Dynamic Portfolio Optimization with a Defaultable Security and Regime-Switching

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Abstract

We consider a portfolio optimization problem in a defaultable market with finitely-many economical regimes, where the investor can dynamically allocate her wealth among a defaultable bond, a stock, and a money market account. The market coefficients are assumed to depend on the market regime in place, which is modeled by a finite state continuous time Markov process. By separating the utility maximization problem into a pre-default and post-default component, we deduce two coupled Hamilton-Jacobi-Bellman equations for the post and pre-default optimal value functions, and show a novel verification theorem for their solutions. We obtain explicit constructions of value functions and investment strategies for investors with logarithmic and Constant Relative Risk Aversion (CRRA) utilities, and provide a precise characterization of the directionality of the bond investment strategies in terms of corporate returns, forward rates, and expected recovery at default. We illustrate the dependence of the optimal strategies on time, losses given default, and risk aversion level of the investor through a detailed economic and numerical analysis.

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1 Introduction

Continuous time portfolio optimization problems are among the most widely studied problems in the field of mathematical finance. Since the seminal work of Merton (1969), who explored optimal stochastic control techniques to provide a closed form solution to the problem, a large volume of research has been done to extend Merton's paradigm to other frameworks and portfolio optimization problems (see, e.g., Karatzas et al. (1996), Karatzas and Shreve (1998), and Fleming and Pang (2004)). Most of the models proposed in the literature rely on the assumption that the uncertainty in the asset price dynamics is governed by a continuous process, which is typically chosen to be a Brownian motion.

In recent years, there has been an increasing interest in the use of regime-switching models to capture the macroeconomic regimes affecting the behavior of the market. More specifically, the price of the security evolves with a different dynamics, typically identified by the drift and the diffusion coefficient, depending on the macroeconomic regime in place. Utility maximization problems under regime-switching have been investigated in Sotomayor and Cadenillas (2009), who considered the infinite horizon problem of maximizing the expected utility from consumption and terminal wealth in a market consisting of multiple stocks and a money market account, where both short rate and stock diffusion parameters evolve according to Markov-Chain modulated dynamics. Siu (2011) extended the market to accommodate inflation-linked bonds and solved the optimal portfolio selection problem under a hidden regime-switching model. In the important context of risk management, Elliott and Siu (2010) formulated the risk minimization problem as a stochastic differential game in a regime-switching framework, and provided a verification theorem for the resulting

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Hamilton-Jacobi-Bellman (HJB) equation. Within a similar framework, Elliott and Siu (2011) considered the optimal investment problem of an insurer when the model uncertainty is modeled by a hidden Markov chain. Zhang et al. (2010) solved the portfolio selection problem after completing the continuous-time Markovian regime-switching model with jump securities.

Zariphopoulou (1992) considered an infinite horizon investment-consumption model where the agent can consume and distribute her wealth across a risk-free bond and a stock, while Nagai and Runggaldier (2008) considered a finite horizon portfolio optimization problem for a risk averse investor with power utility, assuming that the underlying Markov chain is hidden. Korn and Kraft (2001) relaxed the assumption of constant interest rate and derived expressions for the optimal percentage of wealth invested in the money market account and stock, under the assumption of a diffusive short rate process with deterministic drift and constant volatility.

Most of the research done on continuous time portfolio optimization has concentrated on markets consisting of a risk-free asset and of securities which only bear market risk. These models do not take into account securities carrying default risk, such as corporate bonds, even though the latter represent a significant portion of the market, comparable to the total capitalization of all publicly traded companies in the United States. In recent years, portfolio optimization problems have started to incorporate defaultable securities, but assuming that the risky factors are modeled by continuous processes and more specifically by Brownian Itô processes. Bielecki and Jang (2006) derived optimal finite horizon investment strategies for an investor with Constant Relative Risk Aversion (CRRA) utility function, who optimally allocates her wealth among a defaultable bond, risk-free account, and stock, assuming constant interest rate, drift, volatility, and default intensity. Bo et al. (2010) considered an infinite horizon portfolio optimization problem, where an investor with logarithmic utility can choose a consumption rate, and invest her wealth across a defaultable perpetual bond, a stock, and a money market. They assume that both the historical intensity and the default premium process depend on a common Brownian factor. Unlike Bielecki and Jang (2006), where the dynamics of the defaultable bond price process was derived from the arbitrage-free bond prices, Bo et al. (2010) postulated the dynamics of the defaultable bond prices partially based on heuristic arguments. Lakner and Liang (2008) analyzed the optimal investment strategy in a market consisting of a defaultable (corporate) bond and a money market account under a continuous time model, where bond prices can jump, and employ duality theory to obtain the optimal strategy. Callegaro et al. (2010) considered a market model consisting of several defaultable assets, which evolve according to discrete dynamics depending on partially observed exogenous factor processes. Jiao and Pham (2010) combined duality and dynamic programming to optimize the utility of an investor with CRRA utility function, in a market consisting of a riskless bond and a stock subject to counterparty risk. Bielecki et al. (2008) developed a variational inequality approach to pricing and hedging of a defaultable game option under a Markov modulated default intensity framework.

In this paper, we consider for the first time finite horizon dynamic portfolio optimization problems in defaultable markets with regime-switching dynamics. We provide a general framework and explicit results on optimal value functions and investment strategies in a market consisting of a money market account, a stock, and a defaultable bond. Similarly to Sotomayor and Cadenillas (2009), we allow the short rate, drift, and volatility of the risky stock to be all regime dependent. For the defaultable bond, we follow the reduced form approach to credit risk, where the global market information, including default, is modeled by the progressive enlargement of a reference filtration representing the default-free information, and the default time is a totally inaccessible stopping time with respect to the enlarged filtration, but not with respect to the reference filtration. We also make the default intensities and loss given default rates to be all regime dependent. The use of regime-switching models for pricing defaultable bonds has proven to be very flexible when fitting the empirical credit spreads curve of corporate bonds as illustrated in, e.g., Jarrow et al. (1997), where the underlying Markov chain models credit ratings.

Our main contributions are discussed next. Using the results on the dynamics of defaultable bond prices obtained in Capponi, Figueroa-López, and Nisen (2012b), we first deduce the Hamilton-Jacobi-Bellman (HJB) equation of the dynamical optimization problem. The HJB equation enables us to separate the utility maximization problem into pre-default and post-default dynamical optimization subproblems, for which novel verification theorems are obtained. We show that the regime dependent pre-default optimal value function and bond investment strategy may be obtained as the solution of a coupled system of nonlinear partial differential equations (satisfied by the pre-default value function) and nonlinear equations (satisfied by the bond investment strategy), each corresponding to a different regime. Moreover, we obtain the interesting feature that the pre-default optimal value function and bond investment strategy depend on the corresponding regime dependent post-default value function. Thirdly, we develop an explicit numerical and economic analysis of value functions and investment strategies for the case of a logarithmic and CRRA investor facing both default and regime-switching risk. For the logarithmic investor, we demonstrate that the computation of the optimal pre-default and post-default value functions amounts to solving a system of ordinary linear differential equations, while the optimal bond strategy may be recovered as the unique solution of a decoupled system of equations, one for each regime. For the CRRA investor, we show that the optimal bond investment strategy and pre-default value function can be uniquely recovered as the solution of a coupled system composed of ordinary differential equations and nonlinear equations. Under mild assumptions, we provide conditions guaranteeing local existence and uniqueness of the solution of the coupled system and show numerically, via a fixed point algorithm, that global convergence is typically achieved. Interestingly, in a different context of liquidity risk, where investors can only trade in stocks at Poisson random times, Pham and Tankov (2009) also found that the optimal control problem leads to solving a coupled system of integro-partial differential equations. We also provide necessary and sufficient conditions under which the logarithmic and CRRA investor go long or short in the defaultable security, and show that these depend on the interplay between corporate bond returns, instantaneous forward rate of the defaultable bond, and expected recovery (the precise statement is given in Section 6.2).

The rest of the paper is organized as follows. Section 2 introduces the market model. Section 3 formulates the dynamic optimization problem. Section 4 gives and proves the two verification theorems associated to the post-default and pre-default case. Section 5 specializes the theorems given therein to the case of investors with logarithmic and CRRA utilities. Section 6 characterizes the directionality of the bond investment strategy in terms of meaningful economic quantities, and numerically illustrates how it behaves as a function of time, risk aversion level of the investor, and loss experienced at default, under a meaningful "realistic" economic scenario. Section 7 summarizes the main conclusions of the paper. The proofs of the main theorems and necessary lemmas are deferred to the appendix.

2 The Model

Assume $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$ is a complete filtered probability space, where $\mathbb{P}$ is the real world probability measure (also called historical probability), $\mathbb{G} := (\mathcal{G}_t)$ is an enlarged filtration given by $\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t$ (the filtrations $\mathcal{F}_t$ and $\mathcal{H}_t$ will be introduced later). Let $\{W_t\}$ be a standard Brownian motion on $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} := (\mathcal{F}_t)_t$ is a suitable filtration satisfying the usual hypotheses of completeness and right continuity. We also assume that the *states of the economy* are modeled by a continuous-time Markov process $\{X_t\}$ defined on $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$ with a finite state space $\{x_1, x_2, \dots, x_N\}$. Without loss of generality, we can identify the state space of $\{X_t\}$ to be a finite set of unit vectors $\{e_1, e_2, \dots, e_N\}$, where $e_i = (0, ..., 1, ...0)^{'} \in \mathbb{R}^N$ and $'$ denotes the transpose. We also assume that $\{X_t\}$ and $\{W_t\}$ are independent. The following semi-martingale representation is well-known (cf. Elliott et al. (1994)):

$$X_t = X_0 + \int_0^t A'(s) X_s ds + M^{\mathbb{P}}(t), \tag{1}$$

where $M^{\mathbb{P}}(t) = (M_1^{\mathbb{P}}(t), \dots, M_N^{\mathbb{P}}(t))'$ is a $\mathbb{R}^N$-valued martingale process under $\mathbb{P}$ and $A(t) = [a_{i,j}(t)]_{i,j=1,\dots,N}$ is the so-called infinitesimal generator of the Markov process. Specifically, denoting $p_{i,j}(t,s) := \mathbb{P}(X_s = e_j | X_t = e_i)$, for $s \geq t$, and $\delta_{i,j} = \mathbf{1}_{i=j}$, we have that

$$a_{i,j}(t) = \lim_{h \to 0} \frac{p_{i,j}(t, t+h) - \delta_{i,j}}{h}; \tag{2}$$

cf. Bielecki and Rutkowski (2001). In particular, $a_{i,i}(t) := -\sum_{j \neq i} a_{i,j}(t)$. For future references, we also introduce the process

$$C_t := \sum_{i=1}^N i \mathbf{1}_{\{X_t = e_i\}}. \tag{3}$$

We consider a frictionless financial market consisting of three instruments: a risk-free bank account, a defaultable bond, and a stock. The dynamics of each of the following instruments will depend on the underlying states of the economy as follows:

Risk-free bank account. The instantaneous market interest rate at time t is $r_t := r(t, X_t) := \langle r, X_t \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^N$ and $r := (r_1, r_2, \ldots, r_N)'$ is a vector of positive constants. This means that, depending on the state of the economy, the interest rate r_t will be different; i.e., if $X_t = e_i$ then $r_t = r_i$. Then, the price process of the risk-free asset associated with risk-free bank account follows the dynamics

$$dB_t = r_t B_t dt. \tag{4}$$

Stock price. We assume that the stock appreciation rate $\{\mu_t\}$ and the volatility $\{\sigma_t\}$ of the stock also depend on the economy regime in place X_t in the following way:

$$\mu_t := \mu(t, X_t) := \langle \mu, X_t \rangle, \qquad \sigma_t := \sigma(t, X_t) := \langle \sigma, X_t \rangle, \tag{5}$$

where $\mu := (\mu_1, \mu_2, \ldots, \mu_N)'$ and $\sigma := (\sigma_1, \sigma_2, \ldots, \sigma_N)'$ are vectors denoting, respectively, the different values that the drift and volatility can take depending on the different economic regimes. Hence, we assume that the stock price process follows the dynamics

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t, \qquad S_0 = s. \tag{6}$$

Risky Bond price. Unlike the previous two securities, whose dynamics have been written under the historical measure, the bond prices are defined under a suitably chosen risk-neutral measure $\mathbb{Q}$ and the historical dynamics of the process (i.e. dynamics under the actual probability measure $\mathbb{P}$) will have to be inferred from the risk-neutral dynamics. Before defining the bond price, we need to introduce a default process.

Let τ be a nonnegative random variable, defined on $(\Omega, \mathcal{G}, \mathbb{P})$, representing the default time of the counterparty selling the bond. Let $\mathcal{H}_t = \sigma(H(u) : u \leq t)$ be the filtration generated by the *default process* $H(t) := \mathbf{1}_{\tau \leq t}$, after completion and regularization on the right, and also let $\mathbb{G} := (\mathcal{G}_t)_t$ be the filtration $\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t$. We use the canonical construction of the default time τ in terms of a given hazard process $\{h_t\}_{t \geq 0}$, which will also be assumed to be driven by the Markov process X. Specifically, throughout the paper we assume that $h_t := \langle h, X_t \rangle$, where $h := (h_1, h_2, \ldots, h_N)'$ are positive constants. For future reference, we now give the details of the construction of the random time τ. We assume the existence of an exponential random variable Θ defined on the probability space $(\Omega, \mathcal{G}, \mathbb{P})$, independent of the process $(X_t)_t$. We define τ by setting

$$\tau := \inf \left\{ t \in \mathbb{R}^+ : \int_0^t h_u du \geq \Theta \right\}. \tag{7}$$

It can be proven that $(h_t)_t$ is the $(\mathbb{F}, \mathbb{G})$-*hazard rate* of τ (see Bielecki and Rutkowski (2001), Section 6.5 for details). That is, $(h_t)_t$ is such that

$$\xi_t^{\mathbb{P}} := H(t) - \int_0^t (1 - H(u^-)) h_u du \tag{8}$$

is a $\mathbb{G}$-martingale under $\mathbb{P}$, where $H(u^-) = \lim_{s \uparrow u} H(s) = \mathbf{1}_{\tau < u}$. Intuitively, Eq. (8) says that the single jump process needs to be compensated for default, prior to the occurrence of the event.

An important consequence of the previous construction is the following so-called H hypothesis. Let us fix $t > 0$ and $\mathcal{F}_\infty = \bigvee_{s \geq 0} \mathcal{F}_s$. For any $u \in \mathbb{R}^+$, we have $\mathbb{P}(\tau \leq u | \mathcal{F}_\infty) = 1 - e^{-\int_0^u h_s ds}$. Therefore, for any $u \leq t$,

$$\mathbb{P}(\tau \leq u | \mathcal{F}_t) = \mathbb{E}^{\mathbb{P}} \left[\mathbb{P}(\tau \leq u | \mathcal{F}_\infty) | \mathcal{F}_t \right] = 1 - e^{-\int_0^u h_s ds} = \mathbb{P}(\tau \leq u | \mathcal{F}_\infty).$$

Plugging $u = t$ inside the above expression, we obtain

$$\mathbb{P}(\tau \leq t | \mathcal{F}_t) = \mathbb{P}(\tau \leq t | \mathcal{F}_\infty). \tag{9}$$

It was proven in Bremaud and Yor (1978) that Eq. (9) is equivalent to saying that any $\mathbb{F}$-square integrable martingale is also a $\mathbb{G}$-square integrable martingale. The latter property is also referred to as the H hypothesis or the martingale invariance principle with respect to $\mathbb{G}$, and we will make use of this property later on. For further details about this property in this context the reader is referred to Sections 8.3.1 and 8.6.1 in Bielecki and Rutkowski (2001).

The final ingredient in the bond pricing formula is the recovery process $(z_t)_t$, i.e., an $\mathbb{F}$-adapted right-continuous with left-limits process to be fully specified below. In terms of $(z_t)_t$, the time-t price of the risky bond with maturity T is given by

$$p(t,T) := \mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^u r_s ds} z_u dH(u) + e^{-\int_t^T r_s ds}(1 - H(T))\middle|\mathcal{G}_t\right], \tag{10}$$

where $\mathbb{Q}$ is the equivalent risk-neutral measure used in pricing. Furthermore, we adopt a pricing measure $\mathbb{Q}$ such that, under $\mathbb{Q}$, W is still a standard Wiener process and X is a continuous-time Markov process (independent of W) with possibly different generator $A^{\mathbb{Q}}(t) := [a^{\mathbb{Q}}_{i,j}(t)]_{i,j=1,2,\dots,N}$.

The existence of the measure $\mathbb{Q}$ in the previous paragraph follows from the theory of change of measures for denumerable Markov processes (see, e.g., Section 11.2 in Bielecki and Rutkowski (2001)). Concretely, for $i \neq j$ and some bounded measurable functions $\kappa_{i,j} : \mathbb{R}_+ \to (-1, \infty)$, define

$$a^{\mathbb{Q}}_{i,j}(t) := a_{i,j}(t)(1 + \kappa_{i,j}(t)), \tag{11}$$

and for $i = j$, define

$$a^{\mathbb{Q}}_{i,i}(t) := -\sum_{k=1,k\neq i}^{N} a^{\mathbb{Q}}_{i,k}(t).$$

We also fix $\kappa_{i,i}(t) = 0$ for $i = 1, \dots, N$. Now, consider the processes

$$M^{i,j}_t := H^{i,j}_t - \int_0^t a_{i,j}(u) H^i_u du, \tag{12}$$

where

$$H^i_t := \mathbf{1}_{\{X_t = e_i\}}, \quad \text{and} \quad H^{i,j}_t := \sum_{0<u\leq t} \mathbf{1}_{\{X_{u^-} = e_i\}} \mathbf{1}_{\{X_u = e_j\}}, \quad (i \neq j). \tag{13}$$

The process $(M^{i,j}_t)_t$ is known to be an $\mathbb{F}$-martingale for any $i \neq j$ (see Lemma 11.2.3 in Bielecki and Rutkowski (2001)) and, since the H-hypothesis holds in our default framework, this is also a $\mathbb{G}$-martingale. Then, by virtue of Proposition 11.2.3 in Bielecki and Rutkowski (2001), the probability measure $\mathbb{Q}$ on $\mathbb{G} = (\mathcal{G}_t)_t$ with Radon-Nikodym density $\{\eta_t\}$ given by

$$\eta_t = 1 + \int_{(0,t]} \sum_{i,j=1}^{N} \eta_{u^-} \kappa_{i,j}(u) dM^{i,j}_u, \tag{14}$$

is such that X is a Markov process under $\mathbb{Q}$ with generator $[a^{\mathbb{Q}}_{i,j}(t)]_{i,j=1,2,\dots,N}$. In particular, note that

$$X_t = X_0 + \int_0^t A^{\mathbb{Q}}(s)' X_s ds + M^{\mathbb{Q}}(t), \tag{15}$$

where $M^{\mathbb{Q}}$ is a $\mathbb{R}^N$-valued martingale under $\mathbb{Q}$, and also,

$$M^{\mathbb{Q}}(t) = M^{\mathbb{P}}(t) + \int_0^t (A(s)' - A^{\mathbb{Q}}(s)') X_s ds. \tag{16}$$

Without loss of generality, $\mathbb{Q}$ can be taken to be such that W is still a Wiener process independent of X under $\mathbb{Q}$. We emphasize that the distribution of the hazard rate process $h_t = \langle h, X_t \rangle$ under the risk-neutral measure differs from the one under the historical measure. Therefore, our framework allows modeling the default risk premium, defined as the ratio between risk-neutral and historical intensities, through the change of measure of the underlying Markov chain.

We now proceed to obtain the bond price dynamics under the historical probability measure. Eq. (10) may be rewritten as

$$p(t,T) = \mathbf{1}_{\tau>t}\mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^u (r_s+h_s)ds} z(u)h_u du \middle| \mathcal{F}_t\right] + \mathbf{1}_{\tau>t}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T (r_s+h_s)ds}\middle| \mathcal{F}_t\right], \tag{17}$$

which follows from Eq. (8), along with application of the following classical identity

$$\mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}_{\tau>s}Y\middle|\mathcal{G}_t\right] = \mathbf{1}_{\tau>t}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^s h_u du}Y\middle|\mathcal{F}_t\right],$$

where $t \leq s$ and Y is a $\mathcal{F}_s$-measurable random variable (see Bielecki and Rutkowski (2001), Corollary 5.1.1, for its proof). We assume the recovery-of-market value assumption, i.e. $z_t := (1-L_t)p(t^-,T)$, where L_t is $\mathbb{F}$-predictable. As with the other factors in our model, we shall assume that L_t is of the form $L_t := \langle L, X_t\rangle$, where $L := (L_1,\ldots,L_N)' \in [0,1]^N$. Under the recovery-of-market value assumption, it follows using a result in Duffie and Singleton (1999) (see also Proposition 8.3.3 in Bielecki and Rutkowski (2001)) that

$$p(t,T) = \mathbf{1}_{\tau>t}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T (r_s+h_sL_s)ds}\middle|\mathcal{F}_t\right]. \tag{18}$$

The following result from Capponi, Figueroa-López, and Nisen (2012b) will play a key role in obtaining the bond price dynamics (see Section 3 therein for its proof):

Lemma 2.1. *Suppose that, for any $i \neq j$, the function $a^{\mathbb{Q}}_{i,j}$ defined in (11) is continuously differentiable in $(0,T)$ and such that*

$$0 < \inf_{s\in[0,T]} |a^{\mathbb{Q}}_{i,j}(s)| \leq \sup_{s\in[0,T]} |a^{\mathbb{Q}}_{i,j}(s)| < \infty \quad \& \quad \sup_{s\in(0,T)}\left|\frac{da^{\mathbb{Q}}_{i,j}(s)}{ds}\right| < \infty. \tag{19}$$

Then, the time-t bond price under the i^{th}-regime given by the formula

$$\psi_i(t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T (r_s+h_sL_s)ds}\middle| X_t = e_i\right], \tag{20}$$

is differentiable for any $t\in(0,T)$.

The following result, stated as a lemma in this paper, gives the dynamics of the defaultable bond price process under the historical measure $\mathbb{P}$. It can be obtained as a special case of the semi-martingale representation formulas for vulnerable claims provided in Capponi, Figueroa-López, and Nisen (2012b), by setting the terminal payoff of the claim equal to one.

Lemma 2.2. *Under the conditions of Lemma 2.1, the pre-default dynamics of the bond price $p(t,T)$ under the historical measure $\mathbb{P}$ is given by*

$$dp(t,T) = p(t^-,T)\left\{[r_t + h_t(L_t-1) + D(t)]\,dt + \frac{\langle\psi(t), dM^{\mathbb{P}}(t)\rangle}{\langle\psi(t), X_{t^-}\rangle} - d\xi^{\mathbb{P}}_t\right\}, \tag{21}$$

where $(M^{\mathbb{P}}(t))_t$ is the N-dimensional $(\mathbb{F},\mathbb{P})$-martingale defined in (1), $(\xi^{\mathbb{P}}_t)_t$ is the $(\mathbb{G},\mathbb{P})$-martingale defined in (8), $\psi(t) = (\psi_1(t),\ldots,\psi_N(t))'$, and $D(t) := \langle(D_1(t),\ldots,D_N(t))', X_t\rangle$ with

$$D_i(t) := \sum_{j=1}^N (a_{i,j}(t) - a^{\mathbb{Q}}_{i,j}(t))\frac{\psi_j(t)}{\psi_i(t)} = \sum_{j\neq i}(a_{i,j}(t) - a^{\mathbb{Q}}_{i,j}(t))\left(\frac{\psi_j(t)}{\psi_i(t)} - 1\right). \tag{22}$$

3 Optimal Portfolio Problem

We consider an investor who wants to maximize her wealth at time $R \leq T$ by dynamically allocating her financial wealth into the risk-free bank account, the risky asset, and the defaultable bond defined in Section 2. The investor does not have intermediate consumption nor capital income to support her purchase of financial assets. Let us denote by ν_t^B the number of shares of the risk-free asset B that the investor buys ($\nu_t^B > 0$) or sells ($\nu_t^B < 0$) at time t. Similarly, ν_t^S and ν_t^P denote the investor's portfolio positions in the stock and risky bond at time t, respectively. The process $(\nu_t^B, \nu_t^S, \nu_t^P)$ is called a *portfolio process*. We denote $V_t(\nu)$ the wealth of the portfolio process $\nu = (\nu^B, \nu^S, \nu^P)$ at time t, i.e.

$$V_t(\nu) = \nu_t^B B_t + \nu_t^S S_t + \nu_t^P p(t,T).$$

As usual, we require the processes ν_t^B, ν_t^S, and ν_t^P to be $\mathbb{F}$-predictable. We also assume the following self-financing condition:

$$dV_t = \nu_t^B dB_t + \nu_t^S dS_t + \nu_t^P dp(t,T).$$

Given an initial state configuration $(x, z, v) \in \mathbb{E} := \{e_1, e_2, \ldots, e_N\} \times \{0,1\} \times (0, \infty)$, we define the class of admissible strategies $\mathcal{A} := \mathcal{A}(v, i, z)$ to be a set of (self-financing) portfolio processes ν such that $V_t(\nu) \geq 0$ for all $t \geq 0$ when $X_0 = x$, $H_0 = z$, and $V_0 = v$. Let $\pi := (\pi_t^B, \pi_t^S, \pi_t^P)$ be defined as

$$\pi_t^B := \frac{\nu_t^B B_t}{V_{t-}(\nu)}, \quad \pi_t^S := \frac{\nu_t^S S_t}{V_{t-}(\nu)}, \quad \pi_t^P = \frac{\nu_t^P p(t^-,T)}{V_{t-}(\nu)}, \tag{23}$$

if $V_{t-}(\nu) > 0$, while $\pi_t^B = \pi_t^P = \pi_t^S = 0$, when $V_{t-}(\nu) = 0$. The vector $\pi := (\pi_t^B, \pi_t^S, \pi_t^P)$, called a *trading strategy*, represents the corresponding fractions of wealth invested in each asset at time t. Note that if π is admissible, then the dynamics of the resulting wealth process V^π can be written as

$$dV_t^\pi = V_{t^-}^\pi \left\{ \pi_t^B \frac{dB_t}{B_t} + \pi_t^S \frac{dS_t}{S_t} + \pi_t^P \frac{dp(t,T)}{p(t^-,T)} \right\},$$

under the convention that $0/0 = 0$. This convention is needed to deal with the case when default has occurred ($t > \tau$), so that $p(t^-,T)$=0 and we fix $\pi_t^P = 0$. Using the dynamics derived in Proposition 2.2 and that $\pi^B + \pi^P + \pi^S = 1$, we have the following dynamics of the wealth process

$$dV_t^\pi = V_{t^-}^\pi \left[\left\{ r_t + \pi_t^S(\mu_t - r_t) + \pi_t^P[h_t(L_t - 1) + D(t)] \right\} dt + \pi_t^S \sigma_t dW_t + \pi_t^P \frac{\langle \psi(t), dM^{\mathbb{P}}(t) \rangle}{\langle \psi(t), X_{t^-} \rangle} - \pi_t^P d\xi_t^{\mathbb{P}} \right], \tag{24}$$

under the historical probability $\mathbb{P}$.

3.1 The utility maximization problem

For an initial value $(x, z, v) \in \mathbb{E}$ and an admissible strategy $\pi = (\pi^B, \pi^S, \pi^P) \in \mathcal{A}(x, z, v)$, let us define the objective functional to be

$$J_R(x, z, v; \pi) := \mathbb{E}^{\mathbb{P}} \left[U(V_R^\pi) \middle| X_0 = x, H_0 = z, V_0 = v \right]; \tag{25}$$

i.e. we are assuming that the investor starts with v dollars (its initial wealth), that the initial default state is z ($z = 0$ means that no default has occurred yet), and the initial value for the underlying state of the economy is x. The constraint $V_0 = v$ is also called the budget constraint. As usual, we assume that the utility function $U : [0, \infty) \to \mathbb{R} \cup \{\infty\}$ is strictly increasing and concave.

Our goal is to maximize the objective functional $J(x, z, v; \pi)$ for a suitable class of admissible strategies $\pi_t := (\pi_t^B, \pi_t^S, \pi_t^P)$. Furthermore, we shall focus on feedback or Markov strategies of the form

$$\pi_t = (\pi_{C_{t^-}}^B(t, V_{t^-}, H(t^-)), \pi_{C_{t^-}}^S(t, V_{t^-}, H(t^-)), \pi_{C_{t^-}}^P(t, V_{t^-}, H(t^-)),$$

for some functions $\pi_i^B, \pi_i^P, \pi_i^S : [0,\infty) \times [0,\infty) \times \{0,1\} \to \mathbb{R}$ such that $\pi_i^B(t,v,z) + \pi_i^S(t,v,z) + \pi_i^P(t,v,z) = 1$. Above, we had used the process $(C_t)_t$ defined in (3).

As usual, we consider instead the following dynamical optimization problem:

$$\varphi^R(t,v,i,z) := \sup_{\pi \in \mathcal{A}_t(v,i,z)} \mathbb{E}^{\mathbb{P}}\left[U(V_R^{\pi,t,v}) \middle| V_t = v, X_t = e_i, H(t) = z \right], \tag{26}$$

for each $(v,i,z) \in (0,\infty) \times \{1,2,\dots,N\} \times \{0,1\}$, where

$$\begin{aligned} dV_s^{\pi,t,v} &= V_{s^-}^{\pi,t,v} \Bigg[\left\{ r_s + \pi_s^S(\mu_s - r_s) + \pi_s^P (1 - H(s^-))[h_s(L_s - 1) + D(s)] \right\} ds \\ &\quad + \pi_s^S \sigma_s dW_s + \pi_s^P (1 - H(s^-)) \frac{\langle \psi(s), dM^{\mathbb{P}}(s) \rangle}{\langle \psi(s), X_{s^-} \rangle} - \pi_s^P (1 - H(s^-)) d\xi_s^{\mathbb{P}} \Bigg], \quad s \in [t,R], \\ V_t^{\pi,t,v} &= v. \end{aligned} \tag{27}$$

The class of processes $\mathcal{A}_t(v,i,z)$ is defined as follows:

Definition 3.1. *Throughout, $\mathcal{A}_t(v,i,z)$ denotes a suitable class of $\mathbb{F}$-predictable locally bounded feedback trading strategies*

$$\pi_s := (\pi_s^S, \pi_s^P) := (\pi_{C_{s^-}}^S(s, V_{s^-}^{\pi,t,v}, H(s^-)), \pi_{C_{s^-}}^P(s, V_{s^-}^{\pi,t,v}, H(s^-))), \quad s \in [t,R],$$

such that (27) admits a unique strong solution $\{V_s^{\pi,t,v}\}_{s \in [t,R]}$ and $V_s^{\pi,t,v} > 0$ for any $s \in [t,R]$ when $X_t = e_i$ and $H(t) = z$. Throughout this paper, a trading strategy satisfying these conditions is simply said to be t-admissible (with respect to the initial conditions $V_t = v$, $X_t = e_i$, and $H_t = z$).

Remark 3.1. *As it will be discussed below (see Eqs. (32), (33), and (83)), the jump $\Delta V_s := V_s - V_{s^-}$ of the process (27) at time s is given by*

$$\Delta V_s = V_{s^-} \left(\sum_{i=1}^N \sum_{j \neq i} \pi_i^P(s, V_{s^-}, H(s^-)) \frac{\psi_j(s) - \psi_i(s)}{\psi_i(s)} \Delta H_s^{i,j} - \pi_s^P \Delta H(s) \right). \tag{28}$$

Since for $\{V_s\}_{s \geq t}$ to be strictly positive, it is necessary and sufficient that $\Delta V_s > -V_{s^-}$ for any $s > t$ a.s. (cf. (Jacod and Shiryaev, 2003, Theorem 4.61)), we conclude that in order for π^P to be admissible, it is necessary that,

$$M_i := \max_{j \neq i : \psi_i(s) < \psi_j(s)} \left(-\frac{\psi_i(s)}{\psi_j(s) - \psi_i(s)} \right) < \pi_i^P(s,v,z) < 1, \tag{29}$$

for any $s, v > 0$, $z \in \{0,1\}$, and $i = 1, \dots, N$, where we set $M_i := -\infty$ if $\psi_i(s) \geq \psi_j(s)$ for all $j \neq i$.

4 Verification Theorems

As it is usually the case, we start by deriving the HJB formulation of the value function (26) via heuristic arguments. We then verify that the solution of the proposed HJB equation (when it exists and satisfies other regularity conditions) is indeed optimal. Such a result is called the verification theorem of the optimization problem. Let us assume for now that $\varphi^R(t,v,i,z)$ is C^1 in t and C^2 in v for each i and z. Then, using Itô's rule along the lines of Appendix A, we have that

$$\varphi^R(t, V_t^\pi, C_t, H(t)) = \varphi^R(r, V_r^\pi, C_r, H_r) + \int_r^t \mathcal{L}\varphi^R(s, V_s^\pi, C_s, H_s) ds + \mathcal{M}_t - \mathcal{M}_r,$$

where $(C_t)_t$ is the Markov process defined in (3), $\mathcal{L}$ is the infinitesimal generator of (t, V_t, C_t, H_t) given in Eq. (86), and $(\mathcal{M}_t)_t$ is the martingale given by Eq. (87). Next, if $r < t < R$, by virtue of the dynamic programming principle, we expect that

$$\varphi^R(r, V_r^\pi, C_r, H_r) = \max_\pi \mathbb{E}\left[\varphi^R(t, V_t^\pi, C_t, H(t)) | \mathcal{G}_r \right]. \tag{30}$$

Therefore, we obtain $\mathbb{E}\left[\int_r^t \mathcal{L}\varphi^R(s, V_s^\pi, C_s, H_s)ds\middle|\mathcal{G}_r\right] \le 0$, with the inequality becoming an equality if $\pi = \widetilde{\pi}$, where $\widetilde{\pi}$ denotes the optimum. Now, evaluating the derivative with respect to t, at $t = r$, we deduce the following HJB equation:

$$\max_\pi \mathcal{L}\varphi^R(r, v, i, z) = 0, \tag{31}$$

with boundary condition $\varphi^R(T, v, i, z) = U(v)$. In order to further specify (31), let us first note that the dynamics (27) can be written in the form

$$dV_s = \alpha_{C_s} ds + \vartheta_{C_s} dW_s + \sum_{j=1}^N \beta_{C_{s-},j} dM_j^{\mathbb{P}}(s) - \gamma_{C_{s-}} d\xi_s^{\mathbb{P}}, \quad (t < s < R), \tag{32}$$

with coefficients

$$\begin{aligned}
&\beta_{i,j}(t, v, z) = v\pi_i^P(t, v, z)(1 - z)\frac{\psi_j(t)}{\psi_i(t)}, \quad \gamma_i(t, v, z) = v\pi_i^P(t, v, z)(1 - z),\\
&\alpha_i(t, v, z) = v\left[r_i + \pi_i^S(t, v, z)(\mu_i - r_i) + \pi_i^P(t, v, z)(1 - z)(h_i(L_i - 1) + D_i(t))\right]\\
&\vartheta_i(t, v, z) = \pi_i^S(t, v, z)\sigma_i v,
\end{aligned} \tag{33}$$

where $D_i(t)$ is defined as in (22). Using the expression for the generator in Eq. (86), the notation $\varphi_{i,z}(t, v) := \varphi^R(t, v, i, z)$, and the relationship $\pi^B = 1 - \pi^S - \pi^P$, and dropping the dependence of the strategies from the triple (t, v, z) to lighten notation, (31) can be written as follows for each $i = 1, \ldots, N$:

$$\begin{aligned}
0 = &\frac{\partial\varphi_{i,z}}{\partial t} + vr_i\frac{\partial\varphi_{i,z}}{\partial v} + z\sum_{j\neq i} a_{i,j}(t)\left[\varphi_{j,z}(t, v) - \varphi_{i,z}(t, v)\right]\\
&+ \max_{\pi_i^S}\left\{\pi_i^S(\mu_i - r_i)v\frac{\partial\varphi_{i,z}}{\partial v} + (\pi_i^S)^2\frac{\sigma_i^2}{2}v^2\frac{\partial^2\varphi_{i,z}}{\partial v^2}\right\}\\
&+ (1 - z)\max_{\pi_i^P}\left\{\pi_i^P\theta_i(t)v\frac{\partial\varphi_{i,z}}{\partial v} + h_i\left[\varphi_{i,1}(t, v(1 - \pi_i^P)) - \varphi_{i,z}(t, v)\right]\right.\\
&\left.+ \sum_{j\neq i} a_{i,j}(t)\left[\varphi_{j,z}\left(t, v\left[1 + \pi_i^P\left(\frac{\psi_j(t)}{\psi_i(t)} - 1\right)\right]\right) - \varphi_{i,z}(t, v)\right]\right\},
\end{aligned} \tag{34}$$

where

$$\theta_i(t) := h_iL_i - \sum_{j\neq i} a_{i,j}^{\mathbb{Q}}(t)\left(\frac{\psi_j(t)}{\psi_i(t)} - 1\right). \tag{35}$$

We can consider two separate cases

$$\bar{\varphi}^R(t, v, i) = \varphi_{i,0}(t, v) = \varphi^R(t, v, i, 0), \qquad \text{(pre-default case)} \tag{36}$$

and

$$\underline{\varphi}^R(t, v, i) = \varphi_{i,1}(t, v) = \varphi^R(t, v, i, 1), \qquad \text{(post-default case).} \tag{37}$$

Section 4.1 proves a verification theorem for the post-default case, while Section 4.2 proves a verification theorem for the pre-default case.

4.1 Post-default case

In the post-default case, we have that $p(t, T) = 0$, for each $\tau < t \le T$. Consequently, $\pi_t^P = 0$ for $\tau < t \le T$ and, since $\pi_t^B = 1 - \pi_t^S - \pi_t^P$, we can take $\pi = \pi^S$ as the unique control. Below, $\eta_i := \frac{\mu_i - r_i}{\sigma_i}$ denotes the Sharpe ratio of the risky asset under the i^{th} state of economy and $C_0^{1,2}$ denotes the class of functions $\varpi : [0, R] \times \mathbb{R}_+ \times \{1, \ldots, N\} \to \mathbb{R}_+$ such that

$$\varpi(\cdot, \cdot, i) \in C^{1,2}((0, R) \times \mathbb{R}_+) \cap C([0, R] \times \mathbb{R}_+), \quad \varpi_v(s, v, i) \ge 0, \quad \varpi_{vv}(s, v, i) \le 0,$$

for each $i = 1, \ldots, N$. We have the following verification result, whose proof is reported in Appendix B:

Theorem 4.1. *Suppose that there exists a function $\underline{w} \in C_0^{1,2}$ that solves the nonlinear Dirichlet problem*

$$\underline{w}_t(s,v,i) - \frac{\eta_i^2}{2}\frac{\underline{w}_v^2(s,v,i)}{\underline{w}_{vv}(s,v,i)} + r_i v \underline{w}_v(s,v,i) + \sum_{j\neq i} a_{i,j}(s)\left(\underline{w}(s,v,j) - \underline{w}(s,v,i)\right) = 0, \tag{38}$$

for any $s \in (0,R)$ and $i = 1,\dots,N$, with terminal condition $\underline{w}(R,v,i) = U(v)$. We assume additionally that $\underline{w}$ satisfies

$$\text{(i)}\ \ |\underline{w}(s,v,i)| \leq D(s) + E(s)v, \qquad \text{(ii)}\ \ \left|\frac{\underline{w}_v(s,v,i)}{\underline{w}_{vv}(s,v,i)}\right| \leq G(s)(1+v), \tag{39}$$

for some locally bounded functions $D, E, G : \mathbb{R}_+ \to \mathbb{R}_+$. Then, the following statements hold true:

(1) *$\underline{w}(t,v,i)$ coincides with the optimal value function $\underline{\varphi}^R(t,v,i) = \varphi^R(t,v,i,1)$ in (26), when $\mathcal{A}_t(v,i,1)$ is constrained to the class of t-admissible feedback controls $\pi_s^S = \pi_{C_s}(s,V_s)$ such that $\pi_i(\cdot,\cdot) \in C([0,R]\times\mathbb{R}_+)$ for each $i = 1,\dots,N$ and*

$$|v\pi_i(s,v)| \leq G(s)(1+v), \tag{40}$$

for a locally bounded function G. If the solution $\underline{w}$ is non-negative, then condition (40) is not needed.

(2) *The optimal feedback control $\{\pi_s^S\}_{s\in[t,R)}$, denoted by $\widetilde{\pi}_s^S$, can be written as $\widetilde{\pi}_s^S = \widetilde{\pi}_{C_s}(s,V_s)$ with*

$$\widetilde{\pi}_i(s,v) = -\frac{\eta_i}{\sigma_i}\frac{\underline{w}_v(s,v,i)}{v\underline{w}_{vv}(s,v,i)}. \tag{41}$$

4.2 Pre-default case

In the pre-default case ($z = 0$), we take π^S and π^P as our controls. We then have the following verification result, proven in Appendix B:

Theorem 4.2. *Suppose that the conditions of Theorem 4.1 are satisfied and, in particular, let $\underline{w} \in C_0^{1,2}$ be the solution of (38). Assume that $\bar{w} \in C_0^{1,2}$ and $p_i = p_i(s,v)$, $i = 1,\dots,N$, solve simultaneously the following system of equations:*

$$\theta_i(s)\bar{w}_v(s,v,i) - h_i \underline{\varphi}_v^R(s, v(1-p_i), i) + \sum_{j\neq i} a_{i,j}(s)\left(\frac{\psi_j(s)}{\psi_i(s)} - 1\right)\bar{w}_v\left(s, v\left[1 + p_i\left(\frac{\psi_j(s)}{\psi_i(s)} - 1\right)\right], j\right) = 0, \tag{42}$$

$$\begin{aligned}\bar{w}_t(s,v,i) - \frac{\eta_i^2}{2}\frac{\bar{w}_v^2(s,v,i)}{\bar{w}_{vv}(s,v,i)} + r_i v\bar{w}_v(s,v,i) + \Bigg\{ & p_i\theta_i(t)v\bar{w}_v(s,v,i) + h_i\left[\underline{w}(s,v(1-p_i),i) - \bar{w}(s,v,i)\right] \\ & + \sum_{j\neq i} a_{i,j}(t)\left[\bar{w}\left(s, v\left(1 + p_i\left(\frac{\psi_j(s)}{\psi_i(s)} - 1\right)\right), j\right) - \bar{w}(s,v,i)\right]\Bigg\} = 0, \end{aligned} \tag{43}$$

for $t < s < R$, with terminal condition $\bar{w}(R,v,i) = U(v)$. We also assume that $p_i(s,v)$ satisfies (29) and (40) (uniformly in v and i) and $\bar{w}$ satisfies (39). Then, the following statements hold true:

(1) *$\bar{w}(t,v,i)$ coincides with the optimal value function $\bar{\varphi}^R(t,v,i) = \varphi^R(t,v,i,0)$ in (26), when $\mathcal{A}_t(v,i,0)$ is constrained to the class of t-admissible feedback controls $(\pi_s^S, \pi_s^P) = (\pi^S_{C_{s^-}}(s, V_{s^-}, H(s^-)), \pi^P_{C_{s^-}}(s, V_{s^-}, H(s^-)))$ such that*

$$\pi_i^S(\cdot,\cdot,z), \pi_i^P(\cdot,\cdot,z) \in C([0,R]\times\mathbb{R}_+),$$

for each $i = 1,\dots,N$, π^S satisfies (40) for a locally bounded function G, and π^P satisfies (29) and (40) (uniformly in v,i,z). If the solution $\bar{w}$ is non-negative, then these bound conditions are not needed.

(2) *The optimal feedback controls are given by $\widetilde{\pi}_s^S := \widetilde{\pi}^S_{C_{s^-}}(s, V_s, H(s))$ and $\widetilde{\pi}_s^P := \widetilde{\pi}^P_{C_{s^-}}(t, V_t, H(s))$ with*

$$\widetilde{\pi}_i^S(s,v,z) = -\frac{\eta_i}{\sigma_i}\frac{\bar{w}_v(s,v,i)}{v\bar{w}_{vv}(s,v,i)}(1-z) - \frac{\eta_i}{\sigma_i}\frac{\underline{w}_v(s,v,i)}{v\underline{w}_{vv}(s,v,i)}z, \tag{44}$$

$$\widetilde{\pi}_i^P(s,v,z) = p_i(s,v)(1-z). \tag{45}$$

5 Construction of Explicit Solutions

In this section, we specialize the framework developed above to defaultable regime-switching markets with logarithmic and CRRA investors. Section 5.1 analyzes a logarithmic investor, while section 5.2 considers a CRRA investor.

5.1 Logarithmic investor

We consider an investor with utility function given by $U(v) = \log(v)$. We will show that the coupled system yielding the optimal pre-default value function and bond investment strategy decouples, thereby facilitating the construction of explicit solutions. We start by giving a lemma, which will be used later to characterize the optimal pre-default value functions, as well as the long/short directionality of the bond investment strategy.

Lemma 5.1. *The system of equations*

$$\theta_i(s) - \frac{h_i}{1-p_i} + \sum_{j \neq i} a_{i,j}(s) \frac{\psi_j(s) - \psi_i(s)}{\psi_i(s) + p_i(\psi_j(s) - \psi_i(s))} = 0, \tag{46}$$

for $i = 1, \ldots, N$, admits a unique real solution $p_i(s)$ in the interval $(M_i, 1)$, where $M_i \in [-\infty, 0)$ is defined as in (29). Moreover, if for each $i, j = 1, \ldots, N$, $a_{i,j}$ and $a^{\mathbb{Q}}_{i,j}$ are continuous functions, then $p(s,i)$ is a continuous function of s.

Proof. For fixed i and s, consider the function

$$f(p_i, s, i) := \theta_i(s) - \frac{h_i}{1-p_i} + \sum_{j \neq i} a_{i,j}(s) \frac{\psi_j(s) - \psi_i(s)}{\psi_i(s) + p_i(\psi_j(s) - \psi_i(s))}.$$

We first observe that $f(p_i, i, s)$ is a continuous function of p_i in the interval $(M_i, 1)$. Indeed, we can write the above summation as

$$\sum_{j \neq i: \psi_j(s) > \psi_i(s)} \frac{a_{i,j}(s)}{\frac{\psi_i(s)}{\psi_j(s) - \psi_i(s)} + p_i} + \sum_{j \neq i: \psi_j(s) < \psi_i(s)} \frac{a_{i,j}(s)}{\frac{\psi_i(s)}{\psi_j(s) - \psi_i(s)} + p_i},$$

and since $1 < \frac{-\psi_i(s)}{\psi_j(s) - \psi_i(s)}$ when $\psi_j(s) < \psi_i(s)$, we have $p_i + \frac{\psi_i(s)}{\psi_j(s) - \psi_i(s)} < 0$ for $p_i \in (M_i, 1)$. Moreover, the previous decomposition also shows for each fixed s, $f(p_i, s, i)$ is strictly decreasing in p_i from $(M_i, 1)$ onto $(-\infty, \infty)$. This implies the existence of a unique $p_i(s)$ such that $f(p_i(s), s, i) = 0$, for any $s > 0$. In light of Kumagai (1980) implicit theorem, we will also have that $p_i(s)$ is continuous if we prove that $f(p_i, s, i)$ is continuous in (p_i, s). The latter property follows because, by assumption, $a_{i,j}$ and $a^{\mathbb{Q}}_{i,j}$ are continuous, which implies directly the continuity of the functions θ_i. The continuity of the functions ψ_j will follow from a similar argument to that of Lemma 2.1. □

The following result characterizes the optimal pre/post post default value functions. The proof is reported in Appendix C.1.

Proposition 5.2. *Assume that the $a^{\mathbb{Q}}_{i,j}$'s and $a_{i,j}$'s are continuous in $[0, T]$. Then, the following statements hold:*

(1) *The optimal post-default value function is given by*

$$\underline{\varphi}^R(t, v, i) = \log(v) + K(t, i),$$

where $0 \le t \le R$, and $K(t) = [K(t,1), K(t,2), \ldots K(t,N)]'$ is the unique positive solution of the linear system of first order differential equations

$$K_t(t) = F(t)K(t) + b(t), \qquad K(R) = \mathbf{0}, \tag{47}$$

where $\mathbf{0} = [0, \ldots, 0]' \in \mathbb{R}^N$ and

$$[F(t)]_{i,j} := -a_{i,j}(t), \quad [b(t)]_i := -\left(r_i + \frac{\eta_i^2}{2}\right), \quad i, j = 1, \ldots, N. \tag{48}$$

(2) *The optimal percentage of wealth invested in stock is given by* $\tilde{\pi}^S(t) = [\tilde{\pi}_1^S(t), \tilde{\pi}_2^S(t), \ldots, \tilde{\pi}_N^S(t)]$*, where*

$$\tilde{\pi}_j^S(t) = \frac{\mu_j - r_j}{\sigma_j^2}, \qquad 0 \le t \le R.$$

(3) *The optimal percentage of wealth invested in the defaultable bond is* $\tilde{\pi}_j^P(t) = p_j(t)\mathbf{1}_{\tau>t}$*, while the optimal pre-default value function is*

$$\overline{\varphi}^R(t, v, i) = \log(v) + J(t, i),$$

where $J(t) = (J(t,1), J(t,2), \ldots J(t,N))'$ *is the unique positive solution of the linear system of first order differential equations*

$$J_t(t, i) = G(t)J(t) + d(t), \qquad J(R) = \mathbf{0}, \tag{49}$$

with

$$\begin{aligned} [G(t)]_{i,j} &= -a_{i,j}(t), \quad (i \neq j), \qquad [G(t)]_{i,i} = h_i - a_{i,i}(t), \\ [d(t)]_i &= -\bigg[r_i + \frac{\eta_i^2}{2} + p_i(t)\theta_i(t) + h_i \left(\log(1 - p_i(t)) + K(t, i)\right) \\ &\quad + \sum_{j \neq i} a_{i,j}(t) \log\left(1 + p_i(t)\left(\frac{\psi_j(t)}{\psi_i(t)} - 1\right)\right)\bigg], \end{aligned} \tag{50}$$

and $p(t) = [p_1(t), p_2(t), \ldots, p_N(t)]$ *is the unique continuous solution of the nonlinear system of equations (46).*

We note that the difference between the pre-default and post-default optimal value function lies in the time and regime dependent component. Moreover, the optimal proportion of wealth invested in stocks is constant in every economic regime, and independent on time and current level of wealth, similarly to the findings in Sotomayor and Cadenillas (2009) and Bo et al. (2010), where infinite-time horizon problems are considered. We also find that the optimal proportion of wealth allocated to the defaultable bond depends on time and regime, but not on the current level of wealth. Bo et al. (2010) find that the optimal allocation only depends on time through the default risk premium. We also have the following corollary.

Corollary 5.3. *Assume the generator matrix* A *to be time-invariant or homogenous (i.e.* $a_{i,j}(t) \equiv a_{i,j}$*, for all* t*). Then,*

$$K(t) = -\int_t^R e^{-(s-t)F} b'(s) ds, \quad \text{and} \quad J(t) = -\int_t^R e^{-(s-t)G} d'(s) ds,$$

where F *and* b *are given in Eq. (48) and* G *and* d *in Eq. (50).*

Proof. It follows directly from application of Lemma C.1, part (3), given the equations for K and J given in Eq. (47) and (49), respectively. □

5.2 CRRA investor

In this section, we consider a CRRA investor with utility $U(v) = \frac{v^\gamma}{\gamma}$, with $0 < \gamma < 1$. In contrast to a logarithmic investor, we will see that the system characterizing the optimal bond strategy and pre-default value function does not decouple. Nevertheless, we provide conditions for the existence of solutions. We start giving the expressions for the post-default value function and stock investment strategy, which similarly to the logarithmic case can be computed explicitly.

Proposition 5.4. *Assume that the* $a_{i,j}^{\mathbb{Q}}$*'s and* $a_{i,j}$*'s are continuous in* $[0, T]$*. Then, the following statements hold:*

(i) *The optimal post-default value function is given by*

$$\underline{\varphi}^R(t,v,i) = v^\gamma K(t,i), \quad (0 \le t \le R),$$

where $K(t) = [K(t,1), K(t,2), \dots K(t,N)]'$ *is the unique positive solution of the linear system of first order differential equations*

$$K_t(t) = F(t)K(t), \qquad K(R) = \frac{1}{\gamma}\mathbf{1}, \tag{51}$$

with $\mathbf{1} = [1,\dots,1]' \in \mathbb{R}^N$ *and*

$$[F(t)]_{i,j} = \begin{cases} -\left(\gamma r_i - \frac{\eta_i^2}{2}\frac{\gamma}{\gamma-1} + a_{i,i}(t)\right), & \text{if } i = j, \\ -a_{i,j}(t). & \text{if } j \ne i. \end{cases} \tag{52}$$

(ii) *The optimal percentage of wealth invested in stock at time t in a post default scenario is given by*

$$\tilde{\pi}_j^S(t) = \frac{\mu_j - r_j}{\sigma_j^2}\frac{1}{1-\gamma}. \tag{53}$$

Proof. (i) It can be checked by direct substitution that $\underline{\varphi}_t^R(t,v,i) = K(t,i)v^\gamma$ solves the Dirichlet problem (38), with terminal condition $U(v) = v^\gamma/\gamma$, if and only if the functions $K(t,i)$, $i = 1,\dots,N$, $0 \le t \le R$, satisfy the system of ODE's given by Eq. (51). Using the substitution $s = R - t$, we have that the solution $\tilde{K}(s)$ of the initial value problem given by

$$\tilde{K}_s(s) = -F(R-s)\tilde{K}(s) \qquad (0 \le s \le R), \qquad \tilde{K}(0,i) = \frac{1}{\gamma}, \qquad (i = 1,\dots,N), \tag{54}$$

is such that $K(t) = \tilde{K}(R-t)$. Using Lemma C.1, part (1), we have that the unique solution of system (54) can be written as $\tilde{K}(s) = \phi_F(R-s,R)\gamma^{-1}\mathbf{1}$. Therefore, using that $K(t) = \tilde{K}(R-t)$, we obtain that $K(t) = \phi_F(t,R)\gamma^{-1}\mathbf{1} = \phi_{-F}(R,t)\gamma^{-1}\mathbf{1}$. As for all $i \ne j$, and for all t, we have $[F(t)]_{i,j} \le 0$, then $\int_0^t [-F(s)]_{i,j} ds \ge 0$. Therefore, using Lemma C.1, part (2), we obtain that $\phi_{-F}(R,t)$ has all nonnegative entries, and consequently $K(t,i) \ge 0$ for all $0 \le t \le R$ and $i = 1,\dots,N$. Hence, $\underline{\varphi}_t^R(t,v,i) \in C_{1,2}^0$ due to facts that $K(t,i) \ge 0$ and v^γ is concave and increasing in v. Under the choice $D(t) = 0, E(t) = \max_{i=1,\dots,N} K(t,i)$, and $G(t) = |(\gamma-1)^{-1}|$, the function $\underline{\varphi}_t^R(t,v,i)$ satisfies the conditions in (39). Therefore, applying Theorem 4.1, we can conclude that, for each $i = 1,\dots,N$, $\underline{\varphi}_t^R(t,v,i)$ is the optimal post-default value function.

(ii) Plugging the expression for $\underline{\varphi}_t^R(t,v,i)$ inside Eq. (41), we obtain immediately Eq. (53). □

We also have the following corollary .

Corollary 5.5. *Assume the rate matrix* F *defined in (52) to be time invariant. Then, we have that*

(1) *The post-default value function is given by*

$$K(t) = e^{(t-R)F}\frac{1}{\gamma}\mathbf{1}'. \tag{55}$$

(2) *For each* $i \in \{1,\dots,N\}$, $K(t,i)$ *is a decreasing function of* t.

Proof. (1) It follows directly from Lemma C.1, part (3), using the expression $\tilde{K}(s) = \phi_F(R-s,R)$.

(2) It is enough to prove that the time derivative vector $K'(t)$ consists of all negative entries. From Eq. (55), we obtain that $K'(t) = Fe^{(t-R)F}\frac{1}{\gamma}\mathbf{1}'$. Using the well known fact that if two matrices A and B commute, then $Ae^{tB} = e^{tB}A$, we get $K'(t) = e^{(t-R)F}F\frac{1}{\gamma}\mathbf{1}'$. We notice that $F\frac{1}{\gamma}\mathbf{1}'$ is a vector whose entries are negative and given by

$$\left[F\frac{1}{\gamma}\mathbf{1}'\right]_i = -\gamma r_i + \frac{\eta_i^2}{2}\frac{\gamma}{\gamma-1}$$

Since $(t-R)F$ consists of positive off-diagonal entries, from lemma C.1, part (2), we have that $e^{(t-R)F}$ has all nonnegative entries, and consequently $K'(t,i) \leq 0$ for all i and t, thus completing the proof. $\square$

We now consider the pre-default case. The following result gives sufficient conditions for the existence of the pre-default value function, provided that a certain non-linear system of ODE's is well-posed.

Proposition 5.6. *Assume that the $a^{\mathbb{Q}}_{i,j}$'s and $a_{i,j}$'s are continuous in $[0,T]$ and let $K(t) = [K(t,1), K(t,2), \dots K(t,N)]'$ be the unique positive solution of the linear system of first order differential equations (51). Suppose*

$$J(t) = [J(t,1), J(t,2), \dots J(t,N)]', \qquad \text{and} \qquad p(t) = [p_1(t), \dots, p_N(t)]',$$

solve simultaneously the system of equations:

$$J_t(t) = G(t, p(t))J(t) + d(t, p(t)), \qquad J(R) = \frac{1}{\gamma}\mathbf{1}, \tag{56}$$

$$\theta_i(t)J(t,i) - h_i K(t,i)(1-p_i(t))^{\gamma-1} + \sum_{j\neq i} a_{i,j}(t)J(t,j)\left(\frac{\psi_j(t)}{\psi_i(t)} - 1\right)\left(1 + p_i(t)\left(\frac{\psi_j(t)}{\psi_i(t)} - 1\right)\right)^{\gamma-1} = 0, \tag{57}$$

where $G : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^{N\times N}$ and $d : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^{N\times 1}$ are given by

$$\begin{aligned}
[G(t,p)]_{i,j} &= -a_{i,j}(t)\left(1 + p_i\left(\frac{\psi_j(t)}{\psi_i(t)} - 1\right)\right)^{\gamma}, \quad (i \neq j), \\
[G(t,p)]_{i,i} &= -\left(-\frac{\eta_i^2}{2}\frac{\gamma}{\gamma-1} + r_i\gamma + p_i\gamma\theta_i(t) - h_i + a_{i,i}(t)\right), \\
[d(t,p)]_i &= -h_i(1-p_i)^{\gamma}K(t,i), \quad p = [p_1, \dots, p_N]'.
\end{aligned} \tag{58}$$

Then, the optimal pre-default value function is given by

$$\overline{\varphi}^R_t(t,v,i) = v^{\gamma}J(t,i). \tag{59}$$

The optimal percentage of wealth invested in stock in the pre-default scenario is given by

$$\tilde{\pi}^S_j(t) = \frac{\mu_j - r_j}{\sigma_j^2}\frac{1}{1-\gamma},$$

while the optimal percentage of wealth invested in bond is $\tilde{\pi}^P_j(t) = \mathbf{1}_{\tau>t}p_j(t)$.

The proof of the previous proposition follows immediately by plugging the function $\overline{\varphi}^R_t(t,v,i)$ in Eq. (59) inside the coupled system given by Eq. (42) and Eq. (43). The optimal stock strategy follows immediately from Theorem 4.2, item (2), using Eq. (59). Therefore, the optimal demand in stock is myopic and independent from the value functions and from the default event, while the optimal defaultable bond strategy is non-myopic and dependent on the relation between historical and risk neutral regime-switching probabilities. Note that the system (56)-(57) can be formulated as a non-linear system of differential equations on $\mathbb{R}_+ \times \mathbb{R}^N_+$ of the form:

$$J_t(t) = \widehat{G}(t, J(t))J(t) + \widehat{d}(t, J(t)), \qquad J(R) = \frac{1}{\gamma}\mathbf{1}, \tag{60}$$

where $\widehat{G} : \mathbb{R}_+ \times \mathbb{R}^N_+ \to \mathbb{R}^{N\times N}$ and $\widehat{d} : \mathbb{R}_+ \times \mathbb{R}^N_+ \to \mathbb{R}^{N\times N}$ are defined for $J = [J_1, \dots, J_N]' \in \mathbb{R}^N_+$ and $t \geq 0$ as

$$\widehat{G}(t,J) = G(t, p(t,J)), \qquad \text{and} \qquad \widehat{d}(t,J) = d(t, p(t,J)),$$

with G and d given as in Proposition 5.6, and $p(t,J) := [p_1(t,J), \dots, p_N(t,J)]'$ defined implicitly by the system of equations

$$\theta_i(t)J_i - h_i K(t,i)(1-p_i(t,J))^{\gamma-1} + \sum_{j\neq i} a_{i,j}(t)J_j\left(\frac{\psi_j(t)}{\psi_i(t)} - 1\right)\left(1 + p_i(t,J)\left(\frac{\psi_j(t)}{\psi_i(t)} - 1\right)\right)^{\gamma-1} = 0. \tag{61}$$

The following Lemma shows that indeed $p(t,J)$ is well-defined for $(t,J) \in \mathbb{R}_+ \times \mathbb{R}^N_+$.

Lemma 5.7. *Assume $J \in \mathbb{R}_+^N$. The system (61) admits a unique real solution $p_i(t,J)$ in the interval $(M_i(t),1)$, where $M_i(t)$ is defined as in (29). Moreover, if for each $i,j = 1,\dots,N$, $a_{i,j}$ and $a^{\mathbb{Q}}_{i,j}$ are differentiable functions of t, then $(t,J) \to p(t,J)$ is differentiable at each $(t,J) \in \mathbb{R}_+ \times (0,\infty)^N$.*

Proof. For fixed i and s, consider the functions

$$f_i(t,p,J) = \theta_i(t)J_i - h_i K(t,i)(1-p_i)^{\gamma-1} + \sum_{j\neq i} a_{i,j}(t)J_j\left(\frac{\psi_j(t)}{\psi_i(t)} - 1\right)\left(1 + p_i\left(\frac{\psi_j(t)}{\psi_i(t)} - 1\right)\right)^{\gamma-1}, \tag{62}$$

where $J = [J_1,\dots,J_N]'$ and $p = [p_1,\dots,p_N]'$. We observe that $f_i(t,p,J)$ is a continuous function of p_i in the interval $(M_i(t),1)$. Moreover, we know by Proposition 5.4 that $K(t,i) \geq 0$ and, by assumption $J_j \geq 0$, thus implying that $p_i \to f_i(t,p,J)$ is strictly decreasing in $p_i \in (M_i,1)$. We consider two cases: $M_i \in (-\infty,0)$ and $M_i = -\infty$. In the first case, it is easy to check that $\lim_{p\to M_i^+} f_i(t,p,J) = \infty$ and $\lim_{p\to 1^-} f_i(t,p,J) = -\infty$. Therefore, applying the Intermediate Value Theorem, there exists unique $p(t,J) = [p_1(t,J),\dots,p_N(t,J)]'$ such that $f_i(t,p(t,J),J) = 0$, for $i = 1,\dots,N$. The case $M_i = -\infty$ means that $\psi_j(t)/\psi_i(t) \leq 1$ for all $j \neq i$. Then, we have $\lim_{p\to-\infty} f_i(t,p,J) = \theta_i(t)J_i$. By the definition (35), $\theta_i > 0$ (as $h_i, L_i > 0$) and, hence, Intermediate Value Theorem implies again the existence of a unique $p(t,J) = [p_1(t,J),\dots,p_N(t,J)]'$. The differentiability of $p(t,J)$ follows directly from the implicit function theorem. □

Next, we prove that the non-linear system (60) has a unique solution in a local neighborhood $\{(t,J) \in \mathbb{R}_+ \times R_+^N : |t-R| < a, |J_i - 1/\gamma| < b, i = 1,\dots,N\}$ for some $a > 0$ and $b > 0$. For illustration purposes, let us consider in detail the case $N = 2$. In that case, the system (56-57) takes the form:

$$\begin{aligned}
J_t(t,1) &= -a_{1,2}(t)J(t,2)\left(1 + p_1(t,J(t))\left(\frac{\psi_2(t)}{\psi_1(t)} - 1\right)\right)^{\gamma} && (63)\\
&\quad - \left(\xi_1(t) + \gamma\theta_1(t)p_1(t,J(t))\right)J(t,1) - h_1K(t,1)(1-p_1(t,J(t)))^{\gamma},\\
J_t(t,2) &= -a_{2,1}(t)J(t,1)\left(1 + p_2(t,J(t))\left(\frac{\psi_1(t)}{\psi_2(t)} - 1\right)\right)^{\gamma} && (64)\\
&\quad - \left(\xi_2(t) + \gamma\theta_2(t)p_2(t,J(t))\right)J(t,2) - h_2K(t,2)(1-p_2(t,J(t)))^{\gamma},\\
J(R,1) &= J(R,2) = \frac{1}{\gamma}, && (65)
\end{aligned}$$

where $\xi_i(t) := -\frac{\eta_i^2}{2}\frac{\gamma}{\gamma-1} + r_i\gamma - h_i + a_{i,i}(t)$ and the functions $p_1(t,J), p_2(t,J) : \mathbb{R}_+ \times \mathbb{R}_+^2 \to \mathbb{R}$ are defined implicitly by the following equations for any $J := [J_1, J_2]$:

$$0 = \theta_1(t)J_1 - h_1K(t,1)(1-p_1(t,J))^{\gamma-1} + a_{1,2}(t)J_2\left(\frac{\psi_2(t)}{\psi_1(t)} - 1\right)\left(1 + p_1(t,J)\left(\frac{\psi_2(t)}{\psi_1(t)} - 1\right)\right)^{\gamma-1}, \tag{66}$$

$$0 = \theta_2(t)J_2 - h_2K(t,2)(1-p_2(t,J))^{\gamma-1} + a_{2,1}(t)J_1\left(\frac{\psi_1(t)}{\psi_2(t)} - 1\right)\left(1 + p_2(t,J)\left(\frac{\psi_1(t)}{\psi_2(t)} - 1\right)\right)^{\gamma-1}. \tag{67}$$

Note that while the range of one of the functions p_i's is bounded, the other function will be unbounded. For instance, if $\psi_2(t)/\psi_1(t) > 1$, then $p_1(t,J)$ will take values on the bounded domain $(-(\psi_2(t)/\psi_1(t) - 1)^{-1}, 1)$, while $p_2(t,J)$ will take values on $(-\infty, 1)$. In turn, this fact makes the right hand-side of equation (64) potentially unbounded and also is the main reason why it is not possible to obtain global existence without further restrictions (see Remark C.2 in Appendix C.2 for more information). The following result shows the local existence and uniqueness of the solution (the proof of Proposition 5.8 is reported in Appendix C.2).

Proposition 5.8. *Suppose that $a_{i,j}(t)$ and $a^{\mathbb{Q}}_{i,j}(t)$ are differentiable functions. Then, for any $b > \gamma$, there exists an $\alpha := \alpha(b) > 0$ and a unique function $J : (R-\alpha, R] \to [b^{-1}, b]^N$ satisfying (63-64) with terminal condition $J(R,1) = J(R,2) = \gamma^{-1}$.*

Remark 5.9. *Under the conditions of Proposition 5.8, it is known (see, e.g., Theorem 1.263 in Chicone (2006)) that if* $(R - \underline{\alpha}, R + \bar{\alpha})$ *(with* $\bar{\alpha}, \underline{\alpha} \in (0, \infty]$*) is the maximal interval of existence of the solution of (63-65) and* $\underline{\alpha} < \infty$*, then either* $|J(t)| \to \infty$*,* $J(t,1) \to 0$*, or* $J(t,2) \to 0$ *as* $t \to R - \underline{\alpha}$*. Moreover, the solution of (63-65) can be found by the standard Picard's fixed-point algorithm. Hence, for instance, one can show numerically whether the solution is well defined in the whole interval* $[0, R]$ *by analyzing whether the numerical solution blows up or converges to* 0.

Based on the above analysis, we deduce that the pre-default scenario of the power investor differs from the one of the logarithmic investor. In the logarithmic case, the two systems decouple, and the pre-default value function may be obtained in terms of an integral of a matrix exponential for time invariant generators, see Corollary 5.3 for details.

6 Economic Analysis

In this section, we provide a detailed economic analysis of the corporate bond investment strategies and value functions for the type of investors considered in Section 5. The objective is to investigate how the interplay between the historical and risk-neutral generators of the Markov chain, time to maturity, default intensity and loss parameters, affect the directionality of the bond investment strategy. Moreover, we illustrate how the risk aversion level γ of the power utility investor affects the bond investment strategy, including the limiting case of the logarithmic investor. We first introduce the necessary notation and terminology in Section 6.1. Section 6.2 characterizes the "directionality" of the bond investment strategy for CRRA and logarithmic investors in terms of corporate returns, instantaneous forward rates, and expected recovery at default. We present a comparative static analysis under a realistic simulation scenario in Section 6.3.

6.1 Notation and terminology

Throughout, $\mathbb{R}^{n \times m}$ (respectively, $\mathbb{R}_+^{n \times m}$) denotes the set of $n \times m$ (resp., positive) real matrices G. Given $G \in \mathbb{R}^{n \times m}$, $[G]_{i,j}$ denotes its (i,j) entry. Next, we give some definitions, which will be used to characterize the optimal strategies. Let us recall that C_t is given by (3). We notice that the pre-default regime conditioned bond price $\psi_i(t)$, defined in (20) depends on the maturity T. In the following, we will sometimes use the notation $\psi_i(t,T)$ to emphasize this dependence. In all definitions to follow, we assume the macroeconomy to be in the i^{th} regime at t. Let $A^{\Upsilon}(t) = [a_{i,j}^{\Upsilon}(t)]_{i,j=1,\ldots,N}$ be the infinitesimal generator of the Markov process (X_t), under a given equivalent probability measure Υ.

Definition 6.1. *For any* $s < t$*, we have the following terminology:*

- The expected *corporate bond return* per unit time under the measure Υ*,* during the interval $[t, s]$*,* is defined as

$$\mathbb{E}_i^{\Upsilon}(t,s) := \frac{1}{s-t}\mathbb{E}^{\Upsilon}\left[\frac{\psi_{C_s}(s,T) - \psi_i(t,T)}{\psi_i(t,T)}\middle| X_t = e_i\right].$$

- The expected *instantaneous corporate bond return,* under the measure Υ*,* is defined as

$$\mathbb{E}_i^{\Upsilon}(t) := \lim_{s \to t^+} \mathbb{E}_i^{\Upsilon}(t,s). \tag{68}$$

- The *instantaneous forward rate* of the defaultable bond at time t is defined as

$$g_i(t) := -\frac{\partial \log \psi_i(t,T)}{\partial T}\bigg|_{T=t}. \tag{69}$$

Note that the above definitions are meaningful because the function $\psi_i(t)$ is differentiable in time, as it has been shown in Capponi, Figueroa-López, and Nisen (2012b) (Lemma 4.1 therein). We have the following useful results.

Lemma 6.1. *The instantaneous forward rate is given by*

$$g_i(t) = -\Bigg[\sum_{j\neq i} a^{\mathbb{Q}}_{i,j}\frac{\psi_j(t,T)-\psi_i(t,T)}{\psi_i(t,T)}\Bigg], \tag{70}$$

while the instantaneous corporate bond return, under the equivalent measure Υ, is given by

$$\mathbb{E}^{\Upsilon}_i(t) = \sum_{j\neq i} a^{\Upsilon}_{i,j}(t)\frac{\psi_j(t,T)-\psi_i(t,T)}{\psi_i(t,T)}. \tag{71}$$

Proof. For the first identity, a simple calculation shows that

$$\begin{aligned}\frac{\partial\psi_i(t,T)}{\partial T}\bigg|_{T=t} &= \lim_{\Delta t\to 0}\frac{\psi_i(t,T)-\psi_i(t+\Delta t,T)}{\Delta t} = \lim_{\Delta t\to 0}\sum_{j\neq i}\frac{p^{\mathbb{Q}}_{i,j}(t,t+\Delta t)}{\Delta t}\left(\psi_j(t+\Delta t,T)-\psi_i(t+\Delta t,T)\right)\\ &= \sum_{j\neq i} a^{\mathbb{Q}}_{i,j}\left(\psi_j(t,T)-\psi_i(t,T)\right).\end{aligned}$$

Using the previous equation and the definition of instantaneous forward rate given in Eq. (69), we obtain Eq. (70). For the second identity, let $p^{\Upsilon}_{i,j}(t,s)$ denote the probability that the chain with generator $A^{\Upsilon}(t)$ transits to regime j at time $s>t$, given that it is in regime i at time t. Then,

$$\begin{aligned}E^{\Upsilon}_i(t) &= \lim_{s\to t^+}\frac{1}{s-t}\mathbb{E}^{\Upsilon}\left[\frac{\psi_{C_s}(s,T)-\psi_i(t,T)}{\psi_i(t,T)}\bigg| X_t=e_i\right] = \lim_{s\to t^+}\sum_{j\neq i}\frac{p^{\Upsilon}_{i,j}(t,s)}{s-t}\frac{\psi_j(s,T)-\psi_i(t,T)}{\psi_i(t,T)}\\ &= \sum_{j\neq i} a^{\Upsilon}_{i,j}(t)\frac{\psi_j(s,T)-\psi_i(t,T)}{\psi_i(t,T)}.\end{aligned}$$

□

6.2 Characterization of long/short optimal investment strategies

In this section, we provide conditions under which the logarithmic and CRRA investor would go long or short in the defaultable bond. Recall that, under the i^{th} economy regime, the optimal percentage of wealth invested in the defaultable bond is given by $\tilde{\pi}^P_i(t) = \mathbf{1}_{\tau>t}p_i(t)$, where $p_i(t)$ is identified in the Lemma 5.1. We first characterize the directionality of the strategy for a logarithmic investor.

Lemma 6.2. *It holds that $p_i(t)>0$ if and only if*

$$\mathbb{E}^{\mathbb{P}}_i(t) + g_i(t) > h_i(1-L_i). \tag{72}$$

Proof. First, we establish that $p_i(t)>0$ if and only if the following relation holds.

$$\sum_{j\neq i}\left(\frac{\psi_j(t)}{\psi_i(t)}-1\right)\left(a_{i,j}(t)-a^{\mathbb{Q}}_{i,j}(t)\right) > h_i(1-L_i). \tag{73}$$

Using Lemma 5.1 and Eq. (35), we have that the fraction of wealth invested in the bond at time t satisfies the following equation

$$\sum_{j\neq i} a_{i,j}(t)\frac{1}{p_i(t)+\frac{\psi_i(t)}{\psi_j(t)-\psi_i(t)}} = \frac{h_i}{1-p_i(t)} - h_iL_i + \sum_{j\neq i} a^{\mathbb{Q}}_{i,j}(t)\frac{1}{\frac{\psi_i(t)}{\psi_j(t)-\psi_i(t)}}. \tag{74}$$

Notice that, for each fixed t, the left hand side is a strictly decreasing function of $p_i(t)$ from $(M_i,1)$ to $(-\infty,\infty)$. The right hand side, instead, is a strictly increasing function of $p_i(t)$ defined from $(M_i,1)$ to $(0,\infty)$. Moreover, we know from lemma 5.1 that there exists a unique $p_i(t)$ satisfying Eq. (74). Evaluating both left and right hand side at $p_i(t)=0$

leads to the conclusion that $p_i(t) > 0$ if and only if Eq. (73) holds. From the definition of expected instantaneous corporate bond return given in Eq. (68), computed under the historical measure $\Upsilon = \mathbb{P}$, and using Eq. (2), we obtain that $E_i^{\mathbb{P}}(t) = \sum_{j \neq i} a_{i,j}(t) \frac{\psi_j(t)-\psi_i(t)}{\psi_i(t)}$. Using this relation and Eq. (70), we may rewrite Eq. (73) as in Eq. (72). ☐

We say that the *long condition* of the logarithmic investor is satisfied when the relationship (72) holds. The following corollary provides sufficient conditions for the logarithmic investor to always go short in the defaultable security.

Corollary 6.3. *For a logarithmic investor, the following statements hold:*

(i) *If $N = 1$, then for each fixed t, we have $p(t) = 1 - \frac{1}{L_1} < 0$.*

(ii) *For fixed t, i, if $a^{\mathbb{Q}}_{i,j}(t) = a_{i,j}(t)$ for any $j \neq i$, then $p_i(t) < 0$.*

Proof. Both in the case when $N = 1$ and in the case when $a^{\mathbb{Q}}_{i,j}(t) = a_{i,j}(t)$, we have that $E_i^{\mathbb{P}}(t) = -g_i(t)$. Therefore, the long condition in (72) will be never satisfied. Moreover, in case when $N = 1$, we can see directly from Eq. (46) that $p_i(t) = 1 - \frac{1}{L_1}$. This ends the proof. ☐

Corollary 6.3 show that in the mono-regime scenario, or in the case when $a^{\mathbb{Q}}_{i,j}(t) = a_{i,j}(t)$, the corporate bond return gets reduced by the instantaneous forward credit spreads $g_i(t,T)$ by an amount which makes it smaller than the expected recovery at default. This leads the investor to go always short in the security, because the compensation offered by the market is not enough to compensate him for the credit risk incurred. Moreover, item (i) of Corollary 6.3 shows that (1) in case of zero recovery on the defaultable bond ($L_i = 1$), the logarithmic investor would not trade at all in the defaultable security, and (2) the amount of bond units shorted is a decreasing function of the loss incurred at default. Next, we characterize the directionality of the strategy for the power investor. Let us define a measure $\tilde{\mathbb{P}}$, equivalent to the historical measure $\mathbb{P}$, via the generator $A^{\tilde{\mathbb{P}}} = [a^{\tilde{\mathbb{P}}}_{i,j}]$ of the Markov process given by

$$a^{\tilde{\mathbb{P}}}_{i,j}(t) := a_{i,j}(t) \frac{J(t,j)}{J(t,i)}, \quad (j \neq i), \qquad \text{and} \qquad a^{\tilde{\mathbb{P}}}_{i,i}(t) := -\sum_{k=1, k \neq i}^{N} a^{\tilde{\mathbb{P}}}_{i,j}(t), \tag{75}$$

where $J(t,j) > 0$ is the time component of the optimal pre-default value function defined by Eq. (56), (57), and (58). Intuitively, the measure $\tilde{\mathbb{P}}$ is redistributing the mass of the historical distribution $\mathbb{P}$ towards those regimes j with higher values of the pre-default value function with respect to regime i. We next characterize the directionality of the strategy for the CRRA investor, where $p_i(t)$ is coupled with the pre-default value functions as indicated in Proposition 5.6.

Lemma 6.4. *Under the assumptions of Proposition 5.6, we have that $p_i(t) > 0$ if and only if*

$$\mathbb{E}_i^{\tilde{\mathbb{P}}}(t) + g_i(t) > h_i \left(\frac{K(t,i)}{J(t,i)} - L_i \right). \tag{76}$$

Proof. First, we show that $p_i(t) > 0$ if and only if the following relation holds.

$$\sum_{j \neq i} a_{i,j}(t) J(t,j) \frac{\psi_j(t) - \psi_i(t)}{\psi_i(t)} > h_i(K(t,i) - L_i J(t,i)) + \sum_{j \neq i} a^{\mathbb{Q}}_{i,j}(t) J(t,i) \frac{\psi_j(t) - \psi_i(t)}{\psi_i(t)}. \tag{77}$$

We may rewrite Eq. (57) from Proposition 5.6 as

$$\sum_{j \neq i} \frac{a_{i,j}(t) J(t,j)}{\frac{\psi_i(t)}{\psi_j(t)-\psi_i(t)} \left(1 + p_i(t) \frac{\psi_j(t)-\psi_i(t)}{\psi_i(t)}\right)^{1-\gamma}} = \frac{h_i}{(1-p_i(t))^{1-\gamma}} K(t,i) - h_i L_i J(t,i) + \sum_{j \neq i} \frac{a^{\mathbb{Q}}_{i,j}(t) J(t,i)}{\frac{\psi_i(t)}{\psi_j(t)-\psi_i(t)}}, \tag{78}$$

where we have used the expression for $\theta_i(t)$ given in Eq. (35). It can be easily checked that the left hand side of Eq. (78) is a decreasing function of $p_i(t)$ from $(M_i, 1)$ to $(-\infty, \infty)$. The right hand side, instead, is a strictly increasing function of $p_i(t)$, defined from $(M_i, 1)$ to $(0, \infty)$. Since we are assuming that there exists a unique solution $p_i(t)$ to the nonlinear equation (78), then we can evaluate both left and right hand side of Eq. (78) at $p_i(t) = 0$, and obtain that $p_i(t) > 0$ if and only if Eq. (77) holds. From the definition of expected instantaneous corporate bond return given in Eq. (68), computed under the measure $\Upsilon = \tilde{\mathbb{P}}$, we obtain that $\mathbb{E}_i^{\tilde{\mathbb{P}}}(t) = \sum_{j \neq i} a_{i,j}^{\tilde{\mathbb{P}}}(t)(\psi_j(t) - \psi_i(t))/\psi_i(t)$. Using the latter relationship and Eq. (70), we may rewrite Eq. (78) as in Eq. (76). □

In analogy with $h_i(1 - L_i)$, representing the expected recovery rate in the i-th regime, we refer to the quantity $h_i\left(\frac{K(t,i)}{J(t,i)} - L_i\right)$ as the *adjusted expected recovery rate.* Similarly to the case of the logarithmic investor, we say that the long condition is satisfied when the relationship (76) holds. The long condition of the power investor is similar to the one of the logarithmic investor, except that the former computes the expected return of the corporate bond under a probability measure equivalent to the historical measure, but adjusted for default risk through the ratio of pre-default value functions. Then, he decides to go long in the security only if such return plus the instantaneous forward rate is higher than the adjusted expected recovery rate. High values of the instantaneous forward rate indicate high levels of default risk perceived by the market, and consequently lower the bond price. Therefore, if the macroeconomy is in regimes where the bond is cheap but with high expected return (left hand sides of Eq. (72) and Eq. (76) are large), then the the investors would go long as long as such "benefit" is higher than their prescribed thresholds (right hand sides of Eq. (72) and Eq. (76)).

Although it is generally impossible to obtain explicit formulas for the bond investment strategy of a power investor, it is possible to do so in special cases. The rest of this section shows that this is indeed the case, if we consider a square root utility investor, i.e. $\gamma = \frac{1}{2}$, and assume that $N = 1$. Below, we use the notation $L_1^\gamma = L_1\gamma - 1$.

Lemma 6.5. *The optimal investment in the defaultable bond for a square root utility investor is given by*

$$p_1(t) = \frac{(L_1 - 1)\left(-e^{2h_1 t L_1^\gamma} + e^{2h_1 R L_1^\gamma} L_1 \left(L_1^\gamma + \gamma\right)\right)}{e^{2h_1 t L_1^\gamma} L_1(\gamma - 1) + e^{2h_1 R L_1^\gamma}(L_1 - 1)L_1\left(L_1^\gamma + \gamma\right)}. \tag{79}$$

Remark 6.6. *It can be easily checked that* $p_1(t) < 0$*. The numerator of Eq.* (79) *is positive because* $\gamma = \frac{1}{2}$ *and* $0 \leq L_1 \leq 1$*. The denominator is negative because* $2h_1 t L_1^\gamma > 2h_1 R L_1^\gamma$ *and* $|\gamma - 1| > (L_1 - 1)\left(L_1^\gamma + \gamma\right)$*, thus yielding that* $p_1(t) < 0$*.*

While the bond strategy for the logarithmic investor only depends on loss given default in mono-regime scenarios (see item (i) of Corollary 6.3), we can see from Eq. (79) that for the power investor also depends on the default intensity and on time. Similarly to the logarithmic investor, we find that $\lim_{L_1 \to 0} p_1(t) = -\infty$ and $\lim_{L_1 \to 1} p_1(t) = 0$. Therefore, as in the case of the logarithmic investor, the investor does not allocate any wealth to the defaultable bond if the loss $L_1 = 1$. In case of a very low intensity h_1, this may be explained by the fact that, although with high probability default will not occur, in case when it does there is zero recovery, and thus a risk averse investor will tend to avoid the exposure to default risk. In case of very high intensities, this happens because although the investor will likely realize a profit by shorting the bond due to the high default probability, the bond selling price will be close to zero if $L_1 \to 1$ and $h_1 \to \infty$ (see Eq. (10)). This, in turn, will make the realized profit equal to zero. Moreover, we find the asymptotics $\lim_{h_1 \to 0} p_1(t) = 1 - 1/L_1^2$ and $\lim_{h_1 \to \infty} p_1(t) = 2(1 - 1/L_1)$.

6.3 Comparative statics analysis

We describe the simulation scenario in Section 6.3.1 and present the comparative statics results in Section 6.3.2.

6.3.1 The simulation scenario

In order to present a realistic simulation setting, we take the estimates of the historical generator of the Markov chain obtained by Giesecke et al. (2011), who employed a three-state homogenous regime-switching model to examine the effects of an array of financial and macroeconomic variables explaining variations in the realized default rates of the

$a_{i,j}$	1	2	3
1	-0.10474	0.08865	0.01609
2	0.84799	-0.848	0.00001
3	0.69561	0.00001	-0.69562

	h	L
1	0.741%	10%
2	4.261%	40%
3	11.137%	90%

Table 1: Left panel shows the historical generator of the Markov chain obtained in Giesecke et al. (2011). The rows indicate the starting state, while the columns indicate the ending state of the chain. The right panel shows the default intensities as reported in Giesecke et al. (2011) as well as our loss rates given default associated to three regimes.

$a_{i,j}^{\mathbb{Q}}$	1	2	3
1	-0.380313	0.33687	0.043443
2	0.254397	-0.254397	0
3	0.208683	0.000006	-0.208689

Table 2: The generator of the Markov chain under the risk-neutral measure. The rows indicate the starting state, while the columns indicate the ending state of the chain.

U.S. corporate bond market over the course of 150 years. For completeness, we report their value estimates in Table 1. Giesecke et al. (2011) also estimate the annual default rates for each regime. We report them in Table 1, along with the corresponding losses, which we choose to be increasing in the credit riskiness of the regime. Table 1 shows three distinct regimes, hereon referred to as "low", "middle", and "high" default regime. It also indicates that the probability of remaining in a low-default regime is very large, while the other two regimes are much less persistent. Since our objective is to measure the impact of the default event on the optimal strategies, we assume that the annual interest rate is the same across all regimes and equal to 3%. We also assume that the annual stock volatility is equal to 5% in all regimes. We set the drift of the stock equal to $7\%, 5\%$, and 3%, respectively in the low, middle and high default regime. The risk-neutral generator of the Markov chain is given in Table 2, and chosen so that the risk-neutral probability of moving to riskier (safer) regimes is higher (lower) than the corresponding historical probability. This is consistent with empirical findings showing the existence of a positive default risk premium. We take the investment horizon R to be the same as the maturity T of the defaultable bond, and equal to one year.

6.3.2 Numerical results

We present the results obtained under the simulation scenario detailed in Section 6.3.1. We use a fixed point algorithm to solve the coupled system introduced in Proposition 5.6. Namely, the system consists of (1) a system of three ordinary differential equations for the time component of the pre-default value function and (2) a system of three nonlinear equations for the defaultable bond strategy. This algorithm initially sets the pre-default value function equal to the post-default counterpart. Then, it keeps iterating between solving for the time component of the pre-default value function and the bond investment strategy until a desired level of convergence is achieved.

We start showing the behavior of the bond investment strategy for the power utility investor under three different levels of risk aversions, and for the logarithmic investor. Figure 1 shows that the investor always shorts the bond security unless the macroeconomy is in the high risk regime, and the time to maturity is not too small. This is in agreement with the right bottom graph of the figure, showing that the square root investor goes long only if the economy is in the high risk macroeconomic regime and there are still about 2.4 months left to maturity.

In the high risk regime, the corporate bond returns are positive and of larger magnitude than the negative instantaneous forward rate. This is because, any transition from this regime will be to a safer regime and will occur with larger probability under the historical measure, see Table 1 and 2. On the contrary, in the low risk regime, the corporate bond returns are negative and of smaller magnitude than the positive instantaneous forward rate, because,

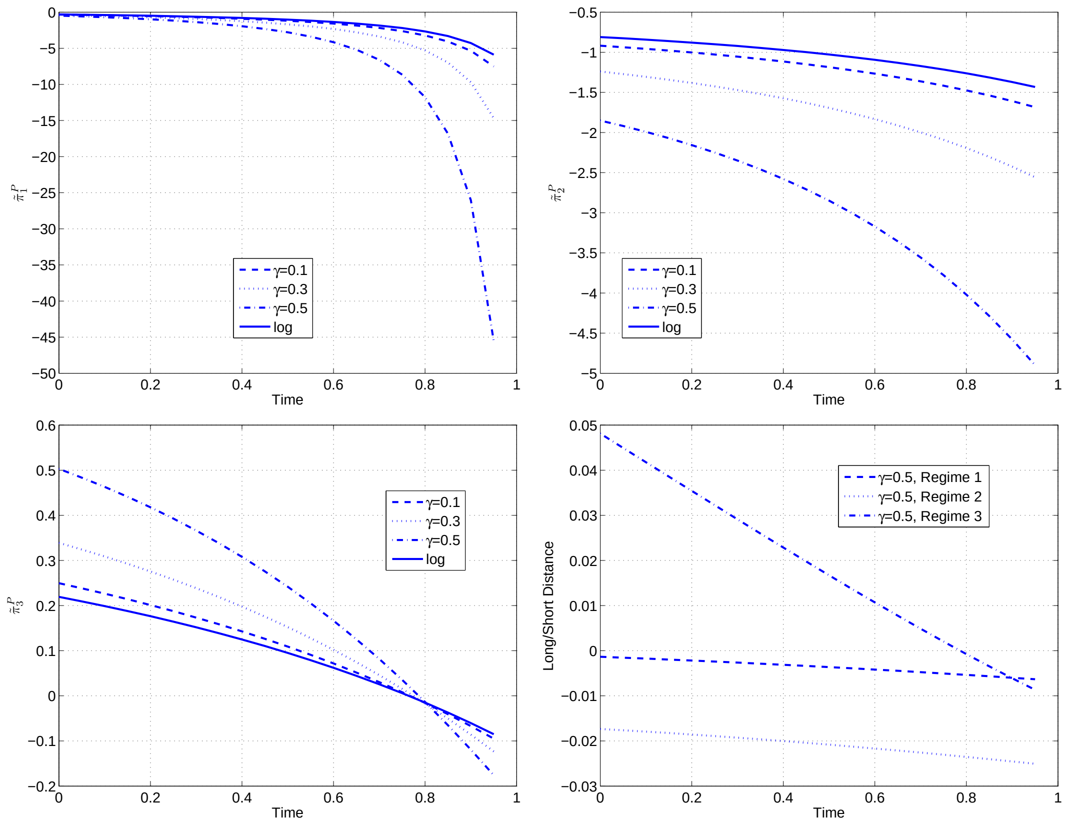

Figure 1: Optimal bond strategy $\tilde{\pi}_1^P$, $\tilde{\pi}_2^P$ and $\tilde{\pi}_3^P$ versus time, for different levels of risk aversion γ, and for the logarithmic investor. The bottom right panel shows the long/short distance for the square root ($\gamma = 0.5$) investor, defined as $\mathbb{E}_i^{\tilde{\mathbb{P}}}(t) + g_i(t) - h_i\left(\frac{K(t,i)}{J(t,i)} - L_i\right)$.

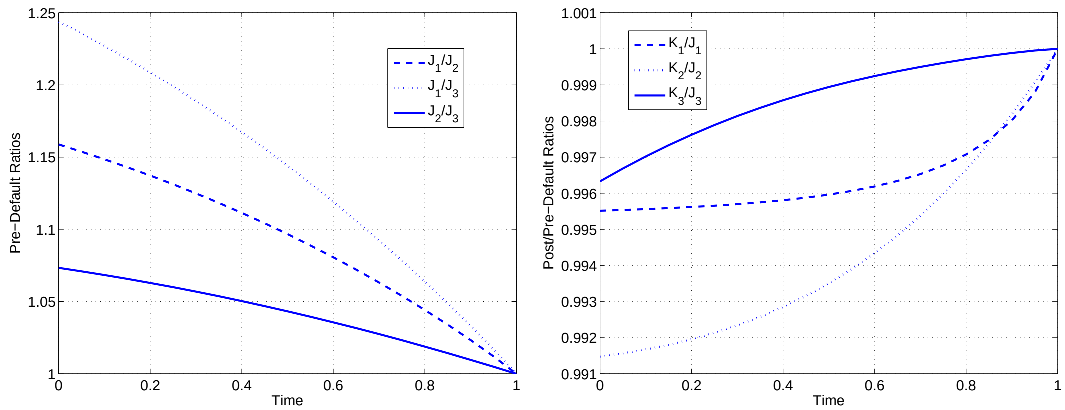

Figure 2: The left panel reports the behavior over time of the pre-default value functions ratio. The right panel reports the behavior over time of the post/pre default value functions ratio. The risk aversion level is $\gamma = 0.5$.

any transition from this regime will be to a riskier regime and will occur with higher probability under the risk neutral measure. Therefore, in both of these cases the left hand sides of Eq. (72) and Eq. (76) and will be positive. However, under the historical measure, transitions from high to low risk regimes occur with probability large enough to satisfy the long condition when the time to maturity is not too small. On the contrary, in the low risk regime, the risk neutral probabilities of transitioning to the riskier regimes are small, and thus do not generate instantaneous forward rates large enough to satisfy the long condition. In the middle risk regime, the historical probability of transitioning to the low risk regime is very high, thus generating a positive corporate bond return. However, the (adjusted) expected recovery is the largest in the middle risk regime (from the right panel in Table 1, we can see that $h_2(1-L_2) > h_i(1-L_i), i = \{1,3\}$), and the long bond condition is never satisfied. All this appears to indicate that, in an unfavorable market situation (which in our model corresponds to the macroeconomy being in the low default regime, from where the macroeconomy can only get worse), it is preferable to go short in defaultable assets. These results are in agreement with Callegaro et al. (2010), who come to similar conclusions using a different model. As the time approaches maturity, the bond prices in the different regimes will get closer and converge to one, therefore the expected return will decrease, until reaching a point where the sum of expected bond return and instantaneous forward rate becomes smaller than the (adjusted) expected recovery, which triggers the investor decision to go short. The bottom left graph of Figure 1 shows that, in the high risk regime, the logarithmic investor changes the directionality of his strategy from long to short before the power utility investor. The reason for that can be understood from Figure 2. Here, we can see from the left graph that the pre-default value function is higher in safer regimes. This means the power utility investor is more "optimistic" than the history, because his expected corporate bond return, computed under the equivalent measure $\tilde{\mathbb{P}}$ given in Eq. (75), is always larger than the one computed by the logarithmic investor under the historical measure $\mathbb{P}$. Moreover, the right graph of Figure 2 shows that the ratio of post vs pre-default value function is always smaller than one in all regimes. This implies that the adjusted expected recovery is smaller than the expected recovery, and thus that the power investor requires a lower threshold to trigger his decision to go long. The conclusion is that when the long bond condition is satisfied for the logarithmic investor, it will be surely satisfied for the power investor. This is expected, because the logarithmic investor is more risk averse than the power investor, and consequently he is more resilient to being exposed to default risk through purchasing of the bond security.

It is evident from Eq. (46) and (57) that the optimal investment strategy in the defaultable bond is time dependent for both the logarithmic and power investor. The graphs of Figure 1 further illustrate that the investor buys (sells) a larger (smaller) number of bond units when the time to maturity is higher. This happens because, for a given level of default probability, the bond price appreciates in value as maturity approaches. As in our scenario, the risk neutral generator is time invariant, the likelihood of a default event happening within a given interval remains the same as time progresses. Therefore, the investor should buy more (sell less) in the defaultable bond when its price is low, that is for longer time to maturity, all else being equal. Similar findings are also obtained from Bielecki and Jang (2006) in a different framework.

Moreover, we can see from Figure 1 that the larger the risk aversion level of the investor, the smaller the number

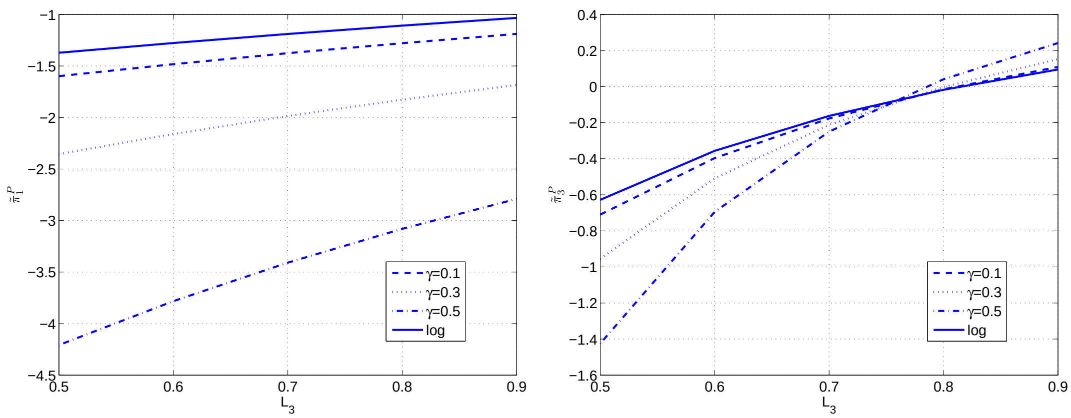

Figure 3: Optimal bond strategy $\tilde{\pi}_1^P$, and $\tilde{\pi}_3^P$ versus the loss given default, for different levels γ of risk aversion, and for the logarithmic investor. The time t is fixed to 0.5.

of bond units traded (with the logarithmic investor trading the least due to his higher level of risk aversion). This is expected because an investor who goes long in the bond security is exposed to the default risk, and therefore buys a smaller number of units with respect to a less risk averse investor. An investor who goes short is instead exposed to regime-switching risk, and thus sells a shorter number of units because this would result in a mark-to-market loss in case the macroeconomy switches to a safer regime.

We conclude the section with an analysis of the behavior of the bond investment strategy as a function of the loss given default in the high risk regime. Figure 3 shows that as the loss increases, the short (long) investor will sell (buy) a smaller (higher) number of bond units. This is expected because, all else being equal, larger losses will translate into cheaper bond prices, and therefore, following the buy low sell high rule, the long investor will buy more and the short investor will sell less. As expected, more risk averse investors will trade a smaller number of bond units to reduce exposure to default or regime-switching risk.

7 Conclusions

We considered the continuous time portfolio optimization problem in a defaultable market, consisting of a stock, a defaultable bond, and a money market account. We assumed that the price dynamics of the assets are governed by a regime-switching model. We have shown that the utility maximization problem may be separated into a pre-default and a post-default optimization subproblems, and proven verification theorems for both cases under the assumption that the solutions are monotonic and concave in the wealth variable v. The post-default verification theorem shows that the optimal value function is the solution of a nonlinear Dirichlet problem with terminal condition. The pre-default verification theorem shows that the optimal pre-default value function and the optimal bond investment strategy can be obtained as the solution of a coupled system of nonlinear partial differential equations with terminal condition (satisfied by the pre-default value function) and nonlinear equations (satisfied by the bond investment strategy). Each equation is associated to a different regime, and the dependence of a regime i from another regime j comes through the Markov transition rates and the ratio between the defaultable bond prices in regime j and regime i. Our results imply that the pre-default optimal value function and the bond investment strategy depend on the optimal post-default value function.

We obtained explicit constructions for pre/post default value functions as well as stock investment strategies for the case of a logarithmic and CRRA investor. We provided a precise characterization of the conditions determining the long/short directionality of the investment strategy in the defaultable bond. We found that investors go long when the macroeconomy is in regimes where bonds are cheap and offer high expected returns. Additionally, we found that the number of bonds units traded by investors increases with their level of risk aversion, and that each investor trades more in the defaultable bond if the planning horizon is higher.

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A Derivation of the generator of (t, V_t, C_t, H_t)

Obviously, from the definition of C_t given in Eq. (3), $\{C_t\}_{t\geq 0}$ is a Markov process with values in $\{1,\dots,N\}$ and infinitesimal generator $A(t) = [a_{i,j}(t)]_{i,j=1,\dots,N}$. In particular, for any function $g : \{0,\dots,N\} \to \mathbb{R}$,

$$M^g(t) := g(C_t) - g(C_0) - \int_0^t (Ag)(C_u, u)du,$$

is a martingale under $\mathbb{P}$, where we have used the notation

$$Ag(i,t) := [\mathbf{g}' A'(t)]_i = \sum_{j=1}^N a_{i,j}(t) g(j), \quad \text{with} \quad \mathbf{g} = (g(1), \dots, g(N))';$$

c.f. Proposition 11.2.2 in Bielecki and Rutkowski (2001). Note that this result follows directly from the semimartingale decomposition (1) by multiplying (from the left) both sides there by $\mathbf{g}'$. In particular, also note that

$$M^g(t) = \mathbf{g}' M^{\mathbb{P}}(t), \quad \text{and} \quad M_j^{\mathbb{P}}(t) = H_t^j - \int_0^t a_{C_u,j}(u)du, \tag{80}$$

where we used notation (13). Let us assume that V admits the following Markov-modulated dynamics:

$$dV_t = \alpha_{C_t} dt + \vartheta_{C_t} dW_t + \sum_{j=1}^N \beta_{C_{t^-},j} dM_j^{\mathbb{P}}(t) - \gamma_{C_{t^-}} d\xi_t^{\mathbb{P}}, \tag{81}$$

where $\alpha_i(\cdot,\cdot,z), \vartheta_i(\cdot,\cdot,z), \beta_{i,j}(\cdot,\cdot,z), \gamma_i(\cdot,\cdot,z)$ are deterministic smooth functions in $[0,\infty)\times\mathbb{R}$ for any $i,j \in \{1,\dots,N\}$ and $z \in \{0,1\}$, and all the coefficients in (81) are evaluated at $(t, V_{t^-}, H(t^-))$. In terms of the processes (12)-(13) and using (80), we first note that

$$\begin{aligned}
\sum_{j=1}^N \beta_{C_{t^-},j} dM_j^{\mathbb{P}}(t) &= \sum_{i,j=1}^N \beta_{i,j}(t, V_{t^-}, H(t^-)) \mathbf{1}_{\{C_{t^-}=i\}} dM_j^{\mathbb{P}}(t) \\
&= \sum_{i,j=1}^N \beta_{i,j}(t, V_{t^-}, H(t^-)) \mathbf{1}_{\{C_{t^-}=i\}} dH_t^j - \sum_{i,j=1}^N \beta_{i,j}(t, V_{t^-}, H(t^-)) H_t^i a_{C_t,j}(t)dt \\
&= \sum_{i=1}^N \sum_{j\neq i} (\beta_{i,j} - \beta_{i,i})(t, V_{t^-}, H(t^-)) dH_t^{i,j} - \sum_{j=1}^N \beta_{C_t,j}(t, V_{t^-}, H(t^-)) a_{C_t,j}(t)dt.
\end{aligned} \tag{82}$$

and, in particular,

$$\Delta V_t = \sum_{i=1}^N \sum_{j\neq i} \beta_{i,j}^0(t, V_{t^-}, H(t^-)) \Delta H_t^{i,j} - \gamma_{C_{t^-}}(t, V_{t^-}, H(t^-)) \Delta H(t), \tag{83}$$

where $\beta^0_{i,j} = \beta_{i,j} - \beta_{i,i}$. Next, let $f(\cdot,\cdot,i,z) \in C^{1,2}([0,\infty)\times\mathbb{R})$, for each $i = 1,\dots,N$ and $z \in \{0,1\}$. We want to find the semimartingale decomposition of $f(t, V_t, C_t, H(t))$. Applying the Itô's formula (seeing C_t as simply a bounded variation process), we have that

$$\begin{aligned} f(t,V_t,C_t,H(t)) = f(0,V_0,C_0,H(0)) &+ \int_0^t f_t(u,V_u,C_u,H(u))du \\ &+ \int_0^t \left\{ \alpha_{C_u} - \sum_{j=1}^N \beta_{C_u,j} a_{C_u,j} + (1-H(u)) h_{C_u}\gamma_{C_u} \right\} f_v du \\ &+ \int_0^t \vartheta_{C_u} f_v dW_u + \frac{1}{2}\int_0^t f_{vv}\vartheta^2_{C_u} du \\ &+ \sum_{0<u\le t} \left\{ f(u,V_u,C_u,H(u)) - f(u,V_{u^-},C_{u^-},H(u^-)) \right\}. \end{aligned} \tag{84}$$

Since τ is not a transition time of C a.s., we can write the last term in the above equation as follows:

$$\begin{aligned} J_t := &\sum_{0<u<t\wedge\tau} \{ f(u,V_u,C_u,0) - f(u,V_{u^-},C_{u^-},0) \} \\ &+ \left\{ f(\tau, V_{\tau^-} - \gamma_{C_{\tau^-}}(\tau, V_{\tau^-},0), C_{\tau^-},1) - f(u,V_{\tau^-},C_{\tau^-},0) \right\} H(t) \\ &+ \sum_{t\wedge\tau<u\le t} \{ f(u,V_u,C_u,1) - f(u,V_{u-},C_{u^-},1) \} \\ = &\sum_{i=1}^N \sum_{j\neq i} \int_0^t \left[f(u,V_{u^-} + \beta^0_{i,j}, j, H(u^-)) - f(u,V_{u-},i,H(u^-)) \right] dH^{i,j}_u \\ &+ \int_0^t \left\{ f(u, V_{u^-} - \gamma_{C_{u^-}}(u,V_{u^-},0), C_{u^-},1) - f(u,V_{u^-},C_{u^-},0) \right\} dH(u). \end{aligned}$$

Next, using the local martingales (8) and (12), we have

$$\begin{aligned} J_t = &\sum_{i=1}^N \sum_{j\neq i} \int_0^t \left[f(u,V_{u^-} + \beta^0_{i,j}, j, H(u^-)) - f(u,V_{u-},i,H(u^-)) \right] dM^{i,j}_u \\ &+ \int_0^t \left\{ f(u, V_{u^-} - \gamma_{C_{u^-}}(u,V_{u^-},0), C_{u^-},1) - f(u,V_{u^-},C_{u^-},0) \right\} d\xi^{\mathbb{P}}_u \\ &+ \int_0^t \sum_{j\neq C_u} a_{C_u,j}(u) \left[f(u,V_u + \beta^0_{C_u,j}, j, H(u)) - f(u,V_{u-},C_u,H(u)) \right] du \\ &+ \int_0^t \left\{ f(u,V_u - \gamma_{C_u}(u,V_u,0), C_{u^-},1) - f(u,V_u,C_u,0) \right\} (1-H(u)) h_{C_u} du, \end{aligned}$$

where we had also used that $V_u = V_{u^-}$, $H(u) = H(u^-)$, and $C_u = C_{u^-}$ a.e. and, hence, the integrands in the last two integrals with respect to du can be evaluated at $(V_u, C_u, H(u)$ instead of $(V_{u^-}, C_{u^-}, H(u^-)$. All together, we have the semimartingale decomposition

$$f(t,V_t,C_t,H(t)) = f(0,V_0,C_0,H(0)) + \int_0^t \mathcal{L}f(u,V_u,C_u,H(u))du + \mathcal{M}_t, \tag{85}$$

where $(\mathcal{M}_t)_t$ is a local martingale and $\mathcal{L}f(t,v,i,z)$ is the so-called generator of $(t, V_t, C_t, H(t))$ defined by

$$
\begin{aligned}
&\frac{\partial f}{\partial t} + \frac{\partial f}{\partial v}\left\{\alpha_i(t,v,z) - \sum_{j=1}^{N}\beta_{i,j}(t,v,z)a_{i,j}(t) + (1-z)h_i\gamma_i\right\} + \frac{\vartheta_i^2(t,v,z)}{2}\frac{\partial^2 f}{\partial v^2}\\
&+\sum_{j\neq i} a_{i,j}(t)\left(f(t, v+\beta^0_{i,j}(t,v,z), j, z) - f(t,v,i,z)\right)\\
&+\left\{f(t, v-\gamma_i(t,v,0), i, 1) - f(t,v,i,0)\right\}(1-z)h_i, \qquad (86)
\end{aligned}
$$

for each $i = 1, \dots, N$. The local martingale component in (85) takes the form:

$$
\begin{aligned}
\mathcal{M}_t := \sum_{i=1}^{N}\Bigg\{&\int_0^t \sum_{j\neq i}\left[f(u, V_{u^-}+\beta^0_{i,j}, j, H(u^-)) - f(u, V_{u^-}, i, H(u^-))\right] dM_u^{i,j}\\
&+\int_0^t \left\{f\left(u, V_{u^-} - \gamma_i(u, V_{u^-}, 0), i, 1\right) - f(u, V_{u^-}, i, 0)\right\}\mathbf{1}_{\{C_{u^-}=i\}} d\xi_u^{\mathbb{P}}\\
&+\int_0^t \vartheta_i \frac{\partial f}{\partial v}(u, V_u, i, H(u))H_u^i dW_u\Bigg\}, \qquad (87)
\end{aligned}
$$

where the functions $\beta_{i,j}$, $\beta^0_{i,j}$, and ϑ_i are evaluated at $(u, V_{u^-}, H(u^-))$ and we used the notation (13).

B Proof of the verification theorems

Proof of Theorem 4.1. We first note that in the post-default case, the process (27) takes the form

$$
\begin{aligned}
dV_s^{\pi,t,v} &= V_s^{\pi,t,v}\left\{\left[r_{C_s} + \pi_s(\mu_{C_s} - r_{C_s})\right]ds + \pi_s\sigma_{C_s}dW_s\right\}, \qquad (88)\\
V_t^{\pi,t,v} &= v, \qquad (t < s < R).
\end{aligned}
$$

Define the process

$$
M_s^\pi := \underline{w}(s, V_s^{\pi,t,v}, C_s), \quad (t \le s \le R), \qquad (89)
$$

for an admissible feedback control $\pi_s := \pi_{C_s}(s, V_s^{\pi,t,v}) \in \mathcal{A}_t(v,i,1)$. For simplicity, through this part we sometimes write V_u or V_u^π instead of $V_u^{\pi,t,v}$. We prove the result through the following steps:

(1) By the semimartingale decomposition (85), it follows that

$$
M_s^\pi = M_t^\pi + \int_t^s R(u, V_u^\pi, C_u, \pi_u)du + \mathcal{M}_s - \mathcal{M}_t,
$$

where

$$
\begin{aligned}
\mathcal{M}_s &= \sum_{i=1}^{N}\Bigg\{\sum_{j\neq i}\int_0^s \left\{\underline{w}(u, V_{u^-}, j) - \underline{w}(u, V_{u^-}, i)\right\} dM_u^{i,j}\\
&\quad+\int_0^s \sigma_i V_u \pi_i(u, V_u)\underline{w}_v(u, V_u, i)H_u^i dW_u\Bigg\}, \qquad (90)\\
R(u,v,i,\pi) &= \underline{w}_u(u,v,i) + \underline{w}_v(u,v,i)v\left(r_i + \pi(\mu_i - r_i)\right)\\
&\quad+\frac{1}{2}\underline{w}_{vv}(u,v,i)v^2\pi^2\sigma_i^2\\
&\quad+\sum_{j\neq i} a_{i,j}(u)\left(\underline{w}(u,v,j) - \underline{w}(u,v,i)\right). \qquad (91)
\end{aligned}
$$

We have that $R(u,v,i,\pi)$ is a concave function in π since, by assumption, $\underline{w}_{vv} < 0$. If we maximize $R(u,v,i,\pi)$ as a function of π for each (u,v,i), we find that the optimum is given by (41). This implies that

$$R(u,v,i,\pi) \le R(u,v,i,\widetilde{\pi}_i(u,v)) = \underline{w}_u(u,v,i) + r_i v w_v(u,v,i) - \eta_i \frac{\underline{w}_v^2(u,v,i)}{\underline{w}_{vv}(u,v,i)} + \sum_{j\neq i} a_{i,j}\left(\underline{w}(u,v,j) - \underline{w}(u,v,i)\right) = 0,$$

where the last equality follows from Eq. (38). Next, let us introduce the stopping times $\tau_{a,b} := \inf\{s \ge t : V_s \ge b^{-1}, \text{ or } V_s \le a\}$, for fixed $0 < a < v < b^{-1} < \infty$. Then, using the notation $\mathbb{E}_t[\cdot] = \mathbb{E}[\cdot|\mathcal{G}_t]$, we get the inequality

$$\mathbb{E}_t\left[M^{\pi}_{s\wedge\tau_{a,b}}\right] \le M^{\pi}_t + \sum_{i=1,j\neq i}^{N} \mathbb{E}_t\left[\int_t^{s\wedge\tau_{a,b}} \{\underline{w}(u,V_{u^-},j) - \underline{w}(u,V_{u^-},i)\}\, dM^{i,j}_u\right] + \sum_{i=1}^{N} \mathbb{E}_t\left[\int_t^{s\wedge\tau_{a,b}} V_u \pi_i(u,V_u)\sigma_i \underline{w}_v(u,V_u,i) H^i_u dW_u\right],$$

with equality if $\pi = \widetilde{\pi}$. Since

$$\sup_{t\le u\le\tau_{a,b}\wedge R} |\underline{w}(u,V_u,i)| \le B_1, \qquad \sup_{t\le u\le\tau_{a,b}\wedge R} |V_u\pi_i(u,V_u)\underline{w}_v(u,V_u,i)|^2 \le B_2,$$

for some constants $B_1, B_2 < \infty$, we conclude that $\mathbb{E}_t\left[M^{\pi}_{R\wedge\tau_{a,b}}\right] \le M^{\pi}_t = \underline{w}(t,v,C_t)$, with equality if $\pi = \widetilde{\pi}$.

(2) In this step, we show that

$$\lim_{a,b\to 0} \mathbb{E}_t\left[\underline{w}(R\wedge\tau_{a,b}, V^{\widetilde{\pi}}_{R\wedge\tau_{a,b}}, C_{R\wedge\tau_{a,b}})\right] = \mathbb{E}_t\left[U(V^{\widetilde{\pi}}_R)\right], \tag{92}$$

where $\widetilde{\pi}_s = \widetilde{\pi}(s, V^{\pi,t,v}_s, C_s)$. Note that (39-i) implies

$$\mathbb{E}\left[\left|\underline{w}(R\wedge\tau_{a,b}, V^{\widetilde{\pi}}_{R\wedge\tau_{a,b}}, C_{R\wedge\tau_{a,b}})\right|^2 \Big| \mathcal{G}_t\right] \le B_1 + B_2\mathbb{E}\left[\left|V^{\widetilde{\pi}}_{R\wedge\tau_{a,b}}\right|^2 \Big| \mathcal{G}_t\right],$$

for some constants $B_1, B_2 < \infty$. Next, we note that $\tilde{\pi}$ satisfies (40) since

$$|v\tilde{\pi}_i(s,v)| = \left|\frac{\mu_i - r_i}{\sigma_i^2}\frac{\underline{w}_v(s,v,i)}{\underline{w}_{vv}(s,v,i)}\right| < G(s)(1+v),$$

in light of (39-ii). Hence, we can apply Lemma B.1 below (with $\pi^P \equiv 0$) and obtain

$$\sup_{0<a<v<b^{-1}<\infty} \mathbb{E}_t\left[\left(V^{\widetilde{\pi}}_{R\wedge\tau_{a,b}}\right)^2\right] \le 2\left(V^{\widetilde{\pi}}_t\right)^2 + 2\mathbb{E}_t\left[\sup_{t\le s\le R}\left(V^{\widetilde{\pi}}_s - V^{\widetilde{\pi}}_t\right)^2\right] < \infty.$$

Using Corollary 7.1.5 in Chow and Teicher (1978), we conclude (92).

(3) Finally, if $\underline{w}$ is non-negative, then Fatou's Lemma implies that

$$\begin{aligned}\mathbb{E}_t\left[U(V^{\pi}_R)\right] &= \mathbb{E}_t\left[\liminf_{a,b\to 0} \underline{w}(R\wedge\tau_{a,b}, V^{\pi}_{R\wedge\tau_{a,b}}, C_{R\wedge\tau_{a,b}})\right] \\ &\le \liminf_{a,b\to 0} \mathbb{E}_t\left[\underline{w}(R\wedge\tau_{a,b}, V^{\pi}_{R\wedge\tau_{a,b}}, C_{R\wedge\tau_{a,b}})\right] \\ &\le \underline{w}(t,v,C_t) = \mathbb{E}_t\left[U(V^{\widetilde{\pi}}_R)\right],\end{aligned}$$

for every admissible feedback control $\pi_s = \pi_{C_s}(s, V_s^{\pi,t,v}) \in \mathcal{A}_t(v, i, 1)$. For a general function $\underline{w}$ (not necessarily non-negative), we proceed along the lines of step (2) above to show

$$\lim_{a,b\to 0} \mathbb{E}_t \left[\underline{w}(R \wedge \tau_{a,b}, V^{\pi}_{R\wedge\tau_{a,b}}, C_{R\wedge\tau_{a,b}})\right] = \mathbb{E}_t \left[U(V_R^{\pi})\right],$$

for any feedback control $\pi_s = \pi_{C_s}(s, V_s^{\pi,t,v}) \in \mathcal{A}_t(v, i, 1)$ satisfying (40). □

Proof of Theorem 4.2. We prove the result through the following steps:

(1) Define the process

$$M_s^{\pi} := \bar{w}(s, V_s^{\pi,t,v}, C_s)(1 - H(s)) + \underline{w}(s, V_s^{\pi,t,v}, C_s)H(s), \tag{93}$$

where $V_s^{\pi,t,v}$ is the solution of Eq. (27) for an admissible feedback control

$$\pi_s := (\pi_s^S, \pi_s^P) := (\pi^S_{C_{s-}}(s, V_{s^-}^{\pi,t,v}), \pi^P_{C_{s^-}}(s, V_{s^-}^{\pi,t,v})) \in \mathcal{A}_t(v, i, 0).$$

For simplicity, we only write $V_s^{\pi} = V_s^{\pi,t,v}$. Using the same arguments as in Eq. (82) and the decomposition (8), the process (27) can be written as

$$\begin{aligned} dV_s^{\pi,t,v} = V_{s-}^{\pi,t,v}\bigg\{ &\left[r_{C_s} + \pi^S_{C_{s-}}(\mu_{C_s} - r_{C_s}) + \pi^P_{C_{s-}}(1 - H(s))\theta_{C_s}(s)\right] ds \\ &+ \pi^S_{C_{s-}}\sigma_{C_s} dW_s + \pi^P_{C_{s^-}}(1 - H(s^-))dH(s) \\ &+ (1 - H(s^-)) \sum_{i=1}^{N} \sum_{j\neq i} \pi_i^P \frac{\psi_j(s) - \psi_i(s)}{\psi_i(s)} dH_s^{i,j} \bigg\}, \end{aligned} \tag{94}$$

for $s \in (t, R)$, with the initial condition $V_t^{\pi,t,v} = v$, where θ_i is defined in (35). By the semimartingale decomposition (85) with the coefficients given by (33) and

$$f(s, v, i, z) := \bar{w}(s, v, i)(1 - z) + \underline{w}(s, v, i)z,$$

it follows that

$$M_s^{\pi} = M_t^{\pi} + \int_t^s R(u, V_u^{\pi}, C_u, \pi^S_{C_u}, \pi^P_{C_u}, H(u))du + \mathcal{M}_s - \mathcal{M}_t.$$

Here, $\mathcal{M}_s - \mathcal{M}_t$ is given by

$$\begin{aligned} \sum_{i=1}^{N} \bigg\{ &\sum_{j\neq i} \int_t^s \left[w\left(u, V_{u^-}\left(1 + \bar{\pi}_i^P \frac{\psi_j(u) - \psi_i(u)}{\psi_i(u)}\right), j\right) - w(u, V_{u^-}, i)\right] dM_u^{i,j} \\ &+ \int_t^s \left[w\left(u, V_{u^-}\left(1 - \bar{\pi}_i^P\right), i\right) - w(u, V_{u^-}, i)\right] \mathbf{1}_{\{C_{u^-}=i\}} d\xi_u^{\mathbb{P}} \\ &+ \int_t^s \sigma_i V_u \pi_i^S(u, V_u) w_v(u, V_u, i) H_u^i dW_u \bigg\}, \end{aligned} \tag{95}$$

where $\bar{\pi}_i^P(u, v) := (1 - H(u))\pi_i^P(u, v)$ and

$$w(u, v, i) = \bar{w}(u, v, i)(1 - H(u^-)) + \underline{w}(u, v, i)H(u^-).$$

Similarly, $R(u, v, i, \pi^S, \pi^P, 1)$ is defined as in (91) with $\pi = \pi^S$, while

$$\begin{aligned} R(u, v, i, \pi^S, \pi^P, 0) := &\bar{w}_u(u, v, i) + \bar{w}_v(u, v, i)v\Big(r_i + \pi^S(\mu_i - r_i) + \pi^P\theta_i(u)\Big) \\ &+ \frac{1}{2}\bar{w}_{vv}(u, v, i)v^2(\pi^S)^2\sigma_i^2 \\ &+ \sum_{j\neq i} a_{i,j}(u) \left[\bar{w}\left(u, v\left(1 + \pi^P \frac{\psi_j(u) - \psi_i(u)}{\psi_i(u)}\right), j\right) - \bar{w}(u, v, i)\right] \\ &+ h_i\left[\underline{w}\left(u, v\left(1 - \pi^P\right), i\right) - \bar{w}\left(u, v, i\right)\right]. \end{aligned} \tag{96}$$

Note that, under our assumptions, $(\pi^S, \pi^P) \to R(u, v, i, \pi^S, \pi^P, 1)$ admits a unique maximal point $(\widetilde{\pi}^S, \widetilde{\pi}^P)$ for each (u, v, i), since (i) $R_{\pi^S\pi^S} \leq 0$, (ii) $R_{\pi^P\pi^P} \leq 0$, and (iii) $R_{\pi^S\pi^P} = 0$. Indeed, (i) follows from our assumption that $\bar{w}_{vv} \leq 0$, while (ii) is evident from the calculation

$$R_{\pi^P\pi^P} = \sum_{j\neq i} a_{i,j}(u) \frac{(\psi_j(u) - \psi_i(u))^2}{\psi_i(u)^2} v^2 \bar{w}_{vv} \left(u, v\left(1 + \pi^P \frac{\psi_j(u) - \psi_i(u)}{\psi_i(u)}\right), j\right) + h_i v^2 \underline{w}_{vv} \left(u, v\left(1 - \pi^P\right), i\right),$$

and the fact that $\underline{w}_{vv} \leq 0$. The optimum is given by Eq. (45) with p defined implicitly by Eq. (42). In light of Eqs. (38)-(43),

$$R(u, v, i, \pi^S, \pi^P, z) \leq R(u, v, i, \widetilde{\pi}_i^S(u, v), \widetilde{\pi}_i^P(u, v), z) = 0.$$

Let

$$\tau_{a,b} := \inf\left\{s \geq t : V_s \geq b^{-1}, \text{ or } V_s \leq a, \text{ or } \pi_{u^-}^P > 1 - a, \text{ or } \pi_{s^-}^P \frac{\psi_j(s) - \psi_i(s)}{\psi_i(s)} < a - 1, \text{ for some } j \neq i\right\},$$

for a small enough $a, b > 0$. Using similar arguments to those in the proof of Theorem 4.1, we can show that

$$\mathbb{E}_t\left[M_{R\wedge\tau_{a,b}}^\pi\right] \leq M_t^\pi = \bar{w}(t, v, C_t)(1 - H(t)) + \underline{w}(t, v, C_t)H(t),$$

with equality if $\pi^S = \widetilde{\pi}^S$ and $\pi^P = \widetilde{\pi}^P$.

(2) We now show that

$$\lim_{a,b\to0} \mathbb{E}_t\left[\bar{w}(R \wedge \tau_{a,b}, V_{R\wedge\tau_{a,b}}^{\widetilde{\pi}}, C_{R\wedge\tau_{a,b}})\bar{H}(R \wedge \tau_{a,b})\right] = \mathbb{E}_t\left[U(V_R^{\widetilde{\pi}})\bar{H}(R)\right] \tag{97}$$

$$\lim_{a,b\to0} \mathbb{E}_t\left[\underline{w}(R \wedge \tau_{a,b}, V_{R\wedge\tau_{a,b}}^{\widetilde{\pi}}, C_{R\wedge\tau_{a,b}})H(R \wedge \tau_{a,b})\right] = \mathbb{E}_t\left[U(V_R^{\widetilde{\pi}})H(R)\right], \tag{98}$$

where $\bar{H}(s) = 1 - H(s)$. Note that (97-98) will imply that

$$\bar{w}(t, v, C_t)\bar{H}(t) + \underline{w}(t, v, C_t)H(t) = \lim_{a,b\to0} \mathbb{E}_t\left[M_{R\wedge\tau_{a,b}}^{\widetilde{\pi}}\right] = \mathbb{E}_t\left[U(V_R^{\widetilde{\pi}})\right].$$

We only prove (97) ((98) can be treated similarly). Note that (39-i) implies

$$\mathbb{E}\left[\left|\bar{w}(R \wedge \tau_{a,b}, V_{R\wedge\tau_{a,b}}^{\widetilde{\pi}}, C_{R\wedge\tau_{a,b}})\bar{H}(R \wedge \tau_{a,b})\right|^2 \middle| \mathcal{G}_t\right] \leq B_1 + B_2\mathbb{E}\left[\left|V_{R\wedge\tau_{a,b}}^{\widetilde{\pi}}\right|^2 \middle| \mathcal{G}_t\right],$$

for some constants $B_1, B_2 < \infty$. Next, we note that $\tilde{\pi}^S$ satisfies (40) since both $\bar{w}$ and $\underline{w}$ satisfies (39-ii) by assumption. Also, $\widetilde{\pi}^P$ satisfies (29) by assumption. Hence, we can applying Lemma B.1 below and obtain

$$\sup_{0<a<v<b^{-1}<\infty} \mathbb{E}_t\left[\left(V_{R\wedge\tau_{a,b}}^{\widetilde{\pi}}\right)^2\right] \leq 2\left(V_t^{\widetilde{\pi}}\right)^2 + 2\mathbb{E}_t\left[\sup_{t\leq s\leq R}\left(V_s^{\widetilde{\pi}} - V_t^{\widetilde{\pi}}\right)^2\right] < \infty.$$

Using Corollary 7.1.5 in Chow and Teicher (1978), we conclude (97).

(3) Finally, if w is non-negative, then Fatou's Lemma implies that

$$\begin{aligned}
\mathbb{E}_t\left[U(V_R^\pi)\right] &= \mathbb{E}_t\left[\liminf_{a,b\to0} \bar{w}(R \wedge \tau_{a,b}, V_{R\wedge\tau_{a,b}}^\pi, C_{R\wedge\tau_{a,b}})\bar{H}(R \wedge \tau_{a,b})\right] \\
&\quad + \mathbb{E}_t\left[\liminf_{a,b\to0} \underline{w}(R \wedge \tau_{a,b}, V_{R\wedge\tau_{a,b}}^\pi, C_{R\wedge\tau_{a,b}})H(R \wedge \tau_{a,b})\right] \\
&\leq \liminf_{a,b\to0} \mathbb{E}_t\Big[\bar{w}(R \wedge \tau_{a,b}, V_{R\wedge\tau_{a,b}}^\pi, C_{R\wedge\tau_{a,b}})\bar{H}(R \wedge \tau_{a,b}) \\
&\qquad + \underline{w}(R \wedge \tau_{a,b}, V_{R\wedge\tau_{a,b}}^\pi, C_{R\wedge\tau_{a,b}})H(R \wedge \tau_{a,b})\Big] \\
&\leq \bar{w}(t, v, C_t)\bar{H}(t) + \underline{w}(t, v, C_t)H(t) = \mathbb{E}_t\left[U(V_R^{\widetilde{\pi}})\right],
\end{aligned}$$

for every admissible feedback control $\pi_s = \pi_{C_{s^-}}(s, V_{s^-}^{\pi,t,v}, H(s^-)) \in \mathcal{A}_t(v, i, 0)$. For a general function w (not necessarily non-negative), we proceed along the lines of step (2) above to show

$$\lim_{a,b\to 0} \mathbb{E}_t \left[\bar{w}(R \wedge \tau_{a,b}, V_{R\wedge\tau_{a,b}}^{\pi}, C_{R\wedge\tau_{a,b}})\bar{H}(R \wedge \tau_{a,b}) + \underline{w}(R \wedge \tau_{a,b}, V_{R\wedge\tau_{a,b}}^{\pi}, C_{R\wedge\tau_{a,b}})H(R \wedge \tau_{a,b}) \right] = \mathbb{E}_t \left[U(V_R^{\pi})\right],$$

for any t-admissible feedback controls $\pi_s = \pi_{C_{s^-}}(s, V_{s^-}^{\pi,t,v}, H(s^-)) \in \mathcal{A}_t(v, i, 0)$ satisfying (40) and (29). □

Lemma B.1. *Let $\pi_i^S(s, v, z)$ and $\pi_i^P(s, v, z)$ be functions such that (94) admits a unique nonnegative solution $(V_s^{\pi})_{s\in[t,R]}$. We also assume that π^S satisfies (40) and π^P satisfies (29). Then, the solution of (94) satisfies the moment condition:*

$$\mathbb{E}_t \left[\sup_{t\le u\le R} |V_u^{\pi} - V_t^{\pi}|^2 \right] \le C_1 (V_t^{\pi})^2 + C_2, \tag{99}$$

for some constants $C_1, C_2 < \infty$.

Proof. For simplicity, we write V_s instead of V_s^{π}. Let us start by recalling that we can write (94) in the form

$$dV_s = \alpha_{C_s} ds + \vartheta_{C_s} dW_s + \sum_{j=1}^{N} \beta_{C_{s^-},j} dM_j^{\mathbb{P}}(s) - \gamma_{C_{s^-}} d\xi_s^{\mathbb{P}}, \tag{100}$$

taking the coefficients as in (33). Due to (29) and (40), we can see that $|\alpha_i(s, v, z)| \le E(s)(1 + v)$ for a locally bounded function E. Hence, by Jensen's inequality and the previous linear growth,

$$\left| \int_t^s \alpha_{C_u}(u, V_u, H(u^-))du \right|^2 \le \kappa\tau \left(\tau + \tau V_t^2 + \int_t^s |V_u - V_t|^2 du \right),$$

for any $s \in [t, R]$, where $\tau := R - t$ and κ denote a generic constant that may change from line to line. Similarly, denoting $\tau_b = \inf\{s \ge t : |V_s| \ge b\}$ $(b > v)$ and using Burkhölder-Davis-Gundy inequality (see Theorem 3.28 in Karatzas and Shreve (1998) or Theorem IV.48 in Protter (1990)) and Jensen's inequality,

$$\mathbb{E}_t \sup_{t\le s\le R\wedge\tau_b} \left| \int_t^s \vartheta_{C_u}(u, V_u, H(u))dW_u \right|^2 \le \kappa\tau \mathbb{E} \int_t^{R\wedge\tau_b} \left|\sigma_{C_u} \pi_u^S V_u\right|^2 du.$$

We can then again use (40) to show that

$$\mathbb{E}_t \sup_{t\le s\le R\wedge\tau_b} \left| \int_t^s \vartheta_{C_u}(u, V_u, H(u))dW_u \right|^2 \le \kappa\tau(\tau + \tau|V_t|^2 + \int_t^{R\wedge\tau_b} |V_u - V_t|^2 du).$$

Next, using again Burkhölder-Davis-Gundy inequality (see, e.g., Theorem 23.12 in Kallenberg (1997)),

$$\mathbb{E}_t \sup_{t\le s\le R\wedge\tau_b} \left| \int_t^s \gamma_{C_{u^-}}(u, V_{u^-}, H(u^-))d\xi_u^{\mathbb{P}} \right|^2 \le \kappa \mathbb{E}_t \int_t^{R\wedge\tau_b} |V_{u^-} \pi_u^P|^2 dH(u).$$

Using (29),

$$\mathbb{E}_t \int_t^{R\wedge\tau_b} |V_{u^-} \pi_u^P|^2 dH(u) \le \kappa \mathbb{E}_t \int_t^{R\wedge\tau_b} |V_{u^-}|^2 dH(u) = \kappa \mathbb{E}_t \int_t^{R\wedge\tau_b} |V_u|^2 h_{C_u}(1 - H(u))du.$$

Then, we can proceed as before to conclude that

$$\mathbb{E}_t \sup_{t\le s\le R\wedge\tau_b} \left| \int_t^s \gamma_{C_{u^-}}(u, V_{u^-}, H(u^-))d\xi_u^{\mathbb{P}} \right|^2 \le \kappa\tau \left(\tau + \tau|V_t|^2 + \int_t^{R\wedge\tau_b} |V_u - V_t|^2 du \right).$$

Using a similar argument, we can also obtain that

$$\mathbb{E}_t \sup_{t\le s\le R\wedge\tau_b} \left|\int_t^s \beta_{C_{u^-},j}(u, V_{u^-}, H(u^-))dM_j^{\mathbb{P}}(u)\right|^2 \le \kappa\tau\left(\tau + \tau|V_t|^2 + \int_t^{R\wedge\tau_b} |V_u - V_t|^2 du\right).$$

Putting together the previous estimates, we conclude that the function $\gamma_b(r) := \mathbb{E}_t \sup_{t\le s\le r\wedge\tau_b} |V_s - V_t|^2$ can be bounded as follows: $\gamma_b(r) \le \kappa\tau(1+v^2) + \kappa\int_t^r \gamma_b(u)du$. By Gronwall inequality, we have

$$\gamma_b(R) \le \kappa(R-t)(1+v^2)e^{\kappa(R-t)},$$

and (99) is obtained by making $b\to\infty$. □

C Proofs related to construction of solutions

Here, we provide proofs related to Section 5. We first recall a useful lemma, which will be needed for following proofs.

Lemma C.1 (Coddington and Levinson (1955), Kaczorek (2001))**.** *Let $F:\mathbb{R}_+\to\mathbb{R}^{N\times N}$ and $b:\mathbb{R}_+\to\mathbb{R}^{N\times 1}$. For a fix $\varsigma\in\mathbb{R}^N$, consider the time varying linear system*

$$x_t(t) = F(t)x(t) + b(t), \qquad x(t_0) = \varsigma. \tag{101}$$

Then, the following statements hold true:

(1) *The system (101) admits a unique solution given by*

$$x(t) = \phi_F(t,t_0)\varsigma + \int_{t_0}^t \phi_F(s,t_0)b(s)ds, \tag{102}$$

where $\phi_F(t,t_0)$ is defined by the Peano-Baker series

$$\phi_F(t,t_0) = I_N + \int_{t_0}^t F(s)ds + \int_{t_0}^t F(s)\int_{t_0}^s F(y)dyds + \int_{t_0}^t F(s)\int_{t_0}^s F(y)\int_{t_0}^y F(z)dzdyds + \dots,$$

with I_N denoting the N dimensional identity matrix.

(2) *For all $t\ge t_0$, $\phi_F(t,t_0)$ has all nonnegative entries if and only if the off-diagonal entries of $F(t)$ satisfy the condition $\int_{t_0}^t [F(s)]_{i,j}ds \ge 0$, for $i\ne j$ and all $t\ge t_0$.*

(3) *If the matrix $F(t)$ is time invariant (i.e. $F(t)\equiv F$ for any t and some matrix F), then $\phi_F(t,t_0) = \exp((t-t_0)F)$, where $\exp(B)$ denotes the exponential of a square-matrix B.*

C.1 Proofs related to Section 5.1

Proof of Proposition 5.2.

(1) It can be checked that the function $\underline{\varphi}_t^R(t,v,i) = \log(v) + K(t,i)$ solves the Dirichlet problem (38), if the functions $K(t,i)$, $i=1,\dots,N$, satisfy the time varying nonhomogeneous system given by Eq. (47). Let ϱ denote the N dimensional zero vector. Using the substitution $s = R-t$, we have that the solution $\tilde{K}(s)$ of the initial value problem given by

$$\begin{aligned} \tilde{K}_s(s) &= -F(R-s)\tilde{K}(s) - b(R-s) \qquad 0\le s\le R \\ \tilde{K}(0,i) &= \varrho \qquad i=1,\dots,N \end{aligned} \tag{103}$$

is such that

$$\tilde{K}(t) = \tilde{K}(R-t) \tag{104}$$

As $a_{i,j}$ is continuous in $[0,T]$ by hypothesis, using Lemma C.1, part (1), we have that the unique solution of system (103) may be written as

$$\tilde{K}(s) = \phi_{-F}(R, R-s)\varrho - \int_0^s \phi_{-F}(R-q, R-s)b(R-q)dq = -\int_{R-s}^R \phi_{-F}(z, R-s)b(z)dz$$

since ϱ has all zero entries. Therefore, using Eq. (104), we obtain that

$$K(t) = -\int_t^R \phi_{-F}(z,t)b(z)dz \tag{105}$$

As for all $i \neq j$, and for all t, we have $[F(t)]_{i,j} \leq 0$, then $\int_0^t [-F(s)]_{i,j} ds \geq 0$. Therefore using Lemma C.1, part (2), we obtain that $\phi_{-F}(z,t)$ has all nonnegative entries, for each $z \geq t$. Since $b(z)$ has all negative entries, we obtain that the integrand in Eq. (105) is negative, and consequently $K(t,i)$ is a positive function of t, for all $0 \leq t \leq R$, and $i = 1, \ldots, N$. Moreover, $\underline{\varphi}_t^R(t,v,i) \in C_{1,2}^0$ due to concavity and increasingness of the logarithmic function, and, under the choice $D(t) = \max_{i=1,\ldots,N} K(t,i)$ and $G(t) = 1$, the function $\underline{\varphi}_t^R(t,v,i)$ satisfies the conditions in (39). Therefore, applying Theorem 4.1, we can conclude that, for each $i = 1, \ldots, N$, $\underline{\varphi}_t^R(t,v,i)$ is the optimal post-default value function.

(2) Plugging the expression for $\underline{\varphi}_t^R(t,v,i)$ inside Eq. (41), we can conclude immediately that $\tilde{\pi}^S(t,j) = \frac{\mu_j - r_j}{\sigma_j^2}$.

(3) It can be checked that the vector of functions $[\overline{\varphi}_t^R(t,v,1), \ldots, \overline{\varphi}_t^R(t,v,N)]$, where $\overline{\varphi}_t^R(t,v,i) = \log(v) + J(t,i)$, and the vector $p(t)$ solving the nonlinear system of equations (46) simultaneously satisfy the system composed of Eq. (42) and Eq. (43) if the vector of functions $J(t) = (J(t,1), J(t,2), \ldots, J(t,N))$ solves the system (49). Using the same argument as for $K(t)$, we obtain that

$$J(t) = -\int_t^R \phi_{-G}(z,t)d(z)dz \tag{106}$$

From Eq. (50), we can see that, for all $i \neq j$, and for all t, $[G(t)]_{i,j} \leq 0$. Therefore, $\int_0^t [-G(s)]_{i,j} ds \geq 0$, and using Lemma C.1, part (2), we obtain that $\phi_{-G}(z,t)$ has all nonnegative entries, for each $z \geq t$. We next show that the vector $d(t)$ defined in Eq. (50) consists of all negative entries. Using (46), we obtain that the i-th component of $d(t)$ may be written as

$$[d(t)]_i = -h_i\left(\frac{p_i(t)}{1-p_i(t)} + \log(1-p_i(t))\right) + \sum_{j \neq i} a_{i,j}(t)\left(p_i(t)\frac{\frac{\psi_j(t)}{\psi_i(t)} - 1}{1 + p_i(t)\left(\frac{\psi_j(t)}{\psi_i(t)} - 1\right)} - \log\left(1 + p_i(t)\left(\frac{\psi_j(t)}{\psi_i(t)} - 1\right)\right)\right) \tag{107}$$

It is easily seen that the term multiplying h_i is positive due to the well known inequality $\frac{x}{1-x} + \log(1-x) > 0$ when $x \neq 0$ and $x < 1$, and each term multiplying $a_{i,j}(t)$ in Eq. (107) is negative due to that $\frac{x}{1+x} - \log(1+x) < 0$ when $x > -1$. As $\phi_{-G}(z,t)$ has all nonnegative entries and $d(z) < 0$, for all $t \leq z \leq R$, we obtain that $J(t,i) \geq 0$ for all $i = 1, \ldots, N$. Moreover, under the choice of $D(t) = \max_{i=1\ldots,N} J(t,i)$ and $G(t) = 1$, we have that $\underline{\varphi}_t^R(t,v,i)$ satisfies the conditions in (39). As the logarithmic function is increasing and concave in v, then $\overline{\varphi}_t^R(t,v,i) \in C_0^{1,2}$, therefore it must be the optimal pre-default value function by Theorem 4.2. □

C.2 Proofs related to Section 5.2

Proof of Proposition 5.8.

Let us denote $f_1(t, J(t))$ and $f_2(t, J(t))$, with $J(t) = [J(t,1), J(t,2)]'$, the right-hand side of the differential equations (63) and (64), respectively. Similarly, $g_1(t, p_1(t,J), J)$ and $g_2(t, p_2(t,J), J)$, with $J = [J_1, J_2]'$, denote the right-hand side of the equations (66) and (67), respectively. First, we show that for any fixed $b > 1$ and $t \in [0, R]$, the functions f_1, f_2 are bounded and Lipschitz on the domain $\mathcal{R} = [0, R] \times [b^{-1}, b]^2$. Let us assume $\psi_2(t)/\psi_1(t) > 1$ (the case $\psi_2(t)/\psi_1(t) < 1$ can be treated similarly). In light of (66), $p_1(t, J)$ will take values on $(-(\psi_2(t)/\psi_1(t) - 1)^{-1}, 1)$ and hence, $f_1(t, J)$ is uniformly bounded as follows:

$$\sup_{b^{-1} < J_1, J_2 < b} |f_1(t, J)| \leq |\xi_1(t)| + b \left(|a_{12}||\psi_2/\psi_1|^\gamma + K(t, 2)| \frac{\psi_2}{\psi_2 - \psi_1} |^\gamma \right).$$

where for simplicity we had omitted the dependence on t of the functions appearing on the right-hand. We conclude that for a bounded function $C_1(t) > 0$,

$$\sup_{b^{-1} < J_1, J_2 < b} |f_1(t, J)| \leq C_1(t) b.$$

In order to estimate, f_2 on the given domain, let us proceed as follows. Noticed the following inequality (still assuming $\psi_2(t)/\psi_1(t) > 1$), valid for $p_2 < 0$:

$$g_2(t, p, J) \geq \theta_2 J_2 + \left[a_{21} J_1 \left(\frac{\psi_1}{\psi_2} - 1 \right) - h_2 K_2 \right] \left(1 + p_2 \left(\frac{\psi_1}{\psi_2} - 1 \right) \right)^{\gamma - 1}.$$

Therefore, we can lower bound the root of $g(t, p, J)$ with the root of the right-hand side in the previous inequality. Then,

$$p_2(t, J) \geq -\frac{\psi_2}{\psi_2 - \psi_1} \left([a_{1,2}(1 - \frac{\psi_1}{\psi_2}) J_1 + h_2 K_2] / \theta_2 J_2 \right)^{1/(1-\gamma)}. \tag{108}$$

In terms of b, we have $p_2(t, J) \geq -C(t) b^{2/(1-\gamma)}$, for some bounded function $C(t) > 0$. This fact will imply that

$$\sup_{b^{-1} < J_1, J_2 < b} |f_2(t, J)| \leq C_2(t) b^{(3-\gamma)/(1-\gamma)},$$

for a bounded function $C_2(t)$. In order to show that f_1, f_2 satisfy a Lipschitz condition in the Region $\mathcal{R}$, it suffices to show that each $p_i(t, J)$ remains bounded away from $M_i(t)$ and 1 when $(t, J) \in \mathcal{R}$. Indeed, suppose for instance that $\psi_2(t)/\psi_1(t) > 1$. By definition,

$$0 = \theta_1(t) J_1 - h_1 K(t, 1)(1 - p_1(t, J))^{\gamma - 1} + a_{1,2}(t) J_2 \left(\frac{\psi_2(t)}{\psi_1(t)} - 1 \right) \left(1 + p_1(t, J) \left(\frac{\psi_2(t)}{\psi_1(t)} - 1 \right) \right)^{\gamma - 1}.$$

Hence, if p_1 is either close to 1 or $M_1(t) = -(\psi_2(t)/\psi_1(t) - 1)^{-1}$, J_1 will be arbitrarily large when J_2 is bounded. Similarly, $p_2(t, J)$ cannot be arbitrarily close to 1. Also, when $J_1, J_2 \in (b^{-1}, b)$, $p_2 > C > -\infty$ for some C in light of (108). Finally, we use the classical existence theorems (see, e.g., Theorem I-1-4 in Hsieh and Sibuya (2009)) to conclude the existence of the solution for $t \in [R - \alpha, R]$. □

Remark C.2. *One of the consequences of the Theorem I-1-4 in Hsieh and Sibuya (2009) is that one can take* $\alpha = \min\{R, b/M(b)\}$, *where* $M(b) := \max_i \sup_{(t,J) \in \mathcal{R}} |f_i(t, J)|$, *where as before* $\mathcal{R} := [0, R] \times (b^{-1}, b)^2$. *Typically,* $M(b)$ *increases when b increase and hence,* α *might decrease. As seen in the proof,* $M(b)$ *seems to increase as* $b^{(3-\gamma)/\gamma}$, *and hence,* $b/M(b) \to 0$ *as* $b \to \infty$. *We can however take* $\alpha = 0$ *if, e.g.,* K, $a_{i,j}$, $|\psi_2/\psi_1|$, *etc. are made small enough.*

D Proofs related to Economic Analysis

We provide the proof yielding the bond investment strategy for the square root investor in the case of a market consisting of one regime.

D.1 Proofs related to Section 6.2

Proof of Lemma 6.5. Using Eq. (57), we obtain that

$$p_1(t) = 1 - \left(\frac{L_1 J(t,1)}{K(t,1)} \right)^{-2}. \tag{109}$$

Plugging Eq. (109) into Eq. (56), we obtain

$$J_t(t,1) = \left(\frac{\eta_1^2}{2} \frac{\gamma}{\gamma - 1} - r_1 \gamma + h_1 \right) J(t,1) + h_1 L_1 J(t,1) \left(\frac{L_1 J(t,1)}{K(t,1)} \right)^{-2} (\gamma - 1) - h_1 L_1 \gamma J(t,1). \tag{110}$$

Moreover, notice that the post-default time component of the solution $K(t,1)$ satisfies

$$K_t(t,1) = -\left(\gamma r_1 - \frac{\eta_1^2}{2} \frac{\gamma}{\gamma - 1} \right) K(t,1), \qquad K(R,1) = \frac{1}{\gamma}, \tag{111}$$

leading to $K(t,1) = \frac{1}{\gamma} \exp\left\{ \gamma_1 r_1 - \frac{\eta_1^2}{2} \frac{\gamma}{\gamma-1} \right\}$. Let us denote by

$$a := \frac{\eta_1^2}{2} \frac{\gamma}{\gamma - 1} - r_1 \gamma + h_1 - h_1 L_1 \gamma, \qquad b(t) := \frac{h_1}{L_1} (\gamma - 1) K^2(t,1). \tag{112}$$

Then, we find that Eq. (110) admits two solutions, given by

$$J(t,1) = \pm \frac{1}{\gamma} e^{a(t-R)} \left(1 - 2e^{2aR} \gamma^2 \int_t^R e^{-2as} b(s) ds \right)^{1/2} \tag{113}$$

which can be evaluated, using Eq. (112), to obtain

$$J(t,1) = \pm \frac{\exp\left\{ -\frac{(R-t)\gamma(\eta^2 - 2r_1(\gamma-1))}{2(\gamma-1)} - h_1 t L_1^\gamma \right\}}{L_1^{1/2} \gamma L_1^{\gamma/2}} \sqrt{\exp\left\{2h_1 t L_1^\gamma\right\} (\gamma - 1) + \exp\left\{2h_1 R L_1^\gamma\right\} (L_1 - 1)(L_1^\gamma + \gamma)}, \tag{114}$$

where we recall that $L_1^\gamma = L_1 \gamma - 1$. Note that the expression under the square root is nonnegative, thus $J(t,1)$ is well defined. Moreover, we know that the post-default value function is concave, which means that we only need to consider the positive solution in (113). Plugging Eq. (111) and Eq. (114) into Eq. (109), we can conclude that $p_1(t)$ evaluates to the expression given in Eq. (79). □

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