frac{1}{h} \).

We finally reach the uniqueness of the optimal threshold \( \bar{\varepsilon} \) as a consequence of the following result, whose proof is in the Appendix. We remark that the asymptotic behavior of \( \bar{\varepsilon} \) described in Corollary 3 is obtained after having proved just before (40) that it has to satisfy \( \bar{\varepsilon}_h \sim 4\sigma \frac{w_h}{\sqrt{h}} \) as \( h \to 0 \).

Proposition 6. The first derivative \( \frac{d}{d\varepsilon} F(\varepsilon) \) of \( F \) is such that, when evaluated at a function \( \varepsilon_h \) of \( h \) satisfying \( \varepsilon_h \to 0, \frac{\varepsilon_h}{\sqrt{h}} \to +\infty \), and \( \varepsilon_h = 4\sigma \frac{w_h}{\sqrt{h}} + h.o.t., \) then, as \( h \to 0 \),

\[
F'(\varepsilon_h) = F_1(\varepsilon_h) + h.o.t., \quad \text{as} \quad h \to 0, \quad \text{where} \quad F_1(\varepsilon_h) = \frac{4}{\sigma^2} e^{-\frac{\varepsilon_h^2}{\sigma^2}} \frac{\varepsilon_h^2}{h^3}.
\]

Remark 6. Uniqueness of \( \bar{\varepsilon} \). Since \( F_1(\varepsilon_h) > 0 \) for any \( \varepsilon_h \), we reach that for sufficiently small \( h \) we have \( \frac{d}{d\varepsilon} F(\varepsilon_h) > 0 \) on any sequence \( \varepsilon_h \) as in the above Proposition. That entails that for any sufficiently small \( h \) the cMSE optimal \( \bar{\varepsilon} \) is unique. Indeed, if there existed two optimal \( \bar{\varepsilon}_h^{(1)} < \bar{\varepsilon}_h^{(2)} \), we would necessarily have that \( \bar{\varepsilon}_h^{(i)} \to 0, \frac{\bar{\varepsilon}_h^{(i)}}{\sqrt{h}} \to +\infty \), and \( \bar{\varepsilon}_h^{(i)} = 4\sigma \frac{w_h}{\sqrt{h}} + h.o.t., \) but then, for small \( h \), on such sequences \( F'(\bar{\varepsilon}_h^{(i)}) > 0 \), which is a contradiction, because in order to be optimal both sequences have to satisfy \( F(\bar{\varepsilon}_h^{(i)}) = 0 \).

1We thank Andrey Sarychev for having provided such nice examples.

2We thank Salvatore Federico for having provided such a nice result. The proof is available upon request.
Remark 7. The fact that the asymptotic behavior of the cMSE optimal threshold $\tilde{\varepsilon} = \hat{\varepsilon}(h)$ is the same as the one of the MSE optimal threshold $\varepsilon^*$ under FA jumps is due to the fact that $\tilde{\varepsilon}$ solves $F = 0$, $\varepsilon^*$ solves $G = 0$, $F = F_0 + h.o.t.$, $G = G_0 + h.o.t.$, and the leading terms in $F$ are the ones with $m_i = 0$, which do not depend on $\omega$, thus they are the same as for $G$. It follows that, in the case of Lévy FA jumps, we have $F = F_0 + h.o.t. = E[F_0] + h.o.t. = G + h.o.t.$.

Also, an alternative heuristic justification is that we expect that $F(\varepsilon) = \sum_{i=1}^{n} \frac{a_i g_i}{n} - n \sim n E[a_i g_i]$, thus the asymptotic behavior of the $\varepsilon^*$ satisfying $G = n E[a_i g_i] = 0$ is the same as any $\varepsilon$ satisfying $F(\varepsilon) = 0$.

Remark 8. Comparison with the results in [10]. In [10], a process $X$ with FA jumps is considered, either of Lévy type, with jumps sizes having distribution density satisfying given conditions, or of Itô SM type, with deterministic absolutely continuous local characteristics (additive process). The estimators

$$\hat{J}_n = \sum_{i=1}^{n} \Delta_i X I(|\Delta_i X| > \varepsilon_h), \quad \hat{N}_n = \sum_{i=1}^{n} I(|\Delta_i X| > \varepsilon_h)$$

are considered, and, as $h \to 0$, firstly it is shown that the condition $\frac{\hat{\varepsilon}_h}{\sqrt{h}} \to +\infty$ is necessary and sufficient for the convergence to 0 of both $MSE(\hat{J}_n - J_T)$ (stronger condition implying consistency of $\hat{J}_n$) and $MSE(\hat{N}_n - N_T)$. Secondly, the authors show that

$$MSE(\hat{N}_n - N_T) \to 0 \iff e^{-\frac{\varepsilon^2}{2\sigma^2 h}} \to 0,$$

meaning that in order to have $L^2(\Omega, P)$ convergence to 0 of the estimation error $\hat{N}_n - N_T$ a stronger condition on $\varepsilon_h$ is needed, implying $\frac{\hat{\varepsilon}_h}{\sqrt{h}} \to \infty$. Thirdly, existence and uniqueness of an optimal threshold $\tilde{\varepsilon}(h)$ minimizing

$$E[|\hat{J}_n - J_T|^2 + |\hat{N}_n - N_T|^2]$$

for fixed $h$ is obtained, and the asymptotic expansion in $h$ of $\tilde{\varepsilon}(h)$ has leading term $\sqrt{3\sigma^2 h \log \frac{1}{h}}$. The factor 3 is higher than the factor 2 of the leading terms of $\tilde{\varepsilon}$ and $\varepsilon^*$: that is due to the fact that the minimization criterion for $\tilde{\varepsilon}(h)$ includes also the error on $N_T$, which requires that $\frac{\tilde{\varepsilon}(h)}{\sqrt{h}}$ is higher than $\frac{\varepsilon^*}{\sqrt{h}}$, and thus $\tilde{\varepsilon}(h) > \varepsilon(h)$ is necessary.

5 A NEW METHOD

In this section, we propose a new method for tuning the threshold parameter $\varepsilon := \sqrt{r(\sigma, h)}$ of the TRV introduced in (2). This is based on the conditional mean square error $cMSE(\varepsilon) = E[(\hat{J} - J)^2 | \sigma, J]$ studied in Section 4. We illustrate the method for a driftless FA process with constant volatility $\sigma$. As proved therein, the optimal threshold $\tilde{\varepsilon}$ is such that

$$F(\varepsilon) = \sum_{i=1}^{n} a_i(\varepsilon) g_i(\varepsilon) = 0, \quad g_i(\varepsilon) = \varepsilon^2 + 2 \sum_{j \neq i} b_j(\varepsilon) - 2 n h \sigma^2,$$

where $a_i(\varepsilon)$ and $b_i(\varepsilon)$ are rewritten here for easy reference:

$$a_i(\varepsilon) := a(\varepsilon, m_i, \sigma) := \frac{e^{-\frac{(\varepsilon - m_i)^2}{2 \sigma^2 h}} + e^{-\frac{(\varepsilon + m_i)^2}{2 \sigma^2 h}}}{\sigma \sqrt{2\pi h}},$$

$$b_i(\varepsilon) := b(\varepsilon, m_i, \sigma) := -\frac{\sqrt{h}}{\sqrt{2 \pi}} e^{-\frac{(\varepsilon - m_i)^2}{2 \sigma^2 h}} (\varepsilon + m_i) + e^{-\frac{(\varepsilon + m_i)^2}{2 \sigma^2 h}} (\varepsilon - m_i) + \frac{m_i^2 + \sigma^2 h}{\sqrt{2 \pi}} \int_{\frac{m_i - \varepsilon}{\sigma \sqrt{h}}}^{\frac{m_i + \varepsilon}{\sigma \sqrt{h}}} e^{-x^2/2} dx.$$

It is convenient to set $\mathbf{m} = (m_1, \ldots, m_n)$ and

$$F(\varepsilon; \sigma, \mathbf{m}) := \sum_{i=1}^{n} a(\varepsilon, m_i, \sigma) \left( \varepsilon^2 + 2 \sum_{j \neq i} b(\varepsilon, m_j, \sigma) - 2 n h \sigma^2 \right).$$

The main issue with the optimal threshold $\tilde{\varepsilon}$ lies on the fact that this depends on $\sigma$ and the increments $\mathbf{m} = (m_1, \ldots, m_n)$ of the jump process, which we don’t know. Note also that, for $h$ small enough, each $m_i$ will be either
0 or one of the jumps of the process and a good proxy of \( m \) is actually \((\Delta_n^n X) 1_{|\Delta_n^n X| > \varepsilon}\). The idea is then to iteratively estimating \( \bar{\varepsilon}, \sigma, \) and \( m \) as follows:

1. Start with some initial ‘guesses’ of \( \sigma \) and \( m \), which we call \( \hat{\sigma}_0 \) and \( \hat{m}_0 \). There are different possibilities for these initial values, for instance \( \hat{\sigma}_{RV} \) (defined in item 1 of Section 5.1) or \( \hat{\sigma}_{BV} \) (defined in item 2 of Section 5.1) or a truncated \( \hat{\sigma}_{TRV} \) (defined in item 12 of Section 5.1) with threshold \( \sqrt{2\hat{\sigma}_{TRV}^2 \log(1/\bar{h})} \), for \( \sigma \), and \( \hat{m}_0 = (0, \ldots, 0) \) (no jumps) for \( m \).

2. Using \( \hat{\sigma}_0 \) and \( \hat{m}_0 \), by solving \( F(\varepsilon; \hat{\sigma}_0, \hat{m}_0) = 0 \), we find an initial estimate for the optimum \( \bar{\varepsilon} \) that we denote \( \varepsilon_{NEW} \). For instance with the choice of \( \hat{m}_0 = (0, \ldots, 0) \), \( \varepsilon_{NEW} \) solves the equation:

\[
\varepsilon^2 + 2(n - 1) \left( -\frac{2\hat{\sigma}_0 \sqrt{\bar{h}}}{\sqrt{2\pi}} \varepsilon - \frac{\varepsilon^2}{\hat{\sigma}_0^2 \sqrt{\bar{h}} h} + \frac{\hat{\sigma}_0^2 h}{\sqrt{2\pi}} \int_{-\varepsilon}^{\varepsilon} e^{-x^2/2} dx \right) - 2n \bar{h} \hat{\sigma}_0^2 = 0. \tag{14}
\]

It is easy to see that, in that case, \( \varepsilon_{NEW} \) is of the form \( v_n \hat{\sigma}_0 \sqrt{\bar{h}} \), where \( v_n \) is the unique solution of the equation:

\[
v_n^2 + 4(n - 1) \left( -v_n \frac{1}{\sqrt{2\pi}} e^{-v_n^2} + \frac{1}{\sqrt{2\pi}} \int_{-v_n}^{v_n} e^{-x^2/2} dx \right) - 2n = 0. \tag{15}
\]

Figure 1 shows that \( v_n \) ranges from about 3 to 4 when \( n \) ranges from 100 to 10000.

![Figure 1: The solution \( v_n \) of equation (15) as a function of \( n \).](image)

3. Once we have an initial estimate of \( \bar{\varepsilon} \), we can update our estimates of \( \sigma \) and \( m \) using:

\[
\hat{\sigma}_{NEW}^2 := \frac{1}{T} \sum_{i=1}^{n} (\Delta_i X)^2 1_{|\Delta_i X| \leq \varepsilon_{NEW}}, \quad \hat{m}_1 := ((\Delta_1^n X) 1_{|\Delta_1^n X| \leq \varepsilon_{NEW}}, \ldots, (\Delta_n^n X) 1_{|\Delta_n^n X| \leq \varepsilon_{NEW}}) \tag{16}
\]

4. We continue this procedure iteratively: \( \varepsilon_{NEW,0} := \varepsilon_{NEW} \), \( \hat{\sigma}_{NEW,1} := \hat{\sigma}_{NEW} \), and for \( k \geq 1 \)

\[
\begin{align*}
\text{Find} & \quad \varepsilon_{NEW,k} \quad \text{s.t.} \quad F(\varepsilon_{NEW,k}; \hat{\sigma}_{NEW,k}, \hat{m}_k) = 0, \tag{17} \\
\text{set} & \quad \hat{\sigma}_{NEW,k+1}^2 := \frac{1}{T} \sum_{i=1}^{n} (\Delta_i X)^2 1_{|\Delta_i X| \leq \varepsilon_{NEW,k}}, \tag{18} \\
\hat{m}_{k+1} := & \quad ((\Delta_1^n X) 1_{|\Delta_1^n X| \leq \varepsilon_{NEW,k}}, \ldots, (\Delta_n^n X) 1_{|\Delta_n^n X| \leq \varepsilon_{NEW,k}})). \tag{19}
\end{align*}
\]

The algorithm is stopped when the sequence of estimates \( \hat{\sigma}_{NEW,k} \) stabilizes (e.g., when \( |\hat{\sigma}_{NEW,k+1} - \hat{\sigma}_{NEW,k}| / \hat{\sigma}_{NEW,k} \leq \text{tol} \), for some desired small tolerance \text{tol}).
The previous procedure resembles the one introduced in [10], which is based on choosing the threshold \( \varepsilon \) so to minimize the expected number of jumps miss-classifications:

\[
\text{Loss}(\varepsilon) := E \left[ \sum_{i=1}^{n} \left( 1_{\{\Delta_i \gamma > \varepsilon, \Delta_i \gamma < 0\}} + 1_{\{\Delta_i \gamma \leq \varepsilon, \Delta_i \gamma > 0\}} \right) \right].
\]  

(20)

It was proved therein that, for a Lévy process with FA jumps, the optimal threshold, hereafter denoted \( \varepsilon_{3mc} \), is asymptotically equivalent to \( \sqrt{3\sigma^2 \ln(1/h)} \), as \( h \to 0 \). Using this information, an iterative method was proposed, in which, given an initial estimate \( \hat{\sigma}_{3mc,0} \) of \( \sigma \), we set, for \( k \geq 1 \),

\[
\varepsilon_{3mc,k-1} := \sqrt{3\hat{\sigma}_{3mc,k-1}^2 h \ln \frac{1}{h}}, \quad \hat{\sigma}_{3mc,k}^2 := \frac{1}{T} \sum_{i=1}^{n} (\Delta_i X)^2 1_{\{\Delta_i X \leq \varepsilon_{3mc,k-1}\}}.
\]  

(21)

Since, as proved in Section 4, the optimal threshold \( \bar{\varepsilon} \) that minimizes cMSE for given \( h \) has the asymptotic behavior \( \sqrt{2\sigma^2 \ln(1/h)} \), as \( h \to 0 \), it is natural to consider the following iterative method to estimate \( \varepsilon \): given an initial guess \( \hat{\sigma}_{2mc,0} \) for \( \sigma \), we set

\[
\varepsilon_{2mc,k-1} := \sqrt{2\hat{\sigma}_{2mc,k-1}^2 h \ln \frac{1}{h}}, \quad \hat{\sigma}_{2mc,k}^2 := \frac{1}{T} \sum_{i=1}^{n} (\Delta_i X)^2 1_{\{\Delta_i X \leq \varepsilon_{2mc,k-1}\}}, \quad k \geq 1.
\]  

(22)

We can go one step further and consider, as suggested below Corollary 3, a threshold of the form \( \varepsilon_h = \sigma w_h \sqrt{2h} \), with \( w_h \) given as in (13)\(^3\). This leads us to consider the iterative method:

\[
\varepsilon_{mc2,k-1} := w_h \sqrt{2\hat{\sigma}_{mc2,k-1}^2 h \ln \frac{1}{h}}, \quad \hat{\sigma}_{mc2,k}^2 := \frac{1}{T} \sum_{i=1}^{n} (\Delta_i X)^2 1_{\{\Delta_i X \leq \varepsilon_{mc2,k-1}\}}, \quad k \geq 1.
\]  

(23)

It can be proved that if we take \( \hat{\sigma}_{3mc,0}, \hat{\sigma}_{2mc,0}, \) and \( \hat{\sigma}_{mc2,0} \) equal \( \hat{\sigma}_{RV} \) in (21), (22), and (23), then the obtained sequences of estimates \( \{\hat{\sigma}_{2mc,k}\}_{k \geq 0}, \{\hat{\sigma}_{3mc,k}\}_{k \geq 0}, \{\hat{\sigma}_{mc2,k}\}_{k \geq 0} \) are nonincreasing and, thus, eventually they reach a constant limiting value. So, for these two estimators we can (and will) set the tolerance tol to 0. Even though asymptotically \( w_h \sim \sqrt{\ln(1/h)} \), there are some differences in finite samples. For instance, for the span of 5 minutes used in our simulations (\( h = \frac{1}{252 \times 6.5 \times 12} \)), we have \( w_h = 2.98 \), while \( \sqrt{\ln(1/h)} = 3.14 \), which means that the \( \varepsilon_{mc2,k} \) will be smaller than \( \varepsilon_{2mc,k} \).

### 5.1 Simulation performance: finite activity jumps and constant volatility

We now proceed to assess the methods introduced in this paper and compare them against other popular alternatives. We take a Merton’s log-normal model of the form:

\[
X_t = \sigma W_t + \sum_{j=1}^{N_t} \gamma_j,
\]  

(24)

where \( N \) is a Poisson process with intensity \( \lambda \) and \( \{\gamma_j\}_{j \geq 1} \) is an independent sequence of independent normally distributed random variables with mean and standard deviation \( \mu^{Imp} \) and \( \sigma^{Imp} \), respectively. We consider the following estimators:

1. The Realized quadratic Variation estimator: \( \hat{\sigma}_{RV}^2 := T^{-1} \sum_{i=1}^{n} (\Delta_i X)^2 \);

2. The realized Bipower Variation (BV) estimator of [5]:

\[
\hat{\sigma}_{BV}^2 := \frac{\pi}{27} \sum_{i=1}^{n-1} |\Delta_i X| |\Delta_{i+1} X|;
\]  

In order to obtain \( w_h \), we change variable, as \( x_h = w_h^2 \), in (3)\(^3\) and then we use a fixed-point algorithm to find the solution \( x_h \), starting with \( x_h(0) = -\log(h) \). The algorithm converges very quickly.
3. The MinRV estimator of [3]:
\[
\hat{\sigma}_{\text{MinRV}}^2 := \frac{\pi}{T(\pi - 2)} \frac{n}{n-1} \sum_{i=1}^{n-1} \min\{|\Delta_i X|, |\Delta_{i+1} X|\}^2 ;
\]

4. The MedRV estimator of [3]:
\[
\hat{\sigma}_{\text{MedRV}}^2 := \frac{\pi}{T(\pi + 6 - 4\sqrt{3})} \frac{n}{n-2} \sum_{i=2}^{n-1} \text{median}\{\{|\Delta_{i-1} X|, |\Delta_i X|, |\Delta_{i+1} X|\}\}^2 ;
\]

5. The TRV given in (3) using a threshold of the form \( \varepsilon = 4h^2\sigma_{BV} \) with \( \omega = 0.49 \). This was used in the recent work of Jacod and Todorov [14] and is denoted \( \hat{\sigma}_{TRV,T}^2 \):

6. The estimator \( \hat{\sigma}_{3mc}^2 \) as in (21) with \( k = 1 \), using the initial threshold \( \varepsilon_{3mc,0} = \sqrt{3\hat{\sigma}_{RV}^2 h \log(1/h)} \):

7. The estimator \( \hat{\sigma}_{3mc,k}^2 \) defined by (21) with \( k \geq 1 \) such that \( \hat{\sigma}_{3mc,\ell} = \hat{\sigma}_{3mc,k-1} \), for all \( \ell \geq k \):

8. The estimator \( \hat{\sigma}_{2mc}^2 \) as in (22) with \( k = 1 \), using the initial threshold \( \varepsilon_{2mc,0} = \sqrt{2\hat{\sigma}_{RV}^2 h \log(1/h)} \):

9. The estimator \( \hat{\sigma}_{2mc,k}^2 \) defined by the iterative formulas (22) and with \( k \geq 1 \) such that \( \hat{\sigma}_{2mc,\ell} = \hat{\sigma}_{2mc,\ell-1} \) for all \( \ell \geq k \):

10. The estimator \( \hat{\sigma}_{mcz}^2 \) as in (23) with \( k = 1 \), using the initial threshold \( \varepsilon_{mc2,0} = \sqrt{2\hat{\sigma}_{RV} h} \):

11. The estimator \( \hat{\sigma}_{mcz,k}^2 \) defined by the iterative formulas (23) and with \( k \geq 1 \) such that \( \hat{\sigma}_{mcz,\ell} = \hat{\sigma}_{mcz,\ell-1} \) for all \( \ell \geq k \):

12. The estimator \( \hat{\sigma}_{NEW}^2 \) as defined in (16) where \( \varepsilon_{NEW} \) is such that \( F(\varepsilon_{NEW}; \hat{\sigma}_0, \hat{m}_0) = 0 \), with initial guesses \( \hat{m}_0 = (0, \ldots, 0) \) and \( \hat{\sigma}_{TRV}^2 = T^{-1} \sum_{i=1}^{n} (\Delta_i^X)^2 I_{\{\|\Delta_i^X\| \leq \varepsilon_{as}\}} \), with \( \varepsilon_{as} := \varepsilon_{2mc,0} = \sqrt{2\hat{\sigma}_{RV}^2 h \log(1/h)} \):

13. \( \hat{\sigma}_{NEW,k}^2 \) found with the new method described by the iterative formulas (17)-(19), with initial guesses given as in the previous item and \( k \) determined by the stopping rule \( \|\hat{\sigma}_{NEW,k} - \hat{\sigma}_{NEW,k-1}\| / \hat{\sigma}_{NEW,k-1} \leq \text{tol} = 10^{-5} \):

14. An Oracle type estimator of the form
\[
\hat{\sigma}_{\text{Orc}}^2 := \sum_{i=1}^{n} (\Delta_i X)^2 I_{\{\|\Delta_i X\| \leq \varepsilon_{\text{Orc}}\}},
\]
where \( \varepsilon_{\text{Orc}} \) is such that \( F(\varepsilon_{\text{Orc}}; \sigma, \mathbf{m}) = 0 \), using the true values of the volatility \( \sigma \) and of the jump vector \( \mathbf{m} = (m_1, \ldots, m_n) = (\Delta_1 J, \ldots, \Delta_n J) \):

15. The following estimator based on the Threshold Bipower Variation (TBV):
\[
\hat{\sigma}_{TBV}^2 := \frac{\pi}{2T} \sum_{i=1}^{n-1} |\Delta_i X||\Delta_{i+1} X| I_{\{\|\Delta_i X\| \leq \varepsilon_{TBV}\}} I_{\{\|\Delta_{i+1} X\| \leq \varepsilon_{TBV}\}} ;
\]
using a threshold of the form \( \varepsilon_{TBV} := 4h^2\sigma_{BV} \) with \( \omega = 0.49 \):

16. The iterated TBV estimator given by:
\[
\hat{\sigma}_{TBV,1}^2 := \hat{\sigma}_{TBV}^2 ,
\]
\[
\varepsilon_{TBV,k} := 4h^2\sigma_{TBV,k} \quad \hat{\sigma}_{TBV,k+1}^2 := \frac{\pi}{2T} \sum_{i=1}^{n-1} |\Delta_i X||\Delta_{i+1} X| I_{\{\|\Delta_i X\| \leq \varepsilon_{TBV,k}\}} I_{\{\|\Delta_{i+1} X\| \leq \varepsilon_{TBV,k}\}} ; \quad k \geq 1 ,
\]
using \( \omega = 0.49 \) and \( \hat{\sigma}_{TBV}^2 \) as defined in the previous item.\(^5\) We stop when \( |\hat{\sigma}_{TBV,k} - \hat{\sigma}_{TBV,k-1}| / \hat{\sigma}_{TBV,k-1} \leq \text{tol} = 10^{-5} \).

\(^5\)The estimators in items 15 and 16 were suggested by an anonymous referee.
Remark 9. Different variations of the above estimators, that are not shown here for sake of brevity, were also analyzed in our simulations. For instance, the 3 alternative thresholds \( \varepsilon = h^2 \) and \( \varepsilon = 2h^2 \), with \( \omega = 0.495 \), were implemented; each one of the estimators in items 15 and 16 was also implemented with thresholds \( 3h^2 \hat{\sigma}_{BV} \) and \( 5h^2 \hat{\sigma}_{BV} \). The results of these variations were suboptimal to those shown here. We also implemented the estimators in the items 6 to 11 starting with an initial threshold of the form \( \sqrt{2\sigma^2_{BV} h \log(1/h)} \) (i.e., using \( \hat{\sigma}_{BV} \) rather than \( \hat{\sigma}_{RV} \) as an initial guess for \( \sigma \)), and the same stopping condition as therein: in these cases we obtained the same performances for the liming estimators.

The adopted time unit of measure is 1 year (252 days) and we consider 5 minute observations over a 1 month time horizon with a 6.5 hours per day open market. For our first simulation experiment, we use the following parameters:

\[
\sigma = 0.4, \quad \sigma_{imp} = 3\sqrt{h}, \quad \mu_{imp} = 0, \quad \lambda = 100, \quad h = \frac{1}{252 \times 6.5 \times 12}, \quad (25)
\]

The dependence of \( \sigma_{imp} \) on \( \sqrt{h} \) was done for an easier comparison with the standard deviation of the increments of the continuous component, which is \( 0.4\sqrt{h} \). So, the standard deviation of the jumps is about 7.5 times the standard deviation of the continuous component increment. The parameter values in (25) yield an expected annualized volatility of 0.45, which is reasonable. Table 1 below shows the sample biases, standard deviations, and MSE’s based on 5000 simulations. We also show the sample version of Loss, i.e., the expected number of jump misclassifications as defined by (20), with its standard deviation; the sample average of \( N \), i.e. the number of iterations needed to find the estimator’s value, with its standard deviation; and, for the methods using truncation, the average threshold of the last step of the iteration used to obtain the estimate of \( \sigma \).

As expected, the unfeasible oracle estimator, which is shown as a benchmark for the other estimators, performs the best, followed by the estimators \( \hat{\sigma}_{NEW} \) and \( \hat{\sigma}_{NEW,k} \) based on finding the root of \( F(\varepsilon; \sigma, m) \). The iterative estimators \( \hat{\sigma}_{2mc,k} \) and \( \hat{\sigma}_{mc2,k} \), based on the thresholds \( \sqrt{2\sigma^2 h \ln(1/h)} \) and \( \sqrt{2\sigma^2 h w} \), also have a good performance and significantly improve on the estimator \( \hat{\sigma}_{3mc,k} \) (number 7 above) proposed in [10] and based on \( \sqrt{3\sigma^2 h \ln(1/h)} \). The estimator \( \hat{\sigma}_{TRV,T} \) proposed by Jacod and Todorov [14] also performs quite well in terms of MSE, but the estimation relative error is comparatively large. The estimators based on TBV (namely, the estimators \( \hat{\sigma}_{TBV} \) and \( \hat{\sigma}_{TBV,k} \) of items 15 and 16 above) as well as the MinRV and MedRV are suboptimal for the considered parameters choice.

We now double the intensity of jumps and consider the following parameter setting:

\[
\sigma = 0.4, \quad \sigma_{imp} = 3\sqrt{h}, \quad \mu_{imp} = 0, \quad \lambda = 200, \quad h = \frac{1}{252 \times 6.5 \times 12}, \quad (26)
\]

which yields an expected annualized volatility of 0.5. The results are shown in Table 2. We again notice that the Oracle estimator performs the best followed by the new estimators based on finding the root of \( F(\varepsilon; \sigma, m) \). As before, the estimators based on the MinRV, the MedRV, and \( \hat{\sigma}_{TBV} \) underperform compared to \( \hat{\sigma}_{NEW} \) and \( \hat{\sigma}_{NEW,k}; \)

\( \hat{\sigma}_{TBV,k} \) has a small relative estimation error, but a comparatively high MSE.

6 Extensions

In this section we assess our results on models with stochastic volatility and leverage and on models with infinite activity jumps. We now mention the main ideas that we are pursuing in the theoretical ongoing analysis in the presence of stochastic volatility and then we show on simulated data that the performance of our new methods is promising also in such extended contexts.

In the presence of stochastic volatility without leverage we can deal with cMSE as described in the subsequent paragraph. If also leverage is present, then we can use a similar approach under MSE or, alternatively, we can work at minimizing cMSE by assuming that \( d\sigma_t = \gamma_t dB_t \), with \( B \) a Brownian motion correlated with \( W \), and by
More specifically, we want to consider an estimator of the form $W$ splitting $\mu = \epsilon$ where observations of 5000 paths over a 1 month time horizon. The jump parameters are $\lambda = 100$, $\sigma^{jmp} = 3\sqrt{h}$ and $\mu^{jmp} = 0$.

A popular approach to deal with the case of stochastic volatility is “localization”. Assuming continuity of the paths of $\sigma$, the idea is that the volatility is approximately constant in a small time interval. So we can divide the interval $[t_{i-1}, t_i]$ into $k$ subintervals $[t_{i,\ell-1}, t_{i,\ell}]$, $\ell = 1, \ldots, n_i$, where $t_{i,0} < t_{i,1} < \cdots < t_{i,n_i} = t_i$, $\Delta_i,\ell X := X_{t_{i,\ell}} - X_{t_{i,\ell-1}}$, and the threshold $\epsilon_i$ is uniform on $[t_{i-1}, t_i]$. In the case that $n_i = 1$ for all $i$, we have the extreme case of one different threshold for each subinterval. When $\sigma$ is independent on $W$, we can consider the cMSE of $\hat{\sigma}_n$, denoted by $cMSE(\epsilon)$, and use this to determine the optimal thresholding levels $\epsilon_i$ for the different intervals. We define

$$cMSE_i(\epsilon_i) := E[(\hat{\sigma}_i - \hat{\sigma}_i)^2 | J, \sigma],$$

where $\hat{\sigma}_i = \hat{\sigma}_i h_i$ with $\hat{\sigma}_i$ a random number depending on the path of $\sigma$ over the interval $[t_{i-1}, t_i]$ (e.g., $h^{-1} \int_{t_{i-1}}^{t_i} \sigma_s^2 ds$), and $\hat{\sigma}_i := \sum_{\ell=1}^{n_i} (\Delta_i,\ell X)^2 I(|\Delta_i,\ell X| \leq \epsilon_i)$. Then it turns out that minimizing $cMSE(\epsilon)$, as $\epsilon$ varies while $k$ and $n_1, \ldots, n_k$ are fixed, is asymptotically equivalent to solve the $k$ problems

$$\min_{\epsilon_i} cMSE_i(\epsilon_i), \quad i=1..k,$$

which can be treated at once and justify why we tackled the minimization of cMSE by assuming constant volatility.

---

Table 1: Estimation of the volatility $\sigma = 0.4$ for a log-normal Merton model, based on simulated 5-minutes observations of 5000 paths over a 1 month time horizon. The jump parameters are $\lambda = 100$, $\sigma^{jmp} = 3\sqrt{h}$ and $\mu^{jmp} = 0$.

<table>
<thead>
<tr>
<th>Estimator</th>
<th>mean $\frac{\hat{\sigma}^2 - \sigma^2}{\sigma^2}$</th>
<th>std $\frac{\hat{\sigma}^2 - \sigma^2}{\sigma^2}$</th>
<th>MSE($\hat{\sigma}$) mean std</th>
<th>Loss mean std</th>
<th>mean std($\epsilon$) mean std</th>
<th>$\times10^5$</th>
<th>$\times10^5$</th>
<th>$\times10^3$</th>
<th>$N$</th>
<th>$N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\hat{\sigma}$</td>
<td>0.28625</td>
<td>0.17562</td>
<td></td>
<td>288.7300</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$\hat{\sigma}_{BV}$</td>
<td>0.06664</td>
<td>0.05517</td>
<td></td>
<td>19.1650</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$\hat{\sigma}_{MinRV}$</td>
<td>0.01563</td>
<td>0.05117</td>
<td></td>
<td>7.3287</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$\hat{\sigma}_{MedRV}$</td>
<td>0.01799</td>
<td>0.04593</td>
<td></td>
<td>6.2292</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$\hat{\sigma}_{TRV,T}$</td>
<td>0.00992</td>
<td>0.03712</td>
<td></td>
<td>3.7799</td>
<td>3.825</td>
<td>0.0130</td>
<td>0.34</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>$\hat{\sigma}_{3mc}$</td>
<td>0.02971</td>
<td>0.04262</td>
<td></td>
<td>6.9121</td>
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<td>0.0176</td>
<td>1.20</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>$\hat{\sigma}_{3mc,k}$</td>
<td>0.02033</td>
<td>0.03978</td>
<td></td>
<td>5.1097</td>
<td>4.488</td>
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<td>0.0157</td>
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<td>2.30</td>
</tr>
<tr>
<td>8</td>
<td>$\hat{\sigma}_{2mc}$</td>
<td>0.01500</td>
<td>0.03822</td>
<td></td>
<td>4.3174</td>
<td>4.161</td>
<td>2.060</td>
<td>0.0144</td>
<td>0.98</td>
<td>1</td>
</tr>
<tr>
<td>9</td>
<td>$\hat{\sigma}_{2mc,k}$</td>
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<td>0.03698</td>
<td></td>
<td>3.7127</td>
<td>3.776</td>
<td>1.929</td>
<td>0.0127</td>
<td>0.23</td>
<td>2.30</td>
</tr>
<tr>
<td>10</td>
<td>$\hat{\sigma}_{mc2}$</td>
<td>0.01190</td>
<td>0.03712</td>
<td></td>
<td>3.8920</td>
<td>3.981</td>
<td>1.974</td>
<td>0.0136</td>
<td>0.93</td>
<td>1</td>
</tr>
<tr>
<td>11</td>
<td>$\hat{\sigma}_{mc2,k}$</td>
<td>0.00654</td>
<td>0.03646</td>
<td></td>
<td>3.5133</td>
<td>3.622</td>
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<td>0.0120</td>
<td>0.21</td>
<td>2.31</td>
</tr>
<tr>
<td>12</td>
<td>$\hat{\sigma}_{NEW}$</td>
<td>-0.00046</td>
<td>0.03623</td>
<td></td>
<td>3.3980</td>
<td>3.352</td>
<td>1.812</td>
<td>0.0105</td>
<td>0.46</td>
<td>1</td>
</tr>
<tr>
<td>13</td>
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<td>0.03622</td>
<td></td>
<td>3.3593</td>
<td>3.552</td>
<td>1.819</td>
<td>0.0106</td>
<td>0.54</td>
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</tr>
<tr>
<td>14</td>
<td>$\hat{\sigma}_{Orc}$</td>
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<td></td>
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<td>3.647</td>
<td>2.012</td>
<td>0.0102</td>
<td>0.58</td>
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<tr>
<td>15</td>
<td>$\hat{\sigma}_{TBV}$</td>
<td>0.00185</td>
<td>0.04130</td>
<td></td>
<td>3.3605</td>
<td>3.352</td>
<td>1.812</td>
<td>0.0105</td>
<td>0.46</td>
<td>1</td>
</tr>
<tr>
<td>16</td>
<td>$\hat{\sigma}_{TBV,k}$</td>
<td>0.00110</td>
<td>0.04124</td>
<td></td>
<td>3.3593</td>
<td>3.552</td>
<td>1.819</td>
<td>0.0106</td>
<td>0.54</td>
<td>1.70</td>
</tr>
</tbody>
</table>

---

We thank Alexei Kolokolov for having suggested to consider such an approach.
Table 2: Estimation of the volatility \( \sigma = 0.4 \) for a log-normal Merton model based on simulations of 5-minutes observations of 5000 paths over a 1 month time horizon. The jump parameters are \( \lambda = 200 \), \( \sigma^\text{Jmp} = 3\sqrt{h} \) and \( \mu^\text{Jmp} = 0 \).

Although the theoretical analysis of cMSE under stochastic volatility and leverage and under infinite activity jumps are still ongoing, in the rest of this section, we illustrate on simulated data the behavior of our newly proposed methods. We find in fact that again they outperform the methods currently used in the literature, at least in the realistic scenarios that we considered here.

### 6.1 Simulation performance: stochastic volatility models with leverage

Even though the new method presented in Section 5 was originally designed for a model with constant volatility (and thus no leverage), it can still be applied for the more general stochastic volatility model (1). In this part, we examine by simulations the performance of the same estimators introduced in Section 5.1 in the presence of stochastic volatility and leverage. For the continuous part of the process, we take the popular Heston model [13] and consider:

\[
\begin{align*}
    dX_t &= \mu_t \, dt + \sqrt{V_t} \, dB_t + dJ_t, \quad X_0 = 1, \\
    dV_t &= \kappa (\theta - V_t) \, dt + \xi \sqrt{V_t} \, dW_t, \quad V_0 = \theta,
\end{align*}
\]

where \( B \) and \( W \) are correlated Wiener processes such that \( \mathbb{E}(dB_t \cdot dW_t) = \rho dt \) while, in accordance with our Assumption A1, we take the jump component \( J \) independent of \( (W,B) \). For \( J \) we adopt the Merton’s log-normal model studied in Section 5.1. We consider the following settings, where \( h \) will be set to 5 minutes (i.e., \( h = 1/(252 \times 6.5 \times 12) \)):

<table>
<thead>
<tr>
<th>Continuous Component Parameters</th>
<th>Jump Component</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu_t ) ( \kappa ) ( \xi ) ( \theta ) ( \rho ) ( V_0 ) ( \sigma^\text{Jmp} ) ( \mu^\text{Jmp} ) ( \lambda )</td>
<td>( 3\sqrt{h} ) 0 200</td>
</tr>
</tbody>
</table>

The values of \( \kappa \) and \( \xi \), which are standard in the literature, are the same as those used in [21], where they also propose \( \rho = -0.5 \) and \( \theta = 0.04 \). We adopt here the value of \( \theta = 0.16 \) for easier comparison with the constant
volatility case of Section 5.1, where σ is taken to be 0.4. We remark however that we checked the performance of all the estimators in the case θ = 0.04, and there are no significant changes, except that it is easier to identify jumps because the variance of the jump part is bigger compared to that of the continuous component, so the MSEs are smaller.

Since the volatility changes from simulation to simulation, to assess the accuracy of the different methods, we compute the relative error,

$$\varepsilon := \frac{\hat{\sigma}_n - \sigma}{\sigma},$$

for each simulated path, where $\sigma$ is given as below Eq. (2) and $\hat{\sigma}_n$ is an estimator of the integrated variance. The sample mean and standard deviation of the error over 5000 simulations for each of the estimators considered in Section 5.1 are reported in Table 3. We also show the sample mean of $(\hat{\sigma}_n - IV)^2$. As in Section 5.1, the “Oracle” is obtained by the formula

$$\hat{\sigma}_{Orc}^2 = \frac{1}{n} \sum_{i=1}^n (\Delta_i X)^2 1_{\{|\Delta_i X| \leq \varepsilon_{Orc}\}},$$

where $\varepsilon_{Orc}$ is such that $F(\varepsilon_{Orc}; \sigma_{\text{Avg}}, m) = 0$, using the true increments of the jump component, $m = (m_1, \ldots, m_n) = (\Delta_1 J, \ldots, \Delta_n J)$, and the true average volatility value $\sigma_{\text{Avg}}^2 := \int_0^T \sigma_s^2 ds$.

The results are consistent with those obtained in Section 5.1. The new estimators $\hat{\sigma}_{NEW}$ and $\hat{\sigma}_{NEW,k}$ based on finding the root of $F(\varepsilon; \sigma, m)$ perform the best. Also, the iterative estimators $\hat{\sigma}_{3mc,k}$ and $\hat{\sigma}_{mc,k}$ based on the threshold $\sqrt{2\sigma^2 h \ln(1/h)}$ and $\sqrt{2\sigma^2 \hat{w}_h}$ perform quite well and significantly improve on the estimator $\hat{\sigma}_{3mc,k}$ proposed in [10] and based on $\sqrt{3\sigma^2 h \ln(1/h)}$. In particular, the leverage factor seems to have a minor effect on the performance of all the estimators, while stochastic volatility seems not to have any adverse effects, compared to the constant volatility case. Thus, for instance, for $\lambda = 200$ and a long-run average volatility level of $\sqrt{\theta} = \sqrt{0.16} = 0.4$, the sample mean and standard deviation of $\varepsilon(\hat{\sigma}_{NEW})$ are 0.00186 and 0.03720, respectively, which are smaller that those attained by $\varepsilon(\hat{\sigma}_{NEW})$ in the constant volatility case of $\sigma = 0.4$ (namely, 0.00389 and 0.03762 as seeing in Table 2).

### 6.2 Simulation performance: infinite jump activity

It is natural to wonder about the robustness of the estimators introduced in this article against jumps of infinite activity (IA). To this end, in this section, we consider one of the most popular models of this kind: the Variance Gamma model (VG) of [6]. Concretely, we assume the model

$$X_t = at + \sigma W_t + J_t := at + \sigma W_t + \sigma_{Jmp} B_{S_t} + \theta S_t,$$

where $W$ and $B$ are independent Wiener processes and $\{S_t\}_{t \geq 0}$ is an independent Lévy subordinator such that $S_t$ is Gamma distributed with scale parameter $\beta := \kappa$ and shape parameter $\alpha := t/\kappa$. Note that, in that case, $\mathbb{E}[S_t] = t$ and $\text{Var}(S_t) = \kappa t$. For the parameter values, we take the following (the time unit is one day):

$$\sigma = \frac{0.2}{\sqrt{252}} = 0.0126, \quad \sigma_{Jmp} = 0.01, \quad \kappa = 0.7; \quad a = \theta = 0.$$

The values of $\sigma_{Jmp}$ and $\kappa$ are consistent with the empirical results of [9].

The results are shown in Table 4. Basically, the estimators that use truncation ($\hat{\sigma}_{3mc,k}, \hat{\sigma}_{2mc,k}, \hat{\sigma}_{mc,k}, \hat{\sigma}_{NEW,k}, \hat{\sigma}_{TBV,k}$, but not $\hat{\sigma}_{RV,T}$) perform better than those without it ($\hat{\sigma}_{RV}, \hat{\sigma}_{BV}, \hat{\sigma}_{MedRV}$, and $\hat{\sigma}_{MinRV}$). As expected, the Oracle estimator performs the best, followed by the estimators $\hat{\sigma}_{2mc,k}$ and $\hat{\sigma}_{mc,k}$, which are based on the respective asymptotic thresholds $\sqrt{2\sigma^2 h \ln(1/h)}$ and $\sqrt{2\sigma^2 \hat{w}_h}$. Indeed, their MSEs are less or equal to a quarter of any other feasible threshold estimators. The iterative estimators $\hat{\sigma}_{NEW,k}$ (based on finding the root of $F(\varepsilon; \sigma, m)$) and $\hat{\sigma}_{TBV,k}$ (based on truncated bipower variation) have similar performances.
7 Conclusions

We consider the problem of estimating the integrated variance $IV$ of a semimartingale model $X$ with jumps for the log price of a financial asset. In view of adopting the truncated realized variance of $X$, we look for a theoretical and practical way to select an optimal threshold in finite samples. We consider the following two optimality criteria: minimization of MSE, the expected quadratic error in the estimation of $IV$; and minimization of cMSE, the expected quadratic error conditional to the realized paths of the jump process $J$ and of the volatility process $(\sigma_s)_s \geq 0$. Under given assumptions, we find that for each criterion an optimal TH exists, is unique and is a solution of an explicitly given equation, the equation being different under the two criteria. Also, under each criterion, an asymptotic expansion with respect to the step $h$ between the observations is possible for the optimal TH. The leading terms of both the two expansions turn out to be proportional to the modulus of continuity of the Brownian motion paths and to the spot volatility of $X$, with proportionality constant $\sqrt{2-Y}$, $Y$ being the jump activity index of $X$. Further, we show that the threshold estimator of $IV$ constructed with the leading term of the optimal TH is consistent, at least in the finite activity jumps case, even if it does not satisfy the classical assumptions. The results obtained for the cMSE criterion allow for a novel numerical way to tuneup the threshold parameter in finite samples. Based on simulated data, we illustrate the superiority of the new method on other broadly used estimators in the literature. Minimization of cMSE in the presence of infinite activity jumps in $X$ and in the presence of stochastic volatility and leverage are object of ongoing research, but the newly proposed estimators are implemented on simulated data under such frameworks, and again are superior.
Table 4: Estimation of the volatility $\sigma = 0.2/\sqrt{252} = 0.0126$ for a Gauss-VG model, based on simulations of 5-minutes observations of 5000 paths for each model, over a 1 month time horizon. The jump parameters are $\sigma_{jmp} = 0.01$ and $\theta = 0$.

8 Appendix: proofs

Proof of Theorem 1. Under $A_1$ we have that conditionally to $(\sigma, J)$ the increment $\Delta_i X = \int_{t_{i-1}}^{t_i} \sigma_s dW_s + \Delta_i J$ is a Gaussian r.v. with law $N(m_i, \sigma_i^2)$, which allows to compute the conditional expectation $E[\hat{V}_n|\sigma, J]$. We have

$$E[\hat{V}_n|\sigma, J] = \sum_{i=1}^{n} b_i(\varepsilon) = \sum_{i=1}^{n} \left( e^{-\frac{(x-m_i)^2}{2\varepsilon^2}} (\varepsilon + m_i) + e^{-\frac{(x+m_i)^2}{2\varepsilon^2}} (\varepsilon - m_i) \right) \frac{\sigma_i}{\sqrt{2\pi}}$$

$$+ \frac{m_i^2 + \sigma_i^2}{\sqrt{\pi}} \left( \int_0^{x-m_i} e^{-\varepsilon^2 dt} + \int_0^{x+m_i} e^{-\varepsilon^2 dt} \right),$$

and

$$E[(\hat{V}_n(\varepsilon))^2|\sigma, J] = \sum_i E[(\Delta_i X_s)^4|\sigma, J] + 2 \sum_i \sum_{j>i} E[(\Delta_i X_s)^2(\Delta_j X_s)^2|\sigma, J]$$

$$= \sum_i \left[ e^{-\frac{(x-m_i)^2}{2\varepsilon^2}} \sigma_i (\varepsilon^3 + m_i \varepsilon^2 + m_i^2 \varepsilon + m_i^3 + 5m_i \sigma_i^2 + 3\sigma_i^2 \varepsilon) ight.$$

$$\left. - e^{-\frac{(x+m_i)^2}{2\varepsilon^2}} \sigma_i (\varepsilon^3 - m_i \varepsilon^2 + m_i^2 \varepsilon - m_i^3 - 5m_i \sigma_i^2 + 3\sigma_i^2 \varepsilon) ight]$$

$$+ \left( \int_0^{x-m_i} e^{-\varepsilon^2 dt} + \int_0^{x+m_i} e^{-\varepsilon^2 dt} \right) \sqrt{2} \left( m_i^4 + 6m_i^2 \sigma_i^2 + 3\sigma_i^4 \right) \frac{1}{\sqrt{2\pi}} + 2 \sum_i \sum_{j>i} b_i b_j, \quad (29)$$

having used that conditionally to $\sigma$ and $J$, $\Delta_i X_s$ and $\Delta_j X_s$ are independent. It follows that

$$MSE(\varepsilon) = E \left[ \sum_i \left[ e^{-\frac{(x-m_i)^2}{2\varepsilon^2}} \sigma_i (\varepsilon^3 + m_i \varepsilon^2 + m_i^2 \varepsilon + m_i^3 + 5m_i \sigma_i^2 + 3\sigma_i^2 \varepsilon) ight.$$

$$\left. - e^{-\frac{(x+m_i)^2}{2\varepsilon^2}} \sigma_i (\varepsilon^3 - m_i \varepsilon^2 + m_i^2 \varepsilon - m_i^3 - 5m_i \sigma_i^2 + 3\sigma_i^2 \varepsilon) ight]$$

$$+ \left( \int_0^{x-m_i} e^{-\varepsilon^2 dt} + \int_0^{x+m_i} e^{-\varepsilon^2 dt} \right) \sqrt{2} \left( m_i^4 + 6m_i^2 \sigma_i^2 + 3\sigma_i^4 \right) \frac{1}{\sqrt{2\pi}} + 2 \sum_i \sum_{j>i} b_i b_j, \right].$$
Proof of Corollary 1.

Let $MSE(\varepsilon)$ be a differentiable function of $\varepsilon$, therefore to find the minimum on $[0, +\infty]$ of $MSE(\varepsilon)$ we can study the sign of its first derivative $MSE'(\varepsilon)$. Since $MSE'(\varepsilon) = \frac{d}{d\varepsilon} E[(IV_n(\varepsilon))^2] - 2IV \frac{d}{d\varepsilon} E[IV_n(\varepsilon)]$, we begin to compute $\frac{d}{d\varepsilon} E[IV_n(\varepsilon)|\sigma, J]$. Note that

$$
\frac{d}{d\varepsilon} b_i(\varepsilon) = \left( e^{-\frac{(e_m - e)^2}{2\sigma_i^2}} + e^{-\frac{(e_m + e)^2}{2\sigma_i^2}} \right) \frac{(\varepsilon + m_i)(\varepsilon - m_i)}{\sigma_i \sqrt{2\pi}}
$$

so that

$$
\frac{d}{d\varepsilon} E[IV_n(\varepsilon)|\sigma, J] = \varepsilon^2 \sum_{i=1}^n a_i(\varepsilon)
$$

(30)

is strictly greater than zero for all values of $\varepsilon > 0$. As $\frac{d}{d\varepsilon} E[(IV_n(\varepsilon))^2|\sigma, J]$, note that the term $2 \sum_i \sum_{j>i} b_i b_j$ in (29) can be written as $\sum_i \sum_{j\neq i} b_i b_j$, so its derivative coincides with $\sum_i \sum_{j\neq i} (\varepsilon^2 a_i b_j + b_i \varepsilon^2 a_j)$, however

$$
\sum_i b_i \sum_{j\neq i} a_j = \left( \sum_i b_i \sum_j a_j - \sum_i b_i a_i \right)
$$

$$
= \left( \sum_i a_i \sum_j b_j - \sum_i a_i b_i \right) = \sum_i a_i \sum_{j\neq i} b_j
$$

so that $\sum_i \sum_{j\neq i} (\varepsilon^2 a_i b_j + b_i \varepsilon^2 a_j) = 2 \sum_{i=1}^n \sum_{j\neq i} \varepsilon^2 a_i b_j$

$$
\frac{d}{d\varepsilon} E[(IV_n(\varepsilon))^2|\sigma, J] = \varepsilon^4 \sum_i a_i(\varepsilon) + 2 \left( \sum_{i=1}^n \sum_{j>i} b_i(\varepsilon) b_j(\varepsilon) \right)'
$$

(31)

$$
= \sum_i \left[ \varepsilon^4 a_i + 2\varepsilon^2 a_i \sum_{j\neq i} b_j \right]
$$

and

$$
\frac{d}{d\varepsilon} MSE(\varepsilon) = \varepsilon^2 \sum_i E\left[ \varepsilon^2 a_i + 2a_i \sum_{j\neq i} b_j - 2IV a_i \right] = \varepsilon^2 \sum_i E\left[ a_i \left( \varepsilon^2 + 2 \sum_{j\neq i} b_j - 2IV \right) \right]
$$

(32)

$$
= \varepsilon^2 G(\varepsilon).
$$

Proof of Corollary 1. Note that $a_i(\varepsilon)$ and $b_i(\varepsilon)$ are continuously differentiable functions of $\varepsilon$, and, with fixed $h = \frac{T}{n}$,

$$
a_i(0) = \frac{2e^{-\frac{m_i^2}{2\sigma_i^2}}}{\sigma_i \sqrt{2\pi}}, \quad b_i(0) = 0.
$$
minimum value

MSE is

Proof of Lemma 1.

Proof of Theorem 3.

By definition,

For

Proof of Theorem 2.

so that

so there exists $\varepsilon_+ > 0$:

$MSE'(\varepsilon) > 0$ on $[\varepsilon_+, +\infty)$. On the compact set $[0, \varepsilon_+]$ the continuous function $MSE$ has necessarily absolute minimum value $MSE_\varepsilon$, and since on $[\varepsilon_+, +\infty)$ $MSE$ is increasing we have that on $[0, +\infty)$ the absolute minimum is $MSE$. $MSE'(\varepsilon)$ is continuous and assumes both negative and positive values, thus equation $G(\varepsilon) = 0$ has a solution. Any minimum point of $MSE$ on $[0, +\infty)$ has to be a stationary point, so it has to solve the equation.

Proof of Theorem 2. For $\varepsilon > 0$ we have $MSE'(\varepsilon) > 0$ if and only if $G(\varepsilon) > 0$, which in turn is true if and only if

$g(\varepsilon) := \varepsilon^2 + 2(n - 1)E[b_1] - 2IV > 0$

where, setting $m := m_1 = \Delta_1J$, we recall that we have

$E[b_1] = E \left[ - \left( e^{-\frac{(\varepsilon-m)^2}{2\sigma^2}} (\varepsilon + m) + e^{-\frac{(\varepsilon+m)^2}{2\sigma^2}} (\varepsilon - m) \right) \frac{\sigma\sqrt{h}}{\sqrt{2\pi}} \right. + \left. m^2 + \sigma^2h \left( \int_0^{\frac{\varepsilon-m}{\sigma\sqrt{h}}} e^{-t^2} dt + \int_{\frac{\varepsilon+m}{\sigma\sqrt{h}}} e^{-t^2} dt \right) \right].$

The sign of $g(\varepsilon)$ is studied as follows:

$g(0) = -2\sigma^2T < 0,$

$\lim_{\varepsilon \to +\infty} g(\varepsilon) = +\infty,$

$g'(\varepsilon) = 2\varepsilon(1 + (n-1)\varepsilon E[a_1])$

so that $g'(\varepsilon) > 0$ for all $\varepsilon > 0$, $n > 1$. That implies that $g(\varepsilon)$ starts at $\varepsilon = 0$ from a negative value and strictly increases towards $+\infty$, as $\varepsilon$ increases, so that there exists a unique $\varepsilon^*$ such that $g(\varepsilon) < 0$ for $\varepsilon \in [0, \varepsilon^*]$, $g(\varepsilon^*) = 0$ and $g(\varepsilon) > 0$ for $\varepsilon \in [\varepsilon^*, +\infty]$. That implies in turn that $MSE(\varepsilon)$ has a unique minimum point in $\varepsilon^*$, which is then the optimal threshold we were looking for: $\varepsilon^*$ is the unique solution of equation (5), corresponding to $g(\varepsilon) = G(\varepsilon) = 0$.

Proof of Theorem 3. By definition,

$E[b_1(\varepsilon)] = E \left[ (\Delta^n_1X)^2 \mathbf{1}_{\{\varepsilon, \Delta^n_1N=0\}} \right] + E \left[ (\Delta^n_1X)^2 \mathbf{1}_{\{\varepsilon, \Delta^n_1N \neq 0\}} \right] =: G + \mathcal{L}.$

By Lemma S.2 and Lemma S.5 with $k = 2$ in [11], provided that $\varepsilon \to 0$, we have

$\mathcal{L} := E \left[ (\Delta^n_1X)^2 \mathbf{1}_{\{\varepsilon, \Delta^n_1N \neq 0\}} \right] \sim \lambda h \frac{\varepsilon^3}{3} C(f), \quad (h \to 0),$

$G := \sigma^2h - \frac{2}{\sqrt{2\pi}} \sigma \varepsilon \sqrt{h} e^{-\frac{\varepsilon^2}{2\sigma^2h}} + O(h^2) + o \left( \varepsilon \sqrt{h} e^{-\frac{\varepsilon^2}{2\sigma^2h}} \right),$

which shows the result.

Proof of Lemma 1. Throughout, $p_t$ denotes the density of $J_t$ and recall that the characteristic function of $J_t$ is of the form $E[e^{ituJ_t}] = e^{-ct|u|^2}$. Let us also recall that the Fourier transform and its inverse are defined by $\mathcal{F}g(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(z)e^{-izx}dz$ and $\mathcal{F}^{-1}G(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} G(z)e^{izx}dz$. In what follows, we set

$h(u) := \left( \mathcal{F}^{-1} \phi \left( \frac{x}{\sigma \sqrt{h}} - \frac{\varepsilon}{\sigma \sqrt{h}} \right) \right)(u) = \frac{1}{\sqrt{2\pi}} \int \phi \left( \frac{x}{\sigma \sqrt{h}} - \frac{\varepsilon}{\sigma \sqrt{h}} \right) e^{iux} dx.$
Let us start by noting that
\[
\mathbb{E} \left[ \phi \left( \frac{\varepsilon}{\sigma \sqrt{h}} - \frac{J_h}{\sigma \sqrt{h}} \right) \right] = \int \phi \left( \frac{x}{\sigma \sqrt{h}} - \frac{\varepsilon}{\sigma \sqrt{h}} \right) p_h(x) dx = \int (F_h)(x) p_h(x) dx = \int h(u) (F p_h)(u) du,
\]
where, since \( J \) is a symmetric stable process, \((F p_h)(u) = (2\pi)^{-1/2} e^{-u |u|^\gamma} \). Therefore, we obtain the representation
\[
\mathbb{E} \left[ \phi \left( \frac{\varepsilon}{\sigma \sqrt{h}} - \frac{J_h}{\sigma \sqrt{h}} \right) \right] = \frac{\sigma h^{1/2}}{2\pi} \int e^{-u |u|^\gamma} - \frac{\varepsilon^2}{2} + i \varepsilon u du.
\]

In order to prove (6), let us make the change of variables \( w = \sigma h^{1/2} u \) and, then, expand in a Taylor’s expansion \( \exp(-c\sigma^{-\gamma} h^{1-\gamma/2} |u|^\gamma) \) as follows:
\[
\frac{1}{2\pi} \int e^{-c\sigma^{-\gamma} h^{1-\gamma/2} |w|^\gamma} - \frac{w^2}{2} + i \frac{\varepsilon}{\sigma h^{1/2}} w dw = \frac{1}{2\pi} \int e^{-\frac{w^2}{2}} + i \frac{\varepsilon}{\sigma h^{1/2}} w dw + \sum_{k=1}^{\infty} I_{k,n},
\]
where
\[
I_{k,n} := \frac{1}{k!} (-c)^k \sigma^{-k} h^{k(1-\gamma/2)} \frac{1}{\sqrt{2\pi}} \int |w|^k e^{-\frac{w^2}{2}} + i \frac{\varepsilon}{\sigma h^{1/2}} w dw
= \frac{1}{k!} (-c)^k \sigma^{-k} h^{k(1-\gamma/2)} \frac{2}{\sqrt{2\pi}} \int_0^\infty w^k e^{-\frac{w^2}{2}} \cos \left( \frac{\varepsilon}{\sigma h^{1/2}} w \right) dw.
\]
The first term of (6) is then clear. For the subsequent terms, let us apply the formula for the cosine integral transformation of \( e^{-w^2/2} \) as well as the asymptotics for the generalized hypergeometric series or Kummer’s function \( M(a, b, z) \):
\[
I_{k,n} = \frac{1}{k!} (-c)^k \sigma^{-k} h^{k(1-\gamma/2)} \frac{2}{\sqrt{2\pi}} \left\{ \frac{1}{2} \Gamma \left( \frac{1}{2} + \frac{k}{2} \right) \frac{\Gamma \left( 1 + \frac{k}{2} \right)}{\Gamma \left( \frac{1}{2} + \frac{k}{2} \right)} \right\} \left( \frac{\varepsilon^2}{2\sigma^2 h} \right)^{\frac{1}{2} - \frac{k Y}{2}} + \frac{\Gamma \left( \frac{1}{2} \right)}{\Gamma \left( \frac{1}{2} + \frac{k Y}{2} \right)} e^{-\frac{\varepsilon^2}{2\sigma^2 h} \frac{k Y}{2}} \right) + \text{h.o.t.}
\]
In the asymptotic formula for the Kummer’s function above, the first term (respectively, second term) vanishes if \( \Gamma(-k Y/2) \) (respectively, \( \Gamma(1/2 + k Y/2) \)) are infinity. This happens when \(-k Y/2 \) or \( 1/2 + k Y/2 \) are nonpositive integers. It is now evident that there exist nonzero constants \( a_k \) and \( b_k \) such that
\[
I_{k,n} = \frac{a_k}{\Gamma \left( \frac{1}{2} + \frac{k Y}{2} \right)} e^{-1 - k Y} h^{k+1} + \frac{b_k}{\Gamma \left( \frac{1}{2} + \frac{k Y}{2} \right)} e^{-\frac{\varepsilon^2}{2\sigma^2 h} \frac{k Y}{2}} \text{e}^{k Y h(1-Y)} + \text{h.o.t.}
\]
Note that
\[
\varepsilon^{-1 - k Y} h^{k+1} \gg \varepsilon^{-1 - (k+1) Y} h^{k+1+\frac{1}{2}} \iff \varepsilon \gg h^{1/2} \iff \varepsilon \gg h^{1/2},
\]
\[
\varepsilon^{-1 - Y} h^{k+1} \gg \varepsilon^{-1 - k Y} h^{k+\frac{1}{2}} \gg e^{-\frac{\varepsilon^2}{2\sigma^2 h} \frac{k Y}{2}} e^{k Y h(1-Y)}.
\]
Therefore, \( \varepsilon^{-1 - Y} h^{1+\frac{1}{2}} \gg I_{k,n} \), for all \( k > 1 \).

We now show (7). Note that
\[
\mathbb{E} [J_h \phi \left( \frac{\varepsilon}{\sigma \sqrt{h}} - \frac{J_h}{\sigma \sqrt{h}} \right)] = \int \phi \left( \frac{x}{\sigma \sqrt{h}} - \frac{\varepsilon}{\sigma \sqrt{h}} \right) x p_h(x) dx = \int h(u) F(x p_h(x)) (u) du,
\]
where
\[
F(x p_h(x)) (u) = \frac{d}{du} (F p_h)(u) = \frac{i}{\sqrt{2\pi}} \frac{d}{du} e^{-u |u|^\gamma} = \frac{i}{\sqrt{2\pi}} e^{-ch |u|^\gamma} Y \text{sign}(u) ch |u|^\gamma - 1.
\]
Therefore, we have the following representation:

$$
\mathbb{E} \left[ J_h \phi \left( \frac{\varepsilon}{\sigma \sqrt{h}} - \frac{J_h}{\sigma \sqrt{h}} \right) \right] = \sigma \frac{-i Y_c}{\sqrt{2\pi}} h^{3/2} \int \text{sign}(u) |u|^{Y-1} e^{-chu} Y - \frac{2u^2}{2} + i \varepsilon u \, du.
$$

Furthermore,

$$
\mathbb{E} \left[ J_h \phi \left( \frac{\varepsilon}{\sqrt{h}} - \frac{J_h}{\sqrt{h}} \right) \right] = 2 \sigma Y_c h^{3/2} \int_0^\infty u^{-1} e^{-chu} Y - \frac{2u^2}{2} \sin(\varepsilon u) \, du
$$

$$
= 2 \sigma^{-Y-1} Y_c h^{1/2} \int_0^\infty u^{-1} e^{-cY h^{1/2} w^{Y} - \frac{u^2}{2}} \sin \left( \sigma^{-1} \varepsilon h^{-1/2} w \right) \, dw.
$$

Next, we expand in a Taylor’s expansion \( \exp(-cY h^{1/2} w^{Y}) \) as follows:

$$
\frac{1}{\sqrt{2\pi}} \int_0^\infty w^{-1} e^{-cY h^{1/2} w^{Y} - \frac{u^2}{2}} \sin \left( \sigma^{-1} \varepsilon h^{-1/2} w \right) \, dw = \sum_{k=0}^\infty I_{k,n},
$$

where

$$
I_{k,n} := \frac{1}{k!} (-c)^k \sigma^{-Y} h^{k(1-Y/2)} \frac{1}{\sqrt{2\pi}} \int_0^\infty w^{(k+1)Y-1} e^{-w^2/2} \sin \left( \varepsilon h^{-1/2} w \right) \, dw.
$$

Then, we again apply the following formula for the sine integral transformation of \( w^{(k+1)Y-1} e^{-w^2/2} \):

$$
I_{k,n} = \frac{1}{k!} (-c)^k \sigma^{-Y} h^{k(1-Y/2)} \frac{1}{\sqrt{2\pi}} \left\{ \frac{1}{2} 2^{k+1} \left( \frac{3}{2} \right)^{1/2} \left( \frac{\varepsilon^2}{h} \right) M \left( \frac{1}{2} + \frac{k+1}{2} \right) \right\}.
$$

Finally, we use the relationship

$$
M \left( \frac{1}{2} + \frac{k+1}{2} ; \frac{3}{2}, \frac{\varepsilon^2}{2h} \right) = \frac{\Gamma \left( \frac{3}{2} \right)}{\Gamma \left( 1 - \frac{k+1}{2} \right)} \frac{\varepsilon^2}{2h} \left( \frac{\varepsilon^2}{2h} \right)^{-\frac{k+1}{2}}
$$

$$
+ \frac{\Gamma \left( \frac{3}{2} \right)}{\Gamma \left( 1 + \frac{k+1}{2} \right)} e^{-\frac{\varepsilon^2}{2h}} \left( \frac{\varepsilon^2}{2h} \right)^{-1 + \frac{k+1}{2}} + \text{h.o.t.},
$$

which, in turn shows that,

$$
I_{k,n} \ll I_{1,n} \ll h \varepsilon^{1-Y}.
$$

We then conclude the result of the Lemma.

**Proof of Lemma 2.** Let

$$
I_n^+ := \mathbb{E} \left[ \Phi \left( \frac{\varepsilon}{\sigma \sqrt{h}} - \frac{J_h}{\sigma \sqrt{h}} \right) 1 \{ \phi \left( \frac{\varepsilon}{\sigma \sqrt{h}} - \frac{J_h}{\sigma \sqrt{h}} \right) \geq 0 \} \right]
$$

For \( I_n^+ \), let us note that for a constant \( K, \Phi(z) \leq K \phi(z) \) for all \( z \geq 0 \) and, thus,

$$
I_n^+ \leq K \mathbb{E} \left[ \phi \left( \frac{\varepsilon}{\sigma \sqrt{h}} - \frac{J_h}{\sigma \sqrt{h}} \right) 1 \{ \phi \left( \frac{\varepsilon}{\sigma \sqrt{h}} - \frac{J_h}{\sigma \sqrt{h}} \right) \geq 0 \} \right] = O \left( \mathbb{E} \left[ \phi \left( \frac{\varepsilon}{\sigma \sqrt{h}} - \frac{J_h}{\sigma \sqrt{h}} \right) \right] \right).
$$

For the other term, we decompose it as follows:

$$
I_n = \int_{\mathbb{R}} \phi(u) 1 \{ 0 \geq \frac{\varepsilon}{\sigma \sqrt{h}} - \frac{J_h}{\sigma \sqrt{h}}, u \geq \frac{\varepsilon}{\sigma \sqrt{h}} - \frac{J_h}{\sigma \sqrt{h}} \} \, du
$$

$$
= \int_0^\infty \phi(u) 1 \{ 0 \geq \frac{\varepsilon}{\sigma \sqrt{h}} - \frac{J_h}{\sigma \sqrt{h}} \} \, du + \int_{-\infty}^0 \phi(u) 1 \{ u \geq \frac{\varepsilon}{\sigma \sqrt{h}} - \frac{J_h}{\sigma \sqrt{h}} \} \, du
$$

$$
= \frac{1}{2} \mathbb{P} \left[ J_1 \geq h^{-\frac{1}{2}} \varepsilon \right] + \int_{-\infty}^0 \phi(u) 1 \{ J_1 \geq h^{-\frac{1}{2}} \varepsilon - \sigma u h^{-\frac{1}{2}} \} \, du.
$$
The first term above is well-known to be $\mathbb{P}[J_1 \geq h^{-1/Y} \varepsilon] = Y^{-1} C (h^{-1/Y} \varepsilon)^{-Y} + O(\varepsilon^{-2Y}h^2)$. For the second term, let us first recall that there exists a constant $K$ such that for all $x > 0$,

$$|\mathcal{E}(x)| := |\mathbb{P}[J_1 \geq x] - \frac{C}{Y}x^{-Y}| \leq Kx^{-2Y}.$$  

Therefore,

$$\int_{-\infty}^{0} \phi(u)\mathbb{P}[J_1 \geq h^{-1/Y} \varepsilon - \sigma uh^{\frac{1}{2} - \frac{1}{Y}}] du = \frac{C}{Y} \int_{-\infty}^{0} \phi(u) \left(h^{-\frac{1}{Y}} \varepsilon - \sigma uh^{\frac{1}{2} - \frac{1}{Y}}\right)^{-Y} du + \int_{-\infty}^{0} \phi(u)\mathcal{E}\left(h^{-\frac{1}{Y}} \varepsilon - \sigma uh^{\frac{1}{2} - \frac{1}{Y}}\right) du.$$

For the first term above, note that

$$\frac{1}{h^{1/Y}} \int_{-\infty}^{0} \phi(u) \left(h^{-\frac{1}{Y}} \varepsilon - \sigma uh^{\frac{1}{2} - \frac{1}{Y}}\right)^{-Y} du = \int_{-\infty}^{0} \phi(u) \left(1 - \sigma u\varepsilon^{-h^{1/Y}}\right)^{-Y} du,$$

which, by the dominated convergence theorem, converges to $1/2$, because $\varepsilon^{-1}h^{1/Y} \to 0$, as $n \to \infty$. Similarly, using (36), we have

$$\left|\int_{-\infty}^{0} \phi(u)\mathcal{E}\left(h^{-\frac{1}{Y}} \varepsilon - \sigma uh^{\frac{1}{2} - \frac{1}{Y}}\right) du\right| \leq K \int_{-\infty}^{0} \phi(u) \left(h^{-\frac{1}{Y}} \varepsilon - \sigma uh^{\frac{1}{2} - \frac{1}{Y}}\right)^{-2Y} du = O(\varepsilon^{-2Y}h^2).$$

Therefore, we finally conclude that $I_n^- = Y^{-1} Ch^{-Y} + O(\varepsilon^{-2Y}h^2)$, which implies (8).

We now show (9). To this end, let us first consider

$$E_{1,h}(\varepsilon) := \mathbb{E}\left[J_n^2 1_{\{0 \leq \sigma w_1 , j_1 \leq \varepsilon, \sigma w_1, j_1 \geq 0, \sigma w_1 \geq 0\}}\right]$$

$$= \frac{h^{-1/Y}}{\gamma} \int_{0}^{\varepsilon^{-1}h^{1/Y}} \phi(x) \int_{0}^{h^{-\frac{1}{Y}} \varepsilon - \sigma xh^{\frac{1}{2} - \frac{1}{Y}}} u^2 p_1(u) du dx$$

$$= h^{2/Y} \int_{0}^{\varepsilon^{-1}h^{1/Y}} \phi(x) \int_{0}^{h^{-\frac{1}{Y}} \varepsilon - \sigma xh^{\frac{1}{2} - \frac{1}{Y}}} u^2 p_1(u) du dx.$$

Let $\mathcal{E}(u) := p_1(u) - Cu^{-Y-1}$ and let us recall that, for a constant $K$, $|\mathcal{E}(u)| \leq K (u^{-Y-1} \wedge u^{-2Y-1}) \leq Ku^{-2Y-1}$, for all $u > 0$. Next,

$$E_{1,h}(\varepsilon) = Ch^{2/Y} \left(\frac{\varepsilon}{\sigma h^{\frac{1}{2}}}\right) \int_{0}^{1} \phi\left(\frac{\varepsilon}{\sigma h^{\frac{1}{2}}} w\right) \int_{0}^{h^{-\frac{1}{Y}} \varepsilon \left(1 - w\right)} u^2 \mathcal{E}(u) du dw$$

For the first term above, note that

$$\frac{1}{2 - Y} \int_{0}^{1} \phi\left(\frac{\varepsilon}{\sigma h^{\frac{1}{2}}} w\right) \left(h^{-\frac{1}{Y}} \varepsilon \left(1 - w\right)\right)^{2-Y} dw = \frac{h^{-2Y} \varepsilon^{2-Y}}{2 - Y} \int_{0}^{1} \phi\left(\frac{\varepsilon}{\sigma h^{\frac{1}{2}}} w\right) \left(1 - w\right)^{2-Y} dw$$

$$\sim 2^{-1} \frac{h^{-2Y} \varepsilon^{2-Y}}{2 - Y} \left(\frac{\sigma h^{\frac{1}{2}}}{\varepsilon}\right).$$

We divide the second term in two cases. If $Y \leq 1$, then

$$\left|\int_{0}^{1} \phi\left(\frac{\varepsilon}{\sigma h^{\frac{1}{2}}} w\right) \int_{0}^{h^{-\frac{1}{Y}} \varepsilon \left(1 - w\right)} u^2 \mathcal{E}(u) du dw\right| \leq K \frac{1}{2 - 2Y} \int_{0}^{1} \phi\left(\frac{\varepsilon}{\sigma h^{\frac{1}{2}}} w\right) \left(h^{-\frac{1}{Y}} \varepsilon \left(1 - w\right)\right)^{2-2Y} dw$$

$$\leq K \frac{h^{-2Y} \varepsilon^{2-2Y}}{2 - 2Y} \int_{0}^{1} \phi\left(\frac{\varepsilon}{\sigma h^{\frac{1}{2}}} w\right) \left(1 - w\right)^{2-2Y} dw$$

$$\sim 2K \frac{h^{-2Y} \varepsilon^{2-2Y}}{2 - 2Y} \left(\frac{\sigma h^{\frac{1}{2}}}{\varepsilon}\right).$$
Note that the last limit is valid provided that \( \int_0^1 (1 - w)^{2-2Y} \, dw < \infty \), which holds true when \( Y \leq 1 \). For \( Y > 1 \), let us first observe that

\[
\int_0^1 u^2 (u^{-Y-1} \wedge u^{-2Y-1}) \, du \leq \frac{1}{2-Y} + 1_{\{z>1\}} \frac{1 - \varepsilon^{2(1-Y)}}{2(Y-1)} \leq \frac{1}{2-Y} + \frac{1}{2(Y-1)}.
\]

Therefore, for a constant \( K \),

\[
\left| \int_0^1 \phi \left( \frac{\varepsilon}{\sigma h^2} w \right) \int_0^{h^{-\frac{1}{Y}} \varepsilon^{(1-w)}} u^2 \mathcal{E}(u) \, du \, dx \right| \leq K \int_0^1 \phi \left( \frac{\varepsilon}{\sigma h^2} w \right) \, dw \sim K \left( \frac{\sigma h^{\frac{1}{Y}}}{\varepsilon} \right).
\]

We conclude that

\[
E_{1,h}(\varepsilon) = \frac{2^{-1}C}{2-Y} h \varepsilon^{2-Y} + O \left( h^2 \varepsilon^{2-2Y} \right) + O \left( h^\frac{2}{Y} \right).
\]

Next, we consider

\[
E_{2,h}(\varepsilon) := E \left[ J_h^2 \mathbb{1}_{\{0 \leq \sigma W_h + J_h \leq \varepsilon, J_h \geq 0, W_h \leq 0\}} \right]
\]

\[
= h^{2/Y} \int_{-\infty}^0 \phi(x) \int_{-\sigma h^{\frac{1}{Y}} - x}^{h^{-\frac{1}{Y}} + \varepsilon - \sigma h^{\frac{1}{Y}} + x} u^2 p_1(u) \, du \, dx
\]

\[
= Ch^{2/Y} \int_{-\infty}^0 \phi(x) \int_{-\sigma h^{\frac{1}{Y}} - x}^{h^{-\frac{1}{Y}} + \varepsilon - \sigma h^{\frac{1}{Y}} + x} u^{1-Y} \, du \, dx
\]

\[
+ h^{2/Y} \int_{-\infty}^0 \phi(x) \int_{-\sigma h^{\frac{1}{Y}} - x}^{h^{-\frac{1}{Y}} + \varepsilon - \sigma h^{\frac{1}{Y}} + x} u^2 \mathcal{E}(u) \, du \, dx.
\]

The first term on the right-hand side above can be written as

\[
\frac{C}{2-Y} h^{2/Y} \left( h^{-\frac{1}{Y}} + \varepsilon \right)^{2-Y} \int_{-\infty}^0 \phi(x) \left\{ \left( 1 - \frac{\sigma h^{\frac{1}{Y}}}{\varepsilon} x \right)^{2-Y} - \left( -\frac{\sigma h^{\frac{1}{Y}}}{\varepsilon} x \right)^{2-Y} \right\} \, dx \sim 2^{-1} \frac{C}{2-Y} h^{2-Y} \varepsilon^{2-Y},
\]

where the last asymptotic relationship follows from dominated convergence theorem and the facts that \( h^{1/2} / \varepsilon \rightarrow 0 \) and \( \int_{-\infty}^0 (1 - x)^{2-Y} \phi(x) \, dx < \infty \). For the second term of \( E_{2,h}(\varepsilon) \), we have two cases. For \( Y \leq 1 \), we have

\[
h^{-\frac{1}{Y}} \int_{-\sigma h^{\frac{1}{Y}} - x}^{h^{-\frac{1}{Y}} + \varepsilon - \sigma h^{\frac{1}{Y}} + x} u^2 \mathcal{E}(u) \, du \, dx \leq Kh^{-\frac{1}{Y}} \int_{-\infty}^0 \phi(x) \int_{-\sigma h^{\frac{1}{Y}} - x}^{h^{-\frac{1}{Y}} + \varepsilon - \sigma h^{\frac{1}{Y}} + x} u^{1-2Y} \, du \, dx
\]

\[
= \frac{K}{2(1-Y)} h^{-\frac{1}{Y}} \left( h^{-\frac{1}{Y}} + \varepsilon \right)^{2-2Y} \int_{-\infty}^0 \phi(x) \left\{ \left( 1 - \frac{\sigma h^{\frac{1}{Y}}}{\varepsilon} x \right)^{2(1-Y)} - \left( -\frac{\sigma h^{\frac{1}{Y}}}{\varepsilon} x \right)^{2(1-Y)} \right\} \, dx \sim Kh^2 \varepsilon^{2-2Y},
\]

where again we used dominated convergence and use the fact that \( \int_{-\infty}^0 \phi(x)(1 - x)^{2(1-Y)} \, dx < \infty \). For \( Y > 1 \), we just use (37) to deduce that

\[
h^{-\frac{1}{Y}} \int_{-\infty}^0 \phi(x) \int_{-\sigma h^{\frac{1}{Y}} - x}^{h^{-\frac{1}{Y}} + \varepsilon - \sigma h^{\frac{1}{Y}} + x} u^2 |\mathcal{E}(u)| \, du \, dx \leq K'h^{-\frac{1}{Y}} \int_{-\infty}^0 \phi(x) \, dx,
\]

for a constant \( K' \). Finally, we conclude that

\[
E_{2,h} = 2^{-1} \frac{C}{2-Y} h \varepsilon^{2-Y} + O \left( h^2 \varepsilon^{2-2Y} \right) + O \left( h^\frac{2}{Y} \right).
\]
Finally, let us consider

\[ E_{3,h}(\varepsilon) := E \left[ J_h^2 1_{(0 \leq \sigma W_h + J_h \leq \varepsilon, J_h \leq 0, W_h \geq 0)} \right] \]

\[ = h^{2/\gamma} \int_0^{\sigma^{-1} h^{-\frac{1}{2}} \varepsilon} \phi(x) \int_0^{\sigma h^{-\frac{1}{2}} + x} u^2 p_1(u) du dx + h^{2/\gamma} \int_{\sigma^{-1} h^{-\frac{1}{2}} \varepsilon}^{\infty} \phi(x) \int_{\sigma h^{-\frac{1}{2}} + x}^{\infty} u^2 p_1(u) du dx. \]

Using the fact that \( p_1(u) \leq K u^{-\gamma - 1} \) for a constant \( K \) and all \( u > 0 \), the first term above is such that

\[ h^{2/\gamma} \int_0^{\sigma^{-1} h^{-\frac{1}{2}} \varepsilon} \phi(x) \int_0^{\sigma h^{-\frac{1}{2}} + x} u^2 p_1(u) du dx \leq K h^{2/\gamma} \int_0^{\sigma^{-1} h^{-\frac{1}{2}} \varepsilon} \phi(x) \int_0^{\sigma h^{-\frac{1}{2}} + x} u^{1-\gamma} du dx \]

\[ = \frac{K}{2 - \gamma} \left( \sigma h^{-\frac{1}{2}} + x \right)^{2 - \gamma} \int_0^{\sigma^{-1} h^{-\frac{1}{2}} \varepsilon} \phi(x) x^{2-\gamma} dx \]

\[ \sim \frac{K}{2 - \gamma} h^{\frac{4-\gamma}{\gamma}} \int_0^{\infty} \phi(x) x^{2-\gamma} dx = o \left( h^{\frac{4-\gamma}{\gamma}} \right). \]

Similarly, the second term can be written as

\[ h^{2/\gamma} \int_{\sigma^{-1} h^{-\frac{1}{2}} \varepsilon}^{\infty} \phi(x) \int_{\sigma h^{-\frac{1}{2}} + x}^{\infty} u^2 p_1(u) du dx \leq \frac{K}{2 - \gamma} \left( \sigma h^{-\frac{1}{2}} + x \right)^{2 - \gamma} \int_{\sigma^{-1} h^{-\frac{1}{2}} \varepsilon}^{\infty} \phi(x) x^{2-\gamma} dx \]

\[ = o \left( h^{\frac{4-\gamma}{\gamma}} \right) = o \left( h^{2-\gamma} \right). \]

Putting together the previous results, we obtain that

\[ E_h(\varepsilon) = 2E \left[ J_h^2 1_{(0 \leq \sigma W_h + J_h \leq \varepsilon)} \right] = 2E_{1,h}(\varepsilon) + 2E_{2,h}(\varepsilon) + 2E_{3,h}(\varepsilon) \]

\[ = \frac{2C}{2 - \gamma} h^{1/\gamma} + O \left( h^{2-\gamma \varepsilon} \right) + O \left( h^{1-\gamma} \right) + O \left( h^{\frac{4-\gamma}{\gamma}} \right). \]

**Proof of Theorem 4.** From Lemmas 1 and 2,

\[ C_h^+(\varepsilon) = E \left[ \left( \frac{\varepsilon}{\sigma h} - \frac{J_h}{\sigma h} \right) \phi \left( \frac{\varepsilon}{\sigma h} - \frac{J_h}{\sigma h} \right) + \Phi \left( \frac{\varepsilon}{\sigma h} - \frac{J_h}{\sigma h} \right) \right] \]

\[ = \frac{\varepsilon}{\sigma \sqrt{h} \sqrt{2\pi}} e^{-\frac{\varepsilon^2}{2h}} - K_1 \varepsilon^{-1-\gamma} h^{\frac{2}{\gamma}} - \frac{1}{\sigma \sqrt{h}} (K_2 \varepsilon^{-1-\gamma}) + \frac{C}{\sqrt{h}} \varepsilon^{-1-\gamma} + h.o.t. \]

\[ = \frac{\varepsilon}{\sigma \sqrt{h} \sqrt{2\pi}} e^{-\frac{\varepsilon^2}{2h}} - K_2 \frac{h^{1/\gamma}}{\gamma} + h.o.t., \]

where above we used that \( \varepsilon^{-\gamma} h \ll h^{1/\gamma} \varepsilon^{-1-\gamma} \). Therefore, using that \( D_h = 0 \) and Lemma 2, with \( K_3 = \frac{2C}{2 - \gamma} \),

\[ E[h_1(\varepsilon)] = E \left[ (\sigma W_h + J_h)^2 1_{(\sigma W_h + J_h \leq \varepsilon)} \right] = C_h(\varepsilon) + D_h(\varepsilon) + E_h(\varepsilon) \]

\[ = \sigma^2 h - 2\sigma^2 \frac{\varepsilon}{\sqrt{h} \sqrt{2\pi}} e^{-\frac{\varepsilon^2}{2h}} - K_2 \frac{h^{1/\gamma}}{\gamma} \varepsilon^{-1-\gamma} + K_3 h^{2-\gamma} + h.o.t. \]

\[ = \sigma^2 h - 2\sigma^2 \frac{\varepsilon}{\sqrt{h} \sqrt{2\pi}} \sqrt{h} e^{-\frac{\varepsilon^2}{2h}} + K_3 h^{2-\gamma} + h.o.t., \]

where above we used that \( h^{2-\gamma} \gg h^{3/2} \varepsilon^{-1-\gamma} \).}

**Proof of Lemma 3.** We show the result by contradiction. Suppose that \( \liminf_{n \to \infty} \frac{\varepsilon^*}{\sqrt{n}} < \infty \). For simplicity and without loss of generality, we further assume that \( \lim_{n \to \infty} \frac{\varepsilon^*}{\sqrt{n}} =: L < \infty \) as all the statements below are valid
on a subsequence \( \{n_k\}_{k \geq 0} \). Let \( M \in (0, \infty) \) be such that \( \sup_n \frac{\epsilon_n}{\sqrt{n}} \leq M \). Also, for simplicity, let us write \( \epsilon_n \) for \( \epsilon_n^* \) and assume that \( T = 1 \) so that \( h_n = 1/n \). Consider the decomposition

\[
E[b_1(\epsilon)] = E \left[ (\sigma W_h + J_h)^2 1_{\{\sigma W_h + J_h \leq \epsilon\}} \right] = \epsilon^2 E[ W_h^2 1_{\{\sigma W_h + J_h \leq \epsilon\}} ] + 2\sigma E[ W_h J_h 1_{\{\sigma W_h + J_h \leq \epsilon\}} ] + \epsilon E[ J_h^2 1_{\{\sigma W_h + J_h \leq \epsilon\}} ] =: c_h(\epsilon) + d_h(\epsilon) + e_h(\epsilon).
\]

Note that dominated convergence implies that

\[
\frac{1}{h_n} c_h(\epsilon_n) = \sigma^2 E\left[ W_1^2 1_{\{\sigma W_1 + h_n^{-1/2} J_{h_n} \leq h_n^{-1/2} \epsilon_n\}} \right] \xrightarrow{n \to \infty} \sigma^2 E\left[ W_1^2 1_{\{|W_1| \leq L/\sigma\}} \right] < \sigma^2,
\]

since \( h_n^{-1/2} J_{h_n} = h_n^{1/2} (\epsilon_n^{-1/2} J_{h_n}) \to 0 \), in probability. For \( d_h \) note that

\[
\sigma |W_1 h_n^{-1/2} J_{h_n}| 1_{\{\sigma W_1 + h_n^{-1/2} J_{h_n} \leq h_n^{-1/2} \epsilon_n\}} \leq \sigma^2 |W_1|^2 + \sigma |W_1| h_n^{-1/2} \epsilon_n \leq \sigma^2 |W_1|^2 + \sigma |W_1| M,
\]

therefore, again by dominated convergence

\[
\frac{1}{h_n} d_h(\epsilon_n) = 2\sigma E\left[ W_1 h_n^{-1/2} J_{h_n} 1_{\{\sigma W_1 + h_n^{-1/2} J_{h_n} \leq h_n^{-1/2} \epsilon_n\}} \right] \xrightarrow{n \to \infty} 0.
\]

Similarly, since \( (h_n^{-1/2} J_{h_n})^2 1_{\{\sigma W_1 + h_n^{-1/2} J_{h_n} \leq h_n^{-1/2} \epsilon_n\}} \leq 2\sigma^2 W_1^2 + 2h_n^{-1} \epsilon_n \leq 2W_1^2 + 2M^2 \),

\[
\frac{1}{h_n} e_h(\epsilon_n) = E\left[ (h_n^{-1/2} J_{h_n})^2 1_{\{\sigma W_1 + h_n^{-1/2} J_{h_n} \leq h_n^{-1/2} \epsilon_n\}} \right] \xrightarrow{n \to \infty} 0.
\]

Finally, let us write the equation

\[
\epsilon^2 + 2(n - 1)E[b_1(\epsilon)] - nh_n \sigma^2 = 0
\]

as

\[
\epsilon^2 + 2(n - 1) \left( \frac{d_h(\epsilon)}{h_n} + \frac{e_h(\epsilon)}{h_n} \right) = 2\sigma^2 - 2 \frac{n - 1}{n} c_h(\epsilon_n) = 0,
\]

The right-hand side of the equation converges to \( 2\sigma^2 \left( 1 - E \left[ W_1^2 1_{\{|W_1| \leq L/\sigma\}} \right] \right) > 0 \), while the left hand side converges to 0 and this leads to a contradiction and therefore \( \lim_{n \to \infty} \frac{\epsilon_n}{\sqrt{h_n}} = \infty. \)

**Proof of Proposition 1.** For simplicity, in what follows we take \( T = 1 \) so that \( h = 1/n \). Again, recall that \( \epsilon^* \) is the solution of

\[(\epsilon^*)^2 + 2(n - 1)E[b_1(\epsilon^*)] - 2n h \sigma^2 = 0.\]

Throughout, we shall use that \( \epsilon^* \gg \sqrt{h} \), as proved in the above lemma. For simplicity, we write \( \epsilon \) instead of \( \epsilon^* \). By the asymptotic behavior of \( E[b_1(\epsilon)] \) described above,

\[
\epsilon^2 + 2(n - 1) \left( \sigma^2 h - \frac{2}{\sqrt{2\pi}} \sigma \epsilon \sqrt{h} e^{-\frac{\epsilon^2}{2h}} + \lambda \frac{\epsilon^3}{3} C(f) + O(h^2) + o \left( \frac{\epsilon \sqrt{h} e^{-\frac{\epsilon^2}{2h}}}{h} \right) \right) - 2n h \sigma^2 = 0,
\]

and, thus, using that \( h = 1/n \),

\[
\epsilon^2 - 2\sigma^2 h - \frac{4}{\sqrt{2\pi}} \sigma \sqrt{h} \epsilon e^{-\frac{\epsilon^2}{2h}} + 2\lambda \frac{\epsilon^3}{3} C(f) + O(h) + o \left( \frac{\epsilon \sqrt{h} e^{-\frac{\epsilon^2}{2h}}}{h} \right) + o(\epsilon^3) = 0.
\]

Now, since \( h = o(\epsilon^2) \) (as assumed at the beginning), we can write the previous equation as

\[
\epsilon^2 - \frac{4}{\sqrt{2\pi}} \sigma \sqrt{h} \epsilon e^{-\frac{\epsilon^2}{2h}} + o \left( \frac{\epsilon \sqrt{h} e^{-\frac{\epsilon^2}{2h}}}{h} \right) + o(\epsilon^3) = 0.
\]

Dividing by \( \epsilon \) and rearranging the terms,

\[
\epsilon (1 + o(1)) = \frac{4}{\sqrt{2\pi}} \sigma \frac{1}{\sqrt{h}} e^{-\frac{\epsilon^2}{2h}} (1 + o(1)).
\]
Then, taking logarithms of both sides and since \( \ln(1 + o(1)) = o(1) \),
\[
\ln \varepsilon + o(1) = -\frac{\varepsilon^2}{2\sigma^2 h} - \frac{1}{2} \ln h + \ln \left( \frac{4\sigma}{\sqrt{2\pi}} \right) + o(1).
\] (41)

which can be written as
\[
\ln \left( \frac{\varepsilon^2}{\sigma^2 h} \right) + o(1) = -\frac{\varepsilon^2}{\sigma^2 h} - 2 \ln h + \ln \left( \frac{8}{\pi} \right) + o(1)
\]
Defining \( \varpi = \varepsilon^2/(\sigma^2 h) \), we can write
\[
-\frac{\ln \varpi}{\varpi} + 2 \frac{\ln \frac{1}{h}}{\varpi} - \frac{\ln \frac{\pi}{\varpi}}{h} + o(1) = 1 + \frac{o(1)}{\varpi}.
\]
Therefore, making \( h \to 0 \) and using that \( \varpi \to \infty \) (since \( \varepsilon \gg \sqrt{h} \)),
\[
\frac{2 \ln \frac{1}{\varpi} h \to 0}{\varpi} \to 1.
\]
Recalling that \( \varpi = \varepsilon^2/(\sigma^2 h) \), we conclude the result.

**Proof of Proposition 2.** For simplicity, we again take \( T = 1 \) so that \( h = 1/n \) and write \( \varepsilon \) instead of \( \epsilon^* \). By the asymptotic behavior of \( \mathbb{E} \left[ b_1(\varepsilon) \right] \) described in Theorem 4, we can write \( (\epsilon^*)^2 + 2(n-1)\mathbb{E}[b_1(\epsilon^*)] - 2nh\sigma^2 = 0 \) as
\[
\varepsilon^2 + 2(n-1) \left( \sigma^2 h - \frac{2\sigma}{\sqrt{2\pi}} \varepsilon \sqrt{h} e^{-\frac{\varepsilon^2}{2\sigma^2 h}} + \frac{2C}{2-Y} h \varepsilon^{2-Y} + \text{h.o.t.} \right) - 2nh\sigma^2 = 0,
\]
and, thus, using that \( h = o(\varepsilon^2) \) and \( \varepsilon^2 = o(\varepsilon^{2-Y}) \), we have
\[
\frac{4C}{2-Y} \varepsilon^{2-Y} - \frac{4}{\sqrt{2\pi}} \frac{\varepsilon}{\sqrt{h}} e^{-\frac{\varepsilon^2}{2\sigma^2 h}} + o \left( \frac{\varepsilon}{\sqrt{h}} e^{-\frac{\varepsilon^2}{2\sigma^2 h}} \right) + o \left( \varepsilon^{2-Y} \right) = 0.
\] (42)
Dividing by \( \varepsilon \) and rearranging the terms,
\[
\varepsilon^{1-Y} (1 + o(1)) = \frac{2-Y}{C \sqrt{2\pi}} \frac{1}{\sqrt{h}} e^{-\frac{\varepsilon^2}{2\sigma^2 h}} \left( 1 + o(1) \right).
\]
Then, taking logarithms of both sides and since \( \ln(1 + o(1)) = o(1) \),
\[
(1 - Y) \ln \varepsilon + o(1) = -\frac{\varepsilon^2}{2\sigma^2 h} - \frac{1}{2} \ln h + \ln \left( \frac{(2-Y)\sigma}{C \sqrt{2\pi}} \right) + o(1),
\]
which can be written as
\[
-\frac{1}{2} \ln \left( \frac{\varepsilon^2}{\sigma^2 h} \right) + \frac{1}{2} \ln (\sigma^2) + \frac{1}{2} \ln (h) + o(1) = -\frac{\varepsilon^2}{2\sigma^2 h} - \frac{1}{2} \ln h + \ln \left( \frac{(2-Y)\sigma}{C \sqrt{2\pi}} \right) + o(1).
\]
Equivalently, writing \( \varpi = \varepsilon^2/(\sigma^2 h) \) and dividing by \( -\varpi \),
\[
-(1 - Y) \ln \frac{\varpi}{\varpi} + \frac{(2-Y) \ln \frac{1}{h}}{\varpi} - \frac{K}{\varpi} = 1 + \frac{o(1)}{\varpi},
\]
and using that \( \varpi \to \infty \) (since \( \varepsilon \gg \sqrt{h} \)), we get
\[
\frac{(2-Y) \ln \frac{1}{h} h \to 0}{\varpi} \to 1.
\]
Recalling that \( \varpi = \varepsilon^2/(\sigma^2 h) \), we conclude the result.

**Proof of Proposition 3.** In order to prove the proposition, we follow and modify the proof of Theorem 1 in [17], in that we show that a.s., for all \( \eta > 0 \), for sufficiently small \( h \), we have
1) \( \forall i = 1, \ldots, n, I_{\{\Delta, N=0\}} \leq I_{\{(\Delta, X)^2 \leq (1+\eta)\tau_i(h)\}} \)
2) \( \forall i = 1, \ldots, n, I_{\{\Delta, N=0\}} \geq I_{\{(\Delta, X)^2 \leq (1+\eta)\tau_i(h)\}} \).

Then the thesis follows.

Call \( \Delta_i X_0 = \int_{t_{i-1}}^{t_i} a_s ds + \int_{t_{i-1}}^{t_i} \sigma_s dW_s, \bar{a} = \sup_{s \in [0,T]} |a_s|, \bar{a} = \sup_{s \in [0,T]} \sigma_s \) and \( \gamma(\omega) = \min_{\Delta N \neq 0} |\gamma_\ell(\omega)| \), and note that under our assumptions \( P(\xi \neq 0) = 1 \). To show 1) a 2) we use the following key fact:

\[
\sup_{i \in \{1, \ldots, n\}} \frac{|\Delta_i X_0|}{\sqrt{2 M_i h \log \frac{1}{h}}} \leq \sup_{i} \frac{\bar{a} \sqrt{h}}{\sqrt{2 M_i \log \frac{1}{h}}} + \sup_{i} \frac{\log \frac{1}{\xi}}{\log \frac{1}{h}} + 1
\]

\[
\leq M_h := \frac{\bar{a} \sqrt{h}}{\sqrt{2 \sigma^2 \log \frac{1}{h}}} + \sqrt{\frac{\log \frac{1}{\xi}}{\log \frac{1}{h}}} + 1
\]

which tends to 1, as \( h \to 0 \).

Now, in order to show 1), we define \( \{J\} = \{i \in \{1, 2, \ldots, n\} : \Delta_i N \neq 0\} \), and it is sufficient to prove that for small enough \( \sup_{\ell \in J} \frac{|\Delta_i X|}{\sqrt{\tau_i(h)}} \leq 1 + \eta \). Indeed, \( \sup_{\ell \in J} \frac{|\Delta_i X|}{\sqrt{\tau_i(h)}} = \sup_{\ell \in J} \frac{|\Delta_i X_0|}{\sqrt{\tau_i(h)}} \leq \sup_{i \in \{1, \ldots, n\}} \frac{|\Delta_i X_0|}{\sqrt{\tau_i(h)}} \leq M_h \to 1 \), thus for all \( \eta > 0 \) for sufficiently small \( h \), it is ensured that \( \sup_{\ell \in J} \frac{|\Delta_i X|}{\sqrt{\tau_i(h)}} < 1 + \eta \), that is: for all \( i \), if \( \Delta_i N = 0 \) then necessarily we have \( |\Delta_i X| < (1 + \eta)\sqrt{\tau_i(h)} \), and 1) follows.

In order to show 2) we prove that, for sufficiently small \( h \), \( \inf_{i \in \{J\}} \frac{|\Delta_i X|}{\sqrt{\tau_i(h)}} > 1 + \eta \). In fact firstly note that for sufficiently small \( h \) all the increments of \( N \) are either 0 or 1. It follows that if \( \Delta_i N \neq 0 \), then \( \Delta_i N = 1 \), and \( \Delta_i J \) coincides with the size, say \( \gamma_\ell_i \), of a single jump \( \Delta_i J = \gamma_\ell_i \). Then

\[
\frac{|\Delta_i X|}{\sqrt{\tau_i(h)}} \geq \frac{\gamma_\ell_i}{\sqrt{2 \sigma^2 h \log \frac{1}{h}}} - \sup_{i \in \{J\}} \frac{|\Delta_i X_0|}{\sqrt{2 M_i h \log \frac{1}{h}}} \geq \frac{\gamma_\ell_i}{\sqrt{2 \sigma^2 \log \frac{1}{h}}} - (1 + \eta)
\]

and this tends to \( +\infty \) when \( h \to 0 \), thus \( \inf_{i \in \{J\}} \frac{|\Delta_i X|}{\sqrt{\tau_i(h)}} > 1 + \eta \), meaning that if \( \Delta_i N \neq 0 \) then necessarily \( |\Delta_i X| > \sqrt{\tau_i(h)(1 + \eta)} \), as we needed.

**Proof of Corollary 2.** The proof of the Corollary is straightforward, in that a.s. we fix any \( \eta > 0 \), and for sufficiently small \( h \) we have

\[
\sum_{i=1}^{n} (\Delta_i X)^2 I_{\{(\Delta_i X)^2 \leq (1+\eta)\tau_i(h)\}} = \sum_{i=1}^{n} (\Delta_i X)^2 I_{\{(\Delta, X)^2 \leq (1+\eta)\tau_i(h)\}} = \sum_{i=1}^{n} (\Delta_i X_0)^2 - \sum_{i=1}^{n} (\Delta_i X)^2 I_{\{(\Delta, X)^2 \leq (1+\eta)\tau_i(h)\}} \overset{P}{\to} IV_T,
\]

since the last term tends to 0 in probability, as \( E[\sum_{i=1}^{n} (\Delta_i X_0)^2 I_{\{(\Delta, X)^2 \leq (1+\eta)\tau_i(h)\}}] \leq N_T O(h) \to 0 \).
Proof of Proposition 4. Recall that \( \varepsilon \) is such that \( \sum_{i=1}^{n} a_i g_i = 0 \), i.e. \( \sum_{i=1}^{n} a_i (\varepsilon^2 + 2 \sum_{j \neq i} b_j - 2IV) = 0 \). For simplicity let us rename \( \varepsilon \) by \( \varepsilon \). If \( \lim_{h \to 0} \frac{\varepsilon(h)}{\sqrt{h}} = L \in [0, +\infty) \) we can find a subsequence such that \( \lim_{h \to 0} \frac{\varepsilon(h)}{\sqrt{h}} = L \). Note that

\[
0 = \sum_{i=1}^{n} a_i \left( \frac{\varepsilon^2}{h} + 2 \frac{\sum_{j \neq i} b_j}{h} - 2\sigma^2 n \right) = \frac{\varepsilon^2}{h} \sum_{i=1}^{n} a_i + \frac{2}{h} \sum_{i=1}^{n} a_i \sum_{j \neq i} b_j - 2\sigma^2 n \sum_{i=1}^{n} a_i.
\]
i.e.

\[
\frac{\varepsilon^2}{h} = 2\sigma^2 n - 2 \sum_{i=1}^{n} a_i \sum_{j \neq i} b_j \sum_{i=1}^{n} a_i = 2n \left[ \sigma^2 - \frac{\sum_{i=1}^{n} a_i \sum_{j \neq i} b_j}{\sum_{i=1}^{n} a_i} \right].
\] (43)
Now we show that \( \sigma^2 - \frac{\sum_{i=1}^{n} a_i \sum_{j \neq i} b_j}{\sum_{i=1}^{n} a_i} \) tends to a strictly positive constant, which in turn means that equality (43) is impossible, since on any sequence \( \varepsilon(h) \) such that \( \frac{\varepsilon(h)}{\sqrt{h}} \to L \) the left term tends to \( L^2 \), while the right one tends to \( +\infty \).

Let us then check that \( \sigma^2 - \frac{\sum_{i=1}^{n} a_i \sum_{j \neq i} b_j}{\sum_{i=1}^{n} a_i} \) tends to a strictly positive constant. Since \( J \) has FA, a.s. we only have finitely many \( \Delta J_t \neq 0 \), and, for small \( h \), \( N_T \) coincides with \( \sum_{i=1}^{n} I_{m_i} \neq 0 \). Recalling the explicit expression of \( b_j \) (also reported below), we have

\[
\sum_{j \neq i} b_j = \sum_{j \neq i, m_j = 0} b_j + \sum_{j \neq i, m_j \neq 0} b_j \leq -(n - N_T) \frac{\sigma \sqrt{h}}{\sqrt{2\pi}} 2\varepsilon e^{-\frac{\varepsilon^2}{2\sigma^2 h}} + (n - N_T) \frac{\sigma^2 h}{\sqrt{2\pi}} \int_{\frac{m_j + \sigma \sqrt{h}}{\sqrt{2\pi}}}^{\frac{m_j - \varepsilon}{\sqrt{2\pi}}} e^{-\frac{x^2}{2\sigma^2 h}} dx
\]

\[
- \sum_{j \neq i, m_j \neq 0} \frac{\sigma \sqrt{h}}{\sqrt{2\pi}} \left( \varepsilon \left( e^{-\frac{\varepsilon^2}{2\sigma^2 h}} + e^{-\frac{(\varepsilon + m_j)^2}{2\sigma^2 h}} \right) - |m_j| \left( e^{-\frac{(\varepsilon - m_j)^2}{2\sigma^2 h}} - e^{-\frac{\varepsilon^2}{2\sigma^2 h}} \right) \right) + \sum_{j \neq i, m_j \neq 0} \frac{m_j^2 + \sigma^2 h}{\sqrt{2\pi}} \int_{\frac{m_j - \varepsilon}{\sqrt{2\pi}}}^{\frac{m_j + \sigma \sqrt{h}}{\sqrt{2\pi}}} e^{-\frac{x^2}{2\sigma^2 h}} dx.
\]
Now, the factors \( \varepsilon e^{-\frac{\varepsilon^2}{2\sigma^2 h}} \) and \( \varepsilon \left( e^{-\frac{(\varepsilon + m_j)^2}{2\sigma^2 h}} + e^{-\frac{\varepsilon^2}{2\sigma^2 h}} \right) \) are strictly positive, so

\[
\sum_{j \neq i} b_j \leq -(n - N_T) \frac{\sigma^2 h}{\sqrt{2\pi}} \int_{\frac{m_j + \sigma \sqrt{h}}{\sqrt{2\pi}}}^{\frac{m_j - \varepsilon}{\sqrt{2\pi}}} e^{-\frac{x^2}{2\sigma^2 h}} dx + \sum_{j \neq i, m_j \neq 0} \frac{m_j^2 + \sigma^2 h}{\sqrt{2\pi}} \int_{\frac{m_j - \varepsilon}{\sqrt{2\pi}}}^{\frac{m_j + \sigma \sqrt{h}}{\sqrt{2\pi}}} e^{-\frac{x^2}{2\sigma^2 h}} dx,
\]
where if \( \frac{\varepsilon(h)}{\sqrt{h}} \to L \) as \( h \to 0 \) then the first term of the rhs above tends to \( \frac{\sigma^2}{\sqrt{2\pi}} \int_{\frac{L}{\sqrt{2\pi}}}^{\frac{L}{\sqrt{2\pi}}} e^{-\frac{x^2}{2\sigma^2}} dx < \sigma^2 \), while each term of the latter finite sum tends to 0, since \( \frac{|m_j|}{\sqrt{h}} \to \infty \), so the finite sum tends to 0. It follows that, for all \( i \), \( \sum_{j \neq i} b_j \leq d + o(1) \), where \( d < \sigma^2 \), so \( \frac{\sum_{i=1}^{n} a_i \sum_{j \neq i} b_j}{\sum_{i=1}^{n} a_i} \geq \sigma^2 - d + o(1) \to \sigma^2 - d > 0 \), as we wanted.

\[\blacksquare\]

Proof of Proposition 5. Let us start by checking the asymptotic behavior of \( b_i(\varepsilon) \) and \( a_i(\varepsilon) \) when \( \varepsilon = \varepsilon(h) \) tends to 0 as \( h \to 0 \) in such a way that \( \frac{\varepsilon(h)}{\sqrt{h}} \to +\infty \). To this end, for fixed \( \sigma \), we define

\[
b(\varepsilon, m, h) := \frac{\sigma \sqrt{h}}{\sqrt{2\pi}} \left( e^{-\frac{(\varepsilon - m)^2}{2\sigma^2 h}} (\varepsilon + m) + e^{-\frac{(\varepsilon + m)^2}{2\sigma^2 h}} (\varepsilon - m) \right) + \frac{m^2 + \sigma^2 h}{\sigma \sqrt{2\pi}} \int_{\frac{m - \varepsilon}{\sqrt{2\pi}}}^{\frac{m + \sigma \sqrt{h}}{\sqrt{2\pi}}} e^{-\frac{x^2}{2\sigma^2 h}} dx
\]

\[
a(\varepsilon, m, h) := \frac{e^{-\frac{(\varepsilon - m)^2}{2\sigma^2 h}} + e^{-\frac{(\varepsilon + m)^2}{2\sigma^2 h}}}{\sigma \varepsilon \sqrt{2\pi}},
\]
and note that \( b_i(\varepsilon) = b(\varepsilon, m_i, h) \) and \( a_i(\varepsilon) = a(\varepsilon, m_i, h) \). Let us also remark that

\[
b(\varepsilon, m, h) = \begin{cases} \sigma^2 h - \frac{2\sigma^2 \varepsilon^2}{\sqrt{2\pi}} + h.o.t., & \text{if } m = 0, \\ \frac{\sigma}{\sqrt{2\pi}} e^{-\frac{|m - \varepsilon|^2}{2\sigma^2 h}} + h.o.t., & \text{if } m \neq 0. \end{cases}
\] (46)

\[
a(\varepsilon, m, h) = \begin{cases} \frac{2}{\sqrt{2\pi}} \frac{1}{\varepsilon \sqrt{h}} e^{-\frac{\varepsilon^2}{2\sigma^2 h}}, & \text{if } m = 0, \\ \frac{1}{\varepsilon \sqrt{2\pi}} e^{-\frac{|m - \varepsilon|^2}{2\sigma^2 h}} + h.o.t., & \text{if } m \neq 0. \end{cases}
\] (47)
The asymptotic behavior for $a(\varepsilon, m, h)$ is direct, while that for $b(\varepsilon, m, h)$ is shown below. For simplicity, in what follows, we omit the dependence on $h$ in the functions $a(\varepsilon, m, h)$ and $b(\varepsilon, m, h)$ defined above. Let us recall that, under Assumption A4', $N_t$ is the number of jumps by time $t$, $\{\gamma_t\}_{t \geq 1}$ are the consecutive jumps of $J$ and $\{J\}_{(n)} := \{ i : \Delta^n_i N \neq 0 \}$. It follows that, for $h$ is small enough, $F(\varepsilon,h) = \sum_{i=1}^{n} a(\varepsilon, m_i) g_i(\varepsilon)$ can be written as

\[
F(\varepsilon,h) = \sum_{i \notin \{J\}} a(\varepsilon, m_i) \left( \varepsilon^2 + 2 \sum_{j \notin i \notin \{J\}} b(\varepsilon, m_j) + 2 \sum_{j \notin i \notin \{J\}} b(\varepsilon, m_j) - 2IV \right) \\
+ \sum_{i \in \{J\}} a(\varepsilon, m_i) \left( \varepsilon^2 + 2 \sum_{j \notin i \notin \{J\}} b(\varepsilon, m_j) + 2 \sum_{j \notin i \notin \{J\}} b(\varepsilon, m_j) - 2IV \right) \\
= (n - N_T) a(\varepsilon, 0) \left[ \varepsilon^2 - 2h\sigma^2(N_T + 1) + 2 \left( \sum_{k=1}^{N_T} b(\varepsilon, \gamma_k) + (n - N_T - 1)(b(\varepsilon, 0) - \sigma^2 h) \right) \right] + \\
+ \sum_{\ell=1}^{N_T} a(\varepsilon, \gamma_\ell) \left[ \varepsilon^2 - 2h\sigma^2 N_T + 2 \left( \sum_{k \neq \ell} b(\varepsilon, \gamma_k) + (n - N_T)(b(\varepsilon, 0) - \sigma^2 h) \right) \right] \\
= (n - N_T) \frac{2}{\sigma \sqrt{h} \sqrt{2\pi}} \varepsilon^2 e^{-\frac{\varepsilon^2}{2\sigma^2 h}} \left[ \varepsilon^2 - 2h\sigma^2(N_T + 1) - 4(n - N_T - 1) \frac{\varepsilon \sqrt{h}}{\sqrt{2\pi}} e^{-\frac{\varepsilon^2}{2\sigma^2 h}} + \right. \\
\left. + 2 \sum_{k=1}^{N_T} \frac{\sigma}{|\gamma_k|} \sqrt{\frac{h}{\pi}} e^{-\frac{(\gamma_k - \varepsilon)^2}{2\sigma^2 h}} \right] + \\
+ \sum_{\ell=1}^{N_T} \frac{1}{\sigma \sqrt{h} \sqrt{2\pi}} e^{-\frac{(|\gamma_\ell - \varepsilon|^2)}{2\sigma^2 h}} \left[ \varepsilon^2 - 2h\sigma^2 N_T - 4(n - N_T) \frac{\varepsilon \sqrt{h}}{\sqrt{2\pi}} e^{-\frac{\varepsilon^2}{2\sigma^2 h}} + \right. \\
\left. + 2 \sum_{k \neq \ell} \frac{\sigma}{|\gamma_k|} \sqrt{\frac{h}{\pi}} e^{-\frac{(\gamma_k - \varepsilon)^2}{2\sigma^2 h}} \right] + \text{h.o.t.}.
\]

In what follows we use the following notation:

\[
v_h = \frac{\varepsilon_h}{\sqrt{h}}, \quad u_{th} = \frac{1}{\sqrt{2\pi}} e^{-\frac{(\gamma_t - \varepsilon)^2}{2\sigma^2 h}}, \quad s_h = \frac{1}{\sqrt{2\pi}} e^{-\frac{\gamma^2_h}{2\sigma^2 h}}, \quad p_{th} = e^{-\frac{(\gamma_t - \varepsilon)^2}{2\sigma^2 h}}.
\]

Now, since $u_{th} = s_h p_{th}$ and $p_{th} \to 0$, as $h \to 0$,

\[
F(\varepsilon,h) = (n - N_T) \frac{2}{\sigma} \sqrt{h} s_h \left[ \varepsilon_h^2 - 2\sigma^2(N_T + 1) + 2\sigma v_h s_h \left( \varepsilon \sum_{k=1}^{N_T} \frac{1}{|\gamma_k|} p_{kh} - 2(n - N_T - 1) \right) \right] + \\
\frac{1}{\sigma} \sqrt{h} s_h \sum_{\ell=1}^{N_T} p_{th} \left[ \varepsilon_h^2 - 2\sigma^2 N_T + 2\sigma v_h s_h \left( \varepsilon \sum_{k \neq \ell} \frac{1}{|\gamma_k|} p_{kh} - 2(n - N_T) \right) \right] + \text{h.o.t.} \\
= (n - N_T) \frac{2}{\sigma} \sqrt{h} s_h \left[ \varepsilon_h^2 - 4\sigma v_h s_h \right] + \frac{1}{\sigma} \sqrt{h} s_h \sum_{\ell=1}^{N_T} p_{th} \left[ \varepsilon_h^2 - 4\sigma v_h s_h \right] + \text{h.o.t.} \\
= \left( n - N_T + \frac{1}{2} \sum_{\ell=1}^{N_T} p_{th} \right) \frac{2}{\sigma} \sqrt{h} s_h \left[ \varepsilon_h^2 - 4\sigma v_h s_h \right] + \text{h.o.t.} \\
= \frac{2n}{\sigma} \sqrt{h} s_h v_h \left[ \varepsilon_h - 4\sigma s_h \right] + \text{h.o.t.} \\
= \frac{2\varepsilon_h}{h} e^{-\frac{\varepsilon_h^2}{2\sigma^2 h}} \left[ \varepsilon_h - \frac{1}{\sqrt{2\pi}} \sqrt{\frac{h}{\pi}} \frac{4\sigma}{\sqrt{2\pi}} \frac{1}{\sigma} \right] + \text{h.o.t.} \quad \Box
\]
Proof of (46). Let
\[ \tilde{N}(x) = \int_x^\infty \phi(z) dz, \quad R(x) = \int_x^\infty \phi(z) dz - \frac{\phi(x)}{x}, \]
and recall that, for \( x > 0 \),
\[ \tilde{N}(x) \leq \frac{1}{x} \phi(x), \quad |R(x)| \leq \frac{\phi(x)}{x^3}. \]
Then, for fixed \( m > 0 \) and \( h \) small such that \( \varepsilon_h < m \), we have
\[
b(\varepsilon, m, h) = \sigma \sqrt{h} \phi \left( \frac{m - \varepsilon}{\sigma \sqrt{h}} \right) \left( \frac{m^2}{m - \varepsilon} - m \right) - \sigma \sqrt{h} \phi \left( \frac{m + \varepsilon}{\sigma \sqrt{h}} \right) \left( \frac{m^2}{m + \varepsilon} - m \right)
- \sigma \sqrt{h} \phi \left( \frac{m - \varepsilon}{\sigma \sqrt{h}} \right) \varepsilon - \sigma \sqrt{h} \phi \left( \frac{m + \varepsilon}{\sigma \sqrt{h}} \right) \varepsilon
+ \sigma^3 h^{3/2} \phi \left( \frac{m - \varepsilon}{\sigma \sqrt{h}} \right) \left( \frac{1}{m - \varepsilon} \right) - \sigma^3 h^{3/2} \phi \left( \frac{m + \varepsilon}{\sigma \sqrt{h}} \right) \left( \frac{1}{m + \varepsilon} \right) \pm (m^2 + \sigma^2 h) R \left( \frac{m \mp \varepsilon_h}{\sigma \sqrt{h}} \right) \varepsilon^3
i = \frac{\sigma}{m(h - \varepsilon)} \sqrt{h} \phi \left( \frac{m - \varepsilon}{\sigma \sqrt{h}} \right) \varepsilon^2 - \sigma \frac{m}{m(h + \varepsilon)} \sqrt{h} \phi \left( \frac{m + \varepsilon}{\sigma \sqrt{h}} \right) \varepsilon^2
+ \frac{\sigma}{m(m - \varepsilon)} \sqrt{h} \phi \left( \frac{m - \varepsilon}{\sigma \sqrt{h}} \right) \varepsilon^3
- \frac{\sigma}{m(m + \varepsilon)} \sqrt{h} \phi \left( \frac{m + \varepsilon}{\sigma \sqrt{h}} \right) \varepsilon^3
+ \frac{\sigma^3}{m - \varepsilon} h^{3/2} \phi \left( \frac{m - \varepsilon}{\sigma \sqrt{h}} \right) \left( \frac{m + \varepsilon}{m + \varepsilon} \right) + (m^2 + \sigma^2 h) R \left( \frac{m \mp \varepsilon_h}{\sigma \sqrt{h}} \right) \varepsilon^3.
\]
It is now clear that (46) holds true. We can similarly deal with the case \( m < 0 \). \( \square \)

Proof of the statement in Remark 5. For nonzero drift, by conditioning also on the drift process \( a \), we have that
\[
F(\varepsilon_h) = \sum_{i \notin \{J\}} a(\varepsilon, \bar{h}^i) \left[ \varepsilon^2 - 2h \sigma^2 (N_T + 1) + 2 \left( \sum_{k=1}^{N_T} b(\varepsilon, \gamma_k + \bar{h}^i_h) + \sum_{j \neq i, j \notin \{J\}} (b(\varepsilon, h\bar{a}_j) - \sigma^2 h) \right) \right] + \sum_{j=1}^{N_T} a(\varepsilon, \gamma_j + \bar{h}^i_h) \left[ \varepsilon^2 - 2h \sigma^2 N_T + 2 \left( \sum_{k \neq \ell} b(\varepsilon, \gamma_k + \bar{h}^i_h) + \sum_{j \neq i, j \notin \{J\}} (b(\varepsilon, h\bar{a}_j) - \sigma^2 h) \right) \right],
\]
where \( \bar{a}_i = \int_{t=1}^{i} a_s ds / h \) and the indices \( i_1 < i_2 < \cdots < i_{N_T} \) are defined such that \( \Delta_{iJ} \neq 0 \), while \( \Delta_{ii} = 0 \) for any other \( j \notin \{i_1, i_2, \ldots, i_{N_T}\} \). Next, we follow the same arguments as those used in the proof of Proposition 5 but, instead of (46)-(47), we exploit the following asymptotics:
\[
a(\varepsilon, \bar{h}^i_h) = \frac{2}{\sigma} h^{-1/2} \phi \left( \frac{\varepsilon}{\sqrt{h}} \right) + \text{h.o.t.}, \quad a(\varepsilon, \gamma_k + \bar{h}^i_h) = \frac{1}{\sigma} h^{-1/2} \phi \left( \frac{|\gamma_k| - \varepsilon}{\sqrt{h}} \right) e^{-\frac{\gamma_k^2 \varepsilon_h}{2h}} + \text{h.o.t.}
\]
\[
b(\varepsilon, \bar{h}^i_h) = \sigma^2 h - 2\varepsilon \sqrt{h} \phi \left( \frac{\varepsilon}{\sqrt{h}} \right) + \text{h.o.t.}, \quad b(\varepsilon, \gamma_k + \bar{h}^i_h) = \frac{\sigma}{|\gamma_k|} \varepsilon^2 \sqrt{h} \phi \left( \frac{|\gamma_k| - \varepsilon}{\sqrt{h}} \right) e^{-\frac{\gamma_k^2 \varepsilon_h}{2h}} + \text{h.o.t.} \quad \square
\]

Proof of Corollary 3. In fact, from Proposition 4 and (48), we have that
\[
F(\varepsilon_h) = \frac{2}{\sigma} n_s \bar{v}_h \sqrt{h} \left( \bar{v}_h - n \bar{s}_h \cdot 4\sigma \right) + \text{h.o.t.} = 0,
\]
where \( \bar{v}_h := \varepsilon_h / \sqrt{h} \) and \( s_h = \frac{\varepsilon_h^2}{2n_s^2} \). Thus,
\[
\bar{v}_h - n_s \bar{s}_h \cdot 4\sigma + \text{h.o.t.} = 0, \quad (49)
\]
or, equivalently,
\[
\varepsilon_h - \frac{\varepsilon_h^2}{2n_s^2} 4\sigma + \text{h.o.t.} = 0.
\]
which is exactly the condition in (40), entailing that as $h \to 0$

$$
\varepsilon_h \sim \sqrt{2\sigma^2 h \ln \frac{1}{h}}.
$$

\hfill \square

**Proof of Proposition 6** We use the same notation as in (11). Let us fix $h$, and $nh = 1$, then $\frac{d}{d\varepsilon} F(\varepsilon) = \sum_{i=1}^n [a'_i g_i + a_i g'_i]$ 

$$
= -\sum_{i=1}^n \frac{1}{\sigma^3 h^2 \sqrt{2\pi}} \left[ e^{-\frac{(x-m_i)^2}{2\sigma^2 h}} (\varepsilon - |m_i|) + e^{-\frac{(x+|m_i|)^2}{2\sigma^2 h}} (\varepsilon + |m_i|) \right] g_i + \frac{2}{\sigma \sqrt{h} \sqrt{2\pi}} \left[ 2\varepsilon + 2 \sum_{j \neq i} \varepsilon^2 a_j \right].
$$

We now evaluate $F'(\varepsilon)$ at $\varepsilon_h$ such that $\varepsilon_h \to 0$ with $\varepsilon_h \gg \sqrt{h}$, as $h \to 0$. Since again when $m_i \neq 0$ we have $e^{-\frac{(x-m_i)^2}{2\sigma^2 h}} \gg e^{-\frac{(x+|m_i|)^2}{2\sigma^2 h}}$ and $\varepsilon \ll m_i$, then $F'(\varepsilon) \gg \frac{1}{\sqrt{2\pi}} \sum_{j \neq i} \frac{1}{\sigma |m_j|} e^{-\frac{(m_j - \varepsilon)^2}{2\sigma^2 h}} = \sum_{j \neq i} \frac{1}{\sigma |m_j|} e^{-\frac{(m_j - \varepsilon)^2}{2\sigma^2 h}}$ is negligible wrt $s_h \ll \frac{1}{\sqrt{2\pi}} \sum_{j \neq i} \frac{1}{\sigma |m_j|} e^{-\frac{(m_j - \varepsilon)^2}{2\sigma^2 h}} = [(n-N_T)I_{(i \in J)} + (n-N_T-1)I_{(i \notin J)}]s_h$.

Then $g_i = \varepsilon^2 + 2 \sum_{j \neq i} b_j - 2IV$, we have that the finite sum $\frac{1}{\sqrt{2\pi}} \sum_{j \neq i} \frac{1}{\sigma |m_j|} e^{-\frac{(m_j - \varepsilon)^2}{2\sigma^2 h}} = \sum_{j \neq i} \frac{1}{\sigma |m_j|} e^{-\frac{(m_j - \varepsilon)^2}{2\sigma^2 h}} = 0$. Therefore

$$
g_i = \varepsilon^2 - \frac{4\sigma}{\sqrt{2\pi}} \varepsilon s_h \varepsilon [(n-N_T)I_{(i \in J)} + (n-N_T-1)I_{(i \notin J)}] - 2\sigma^2 h [N_TI_{(i \in J)} + (N_T + 1)I_{(i \notin J)}] + h.o.t.
$$

Further, $N_T \ll n$ and $h \ll \varepsilon^2$, then for all $i$

$$
g_i = \varepsilon^2 - \frac{4\sigma}{\sqrt{2\pi}} \frac{\varepsilon s_h}{\sqrt{h}} + h.o.t..
$$

Moreover from (27) we reach $\sum_{j \neq i} a_j = \sum_{j \neq i, j \notin J} \frac{2s_h}{\sigma \sqrt{h}} + \sum_{j \neq i, j \notin J} \frac{a_{ij}}{\sigma \sqrt{h}} + h.o.t.$, and again the second sum is negligible wrt the first one, thus, for all $i$

$$
\varepsilon \sum_{j \neq i} a_j = 2\frac{s_h \varepsilon}{\sigma \sqrt{h}} [(n-N_T)I_{(m_i \neq 0)} + (n-N_T-1)I_{(m_i = 0)}] + h.o.t. = \frac{2 \varepsilon s_h}{\sigma h \sqrt{h}} + h.o.t..
$$

Now, using (28), from

$$
\sum_{i \in J} a_i g_i = \frac{1}{\sigma} \sqrt{h} s_h \sum_{\ell=1}^{N_T} p_{th} \left[ v_h^2 - 4\sigma v_h s_h \left( \varepsilon \sum_{k \neq \ell} \frac{1}{|\tau_k|} p_{kh} - 2(n-N_T) \right) \right] + h.o.t. = \frac{1}{\sigma} \sqrt{h} s_h \sum_{\ell=1}^{N_T} p_{th} \left[ v_h^2 - 4\sigma v_h s_h \right] + h.o.t.
$$

we reach that

$$
\sum_{i \in J} a_i g_i \frac{1}{\sigma s_h} = \frac{1}{\sigma} \sqrt{h} s_h \sum_{\ell=1}^{N_T} p_{th} \left[ v_h^2 - 4\sigma v_h s_h \right] + h.o.t.
$$

and from

$$
\sum_{i \notin J} a_i g_i = (n-N_T) \sqrt{h} s_h \left[ v_h^2 - 2\sigma^2 (N_T + 1) + 2\sigma v_h s_h \left( \varepsilon \sum_{k=1}^{N_T} \frac{1}{|\tau_k|} p_{kh} - 2(n-N_T-1) \right) \right] = (n-N_T) \sqrt{h} s_h \left[ v_h^2 - 4\sigma v_h s_h \right] + h.o.t.
$$

we reach that

$$
\sum_{i \notin J} a_i g_i \frac{1}{\sigma s_h} = (n-N_T) \frac{1}{\sigma} \sqrt{h} s_h \left[ v_h^2 - 4\sigma v_h s_h \right] + h.o.t..
$$

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Thus

\[ F'\sqrt{2\pi} = v_h \left[ v_h - 4\sigma s_h n \right] \left[ \frac{1}{\sigma} \sqrt{h} \sum_{\ell=1}^{N_T} u_{\ell h} \frac{|\gamma_{\ell}|}{\sigma^2h} - (n - N_T) \frac{2}{\sigma} \sqrt{h} s_h \frac{\varepsilon}{\sigma^2h} \right] \]

\[ + 2\varepsilon \left( 1 + \frac{2}{\sigma} \frac{s_h \varepsilon}{h \sqrt{h}} \right) \left( \sum_{i \in J} \frac{u_{ih}}{\sigma \sqrt{h}} + \sum_{i \notin J} \frac{2s_h}{\sigma \sqrt{h}} \right) + h.o.t. \]

If now our sequence \( \varepsilon_h \) is such that \( v_h = 4\sigma n s_h + h.o.t. \), and noting that also \( \sum_{j \in J} p_{\varepsilon h} |\gamma_j| \xrightarrow{a.s.} 0 \) and that \( n \varepsilon = n \sqrt{hv_h} = \frac{\varepsilon^2}{h} \rightarrow +\infty \) then

\[ F'(\varepsilon_h)\sqrt{2\pi} = v_h \cdot o(n s_h) \frac{s_h}{\sigma^3 \sqrt{h}} \left[ \sum_{\ell=1}^{N_T} p_{\varepsilon h} |\gamma_{\ell}| - 2n \varepsilon \right] + 2\varepsilon \frac{s_h \varepsilon}{\sigma \sqrt{h}} \left( 1 + \frac{2}{\sigma} \frac{s_h \varepsilon}{h \sqrt{h}} \right) \cdot 2(n - N_T) + h.o.t. \]

\[ = -2n \varepsilon v_h \cdot o(n s_h) \frac{s_h}{\sigma^3 \sqrt{h}} + 4n \varepsilon s_h \frac{2}{\sigma \sqrt{h}} \left( 1 + \frac{2}{\sigma} \frac{s_h \varepsilon}{h \sqrt{h}} \right) + h.o.t. \]

now \( v_h = 4\sigma n s_h + o(v_h) \) means also \( s_h = \varepsilon_h \sqrt{h} + o(\varepsilon_h \sqrt{h}) \), and thus \( \frac{n s_h \varepsilon}{h \sqrt{h}} = \frac{\varepsilon^2}{h} + o(\frac{\varepsilon^2}{h}) \rightarrow +\infty \), therefore

\[ F'(\varepsilon_h)\sqrt{2\pi} = -2\varepsilon \frac{\varepsilon}{\sqrt{h}} \cdot o(n s_h) \frac{s_h}{\sigma^3 \sqrt{h}} + 4n \frac{s_h \varepsilon}{\sigma \sqrt{h}} \frac{2}{\sigma \sqrt{h}} + h.o.t. \]

\[ = \frac{\varepsilon}{\sqrt{h}} \frac{s_h \varepsilon}{\sigma \sqrt{h}} n s_h \frac{8}{\sigma^2} \left( 1 + o(1) \right) + h.o.t. = \frac{8}{\sigma^2} \left( \frac{s_h \varepsilon}{h \sqrt{h}} \right)^2 + h.o.t. \]

\[ \square \]

References


