Small-time expansions for state-dependent local jump-diffusion models with infinite jump activity

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AMS Sectional Meeting
Loyola University, Chicago, IL
Oct 3, 2015

Joint work with Yankeng Luo from VCU
Let $X^{(x)} := \{X^{(x)}_t\}_{t \geq 0}$ be a cádlág Markov process starting at $x$ and having infinitesimal generator

$$L f(x) = b(x)f'(x) + \frac{\sigma^2(x)}{2} f''(x) + \int_{\mathbb{R}_0} \left\{ f(x + \gamma(x, z)) - f(x) - 1_{\{|z| \leq 1\}} \gamma(x, z) f'(x) \right\} \nu(x, z) dz,$$

for $f \in C^2_b(\mathbb{R})$, where $b : \mathbb{R} \to \mathbb{R}$, $\sigma : \mathbb{R} \to [0, \infty)$, $\gamma : \mathbb{R} \times \mathbb{R}_0 \to \mathbb{R}$, and $\nu : \mathbb{R} \times \mathbb{R}_0 \to [0, \infty)$ (throughout, $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$);

**Objective:** Determine the high-order short-term asymptotic behavior of the European OTM digital and call option prices:

$$\Pi_t^{Digital} = \mathbb{P} \left[ X^{(x)}_t \geq \kappa \right], \quad \kappa > x,$$

$$\Pi_t^{Call} = \mathbb{E} \left[ (e^{X^{(x)}_t} - e^{\kappa})_+ \right], \quad \kappa > x.$$
In general, \( dX_t = b(X_t)dt + \sigma(X_t)dW_t + dJ_t, \)
where \( J \) is a pure-jump component controlled by \( \gamma \) and \( \nu \) in that
\[
\mathbb{E} \left[ \# \{ s \in (t, t+\delta) : \Delta X_s \in (a, b) \} \right] = \mathbb{E} \left[ \int_t^{t+\delta} \int 1_{\{\gamma(X_s, z) \in (a, b)\}} \nu (X_s, z) \, dz \, dt \right].
\]

When \( \gamma(y, z) = z, \nu(x, z) \) controls the intensity of jumps with size near \( z \) when the state \( X \) takes the value \( x \);

When \( \nu(x, r) \equiv 0, X \) follows the local volatility model \( dX_t = b(X_t)dt + \sigma(X_t)dW_t \);

When \( \nu(x, r) \equiv \nu(r) \), for a Lévy function \( \nu \), \( X \) is the local jump-diffusion model:
\[
X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s + \sum_{s \leq t} \gamma(s, \Delta Z_s) 1_{\{ |\Delta Z_s| > 1 \}} + \sum_{s \leq t} \gamma(s, \Delta Z_s) 1_{\{ |\Delta Z_s| \leq 1 \}},
\]
where \( Z \) is a Lévy process with Lévy density \( \nu \) and, as usual, \( \Delta Z_s := Z_s - Z_{s-} \).

Suppose \( \nu(x, z) = \lambda(x)\check{\nu}(x, z) \) with \( \int \check{\nu}(x, z)dz = 1, \forall x \), and bounded \( \lambda \),
\[
X_t = x + \int_0^t \check{b}(X_s)ds + \int_0^t \sigma(X_s)dW_s + \sum_{\tau_i \leq t} \gamma \left( X_{\tau_i-}, J_i \right),
\]
where \( \{\tau_i\}_{i \geq 1} \) is a point process in \( \mathbb{R}_+ \) with stochastic intensity \( \lambda_t := \lambda(X_t) \) and \( J_i \sim \check{\nu}(X_{\tau_i-}, \cdot) \).
Intuition Behind the Model

1. In general,
   \[ dX_t = b(X_t)dt + \sigma(X_t)dW_t + dJ_t, \]
   where \( J \) is a pure-jump component controlled by \( \gamma \) and \( \nu \) in that
   \[ E \left[ \# \{ s \in (t, t+\delta) : \Delta X_s \in (a, b) \} \right] = E \left[ \int_t^{t+\delta} \int 1_{\{\gamma(X_t, z) \in (a, b)\}} \nu(X_t, z) \, dz \, dt \right]. \]

2. When \( \gamma(y, z) = z \), \( \nu(x, z) \) controls the intensity of jumps with size near \( z \) when the state \( X \) takes the value \( x \);

3. When \( \nu(x, r) \equiv 0 \), \( X \) follows the local volatility model \( dX_t = b(X_t)dt + \sigma(X_t)dW_t \);

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   where \( Z \) is a Lévy process with Lévy density \( \nu \) and, as usual, \( \Delta Z_s := Z_s - Z_{s-} \).

5. Suppose \( \nu(x, z) = \lambda(x)\tilde{\nu}(x, z) \) with \( \int \tilde{\nu}(x, z)dz = 1 \), \( \forall x \), and bounded \( \lambda \),
   \[ X_t = x + \int_0^t \bar{b}(X_s)ds + \int_0^t \sigma(X_s)dW_s + \sum_{\tau_i \leq t} \gamma\left(X_{\tau_i^-}, J_i\right), \]
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2. When \( \gamma(y, z) = z \), \( \nu(x, z) \) controls the intensity of jumps with size near \( z \) when the state \( X \) takes the value \( x \);

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where \( Z \) is a Lévy process with Lévy density \( \nu \) and, as usual, \( \Delta Z_s := Z_s - Z_{s^-} \).

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where \( \{\tau_i\}_{i \geq 1} \) is a point process in \( \mathbb{R}_+ \) with stochastic intensity \( \lambda_t := \lambda(X_t) \) and \( J_i \sim \tilde{\nu}(X_{\tau_i^-}, \cdot) \).
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   where \( Z \) is a Lévy process with Lévy density \( \nu \) and, as usual, \( \Delta Z_s := Z_s - Z_{s-} \).

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   where \( \{\tau_i\}_{i \geq 1} \) is a point process in \( \mathbb{R_+} \) with stochastic intensity \( \lambda_t := \lambda(X_t) \) and \( J_i \sim \tilde{\nu}(X_{\tau_i-}, \cdot) \).
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where \( \{ \tau_i \}_{i \geq 1} \) is a point process in \( \mathbb{R}_+ \) with stochastic intensity \( \lambda_t := \lambda(X_t) \) and \( J_i \sim \bar{\nu}(X_{\tau_i^-}, \cdot) \).
We look for a thinning type of construction. To that end, we assume
\[ \tilde{\nu}(z) := \sup_{x} \nu(x, z) \in (0, \infty) \text{ s.t. } \int \left( 1 \wedge z^2 \right) \tilde{\nu}(z)dz < \infty. \]

In that case,
\[ X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s + \int_0^t \int_{\mathbb{R}_0} 1_{\{\gamma(X_s - , z)\theta(X_s - , z, u)\}}p(ds, dz, du) \]
\[ + \int_0^t \int_{\mathbb{R}_0} 1_{\{|z| \leq 1\}} \gamma(X_s, z)\theta(X_s - , z, u)\overline{p}(ds, dz, du), \]

where
- \( \theta(x, z, u) := 1_{\left\{ u < \frac{\nu(x, z)}{h(z)} \right\}} \), for a positive Lévy measure \( h \) such that \( \tilde{\nu}(z) \leq h(z) \);
- \( p \) is a Poisson point process on \( \mathbb{R}_+ \times \mathbb{R}_0 \times (0, 1) \) with mean measure \( dsh(z)dzdu \);

Idea of the Construction:
- First generate a pure-jump Lévy process \( Z \) with Lévy density \( h \);
- Independently draw a \( (0, 1) \) uniform \( U_s \) for each jump of \( Z \) at \( s \);

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   \[ \bar{\nu}(z) := \sup_{x} \nu(x, z) \in (0, \infty) \text{ s.t. } \int (1 \wedge z^2) \bar{\nu}(z) dz < \infty. \]

2. In that case,
   \[
   X_t = x + \int_0^t b(X_s) ds + \int_0^t \sigma(X_s) dW_s + \int_0^t \int_{(0,1)} 1_{\{|z|>1\}} \gamma(X_s, z) \theta(X_s, z, u) p(ds, dz, du) \\
   + \int_0^t \int_{\mathbb{R}_0 \times (0,1)} 1_{\{|z|\leq 1\}} \gamma(X_s, z) \theta(X_s, z, u) \bar{p}(ds, dz, du),
   \]
   where
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Construction

1. We look for a **thinning** type of construction. To that end, we assume

\[(\star) \quad \tilde{\nu}(z) := \sup_{x} \nu(x, z) \in (0, \infty) \text{ s.t. } \int \left(1 \wedge z^2\right) \tilde{\nu}(z)dz < \infty.\]

2. In that case,

\[X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s + \int_0^t \int_{\mathbb{R}_0 \times (0,1)} 1\{|z|>1\} \gamma(X_{s-}, z)\theta(X_{s-}, z, u)p(ds, dz, du)\]
\[+ \int_0^t \int_{\mathbb{R}_0 \times (0,1)} 1\{|z|\leq1\} \gamma(X_{s}, z)\theta(X_{s-}, z, u)\bar{p}(ds, dz, du),\]

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Some Related Literature

1. Gatheral, et. al. (2012) gives the 2nd order expansion for a Local Volatility model;

2. F-L, Luo, and Ouyang (2014) obtains the 2nd order expansion for $\Pi_t^{Digital}$ and the 1st order term for $\Pi_t^{Call}$ under a Local jump diffusion ($\nu(x, z) = h(z)$, Lévy);

3. Yu (2007) gives an expansion for the transition densities in the case $\gamma(x, z) = z$ and $\nu(x, z) = \lambda(x)\nu(z)$ with $\int \nu(z)dz = 1$;
Motivation and Idea

1. In the Lévy case \( (\sigma(x) = \sigma, b(x) = b, \text{ and } \nu(x, z) = \nu(z)) \),
   - \( \Pi_t^{Digital} = \mathbb{P}[X_t \geq \kappa] \iff \Pi_t^{Call} = \mathbb{E}(e^{X_t} - \kappa^+) \) (F-L & Forde, 2012);
   - \( \mathbb{E}(e^{X_t} - e^{\kappa})_+ = \mathbb{E}(e^{X_t} - e^\kappa)1\{X_t \geq \kappa\} = \mathbb{P}^*[X_t \geq \kappa] - e^{\kappa}\mathbb{P}[X_t \geq \kappa] \)
     \( \downarrow \text{ with } \mathbb{E}^*[\chi] := \mathbb{E}[\chi e^{X_t}] \);
   - \( \{X_t\}_{t \geq 0} \) is Lévy under \( \mathbb{P}^* \) with Lévy density \( \nu^*(z) = e^z \nu(z) \);

2. What about a jump-diffusion case \( (\nu(x, z) = \nu(z)) \)?
   - The expansion for \( \mathbb{P}[X_t \geq \kappa] \) was obtained in F-L, Luo, & Ouyang (2014);
   - Is \( X_t \) a jump-diffusion process under \( \mathbb{P}^* \)? No...
   - Under \( \mathbb{P}^* \), \( X_t \) has generator \( L^* f(x) \) of the form
     \[
     b^*(x)f'(x) + \frac{\sigma^2(x)}{2}f''(x) + \int_{|z| \leq 1} \{f(x + \gamma(x, z)) - f(x) - 1_{|z| \leq 1}\gamma(x, z)f'(x)\} \nu^*(x, z)dz
     \]
     where
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     \nu^*(x, z) = e^{\gamma(x, z)}\nu(z), \quad b^*(x) = b(x) + \sigma^2(x) + \int_{|z| \leq 1} \left(e^{\gamma(x, z)} - 1\right) \gamma(x, z)\nu(z)dz.
     \]

3. For a general state-dependent jump intensity \( \nu(x, z), X \) is of the same type, under \( \mathbb{P}^* \), with \( \nu^*(x, z) = e^{\gamma(x, z)}\nu(x, z) \);
   So, we only need to find an expansion for \( \Pi_t^{Digital} = \mathbb{P}[X_t \geq \kappa] \).
Motivation and Idea

1. In the Lévy case ($\sigma(x) = \sigma$, $b(x) = b$, and $\nu(x, z) = \nu(z)$),
   
   - $\Pi_t^{Digital} = \mathbb{P}[X_t \geq \kappa] \implies \Pi_t^{Call} = \mathbb{E}(e^{X_t} - \kappa)_+ \quad \text{(F-L & Forde, 2012)}$;
   
   - $\mathbb{E}(e^{X_t} - e^\kappa)_+ = \mathbb{E}(e^{X_t} - e^\kappa) 1_{\{X_t \geq \kappa\}} = \mathbb{P}^* [X_t \geq \kappa] - e^\kappa \mathbb{P} [X_t \geq \kappa]$
     \[\blacktriangleleft \text{with } \mathbb{E}^* [\chi] := \mathbb{E} [\chi e^{X_t^X}]\]
   
   - $\{X_t\}_{t \geq 0}$ is Lévy under $\mathbb{P}^*$ with Lévy density $\nu^*(z) = e^z \nu(z)$;

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     where
     
     \[\nu^*(x, z) = e^{\gamma(x, z)} \nu(z), \quad b^*(x) = b(x) + \sigma^2(x) + \int_{|z| \leq 1} (e^{\gamma(x, z)} - 1) \gamma(x, z) \nu(z) dz.\]

3. For a general state-dependent jump intensity $\nu(x, z)$, $X$ is of the same type, under $\mathbb{P}^*$, with $\nu^*(x, z) = e^{\gamma(x, z)} \nu(x, z)$;
   
   So, we only need to find an expansion for $\Pi_t^{Digital} = \mathbb{P}[X_t \geq \kappa]$. 
Motivation and Idea

1. In the Lévy case \((\sigma(x) = \sigma, b(x) = b, \text{and } \nu(x, z) = \nu(z))\),
   - \(\Pi_t^{\text{Digital}} = \mathbb{P}[X_t \geq \kappa] \implies \Pi_t^{\text{Call}} = \mathbb{E}\left(\left(e^{X_t} - \kappa\right)^+\right)\) (F-L & Forde, 2012);
   - \(\mathbb{E}\left(e^{X_t} - e^{\kappa}\right)^+ = \mathbb{E}\left(e^{X_t} - e^{\kappa}\right) \mathbb{1}_{\{X_t \geq \kappa\}} = \mathbb{P}^* [X_t \geq \kappa] - e^{\kappa} \mathbb{P} [X_t \geq \kappa] \)
     \(\overset{\text{\textless}}{\Leftrightarrow}\) with \(\mathbb{E}^* [\chi] := \mathbb{E} \left[\chi e^{X_t}\right]\);
   - \(\{X_t\}_{t \geq 0}\) is Lévy under \(\mathbb{P}^*\) with Lévy density \(\nu^*(z) = e^z \nu(z)\);

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     \nu^*(x, z) = e^{\gamma(x, z)} \nu(z), \quad b^*(x) = b(x) + \sigma^2(x) + \int_{|z| \leq 1} \left(e^{\gamma(x, z)} - 1\right) \gamma(x, z) \nu(z) dz.
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3. For a general state-dependent jump intensity \(\nu(x, z)\), \(X\) is of the same type, under \(\mathbb{P}^*\), with \(\nu^*(x, z) = e^{\gamma(x, z)} \nu(x, z)\);
   So, we only need to find an expansion for \(\Pi_t^{\text{Digital}} = \mathbb{P}[X_t \geq \kappa]\).
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- \( \mathbb{E} (e^{X_t} - e^{\kappa})_+ = \mathbb{E} (e^{X_t} - e^{\kappa}) \ 1_{\{X_t \geq \kappa\}} = \mathbb{P}^* [X_t \geq \kappa] - e^{\kappa} \mathbb{P} [X_t \geq \kappa] \)
  \( \kern-1em \triangleleft \) with \( \mathbb{E}^* [\chi] := \mathbb{E} [\chi e^{X_t^X}] \);
- \( \{X_t\}_{t \geq 0} \) is Lévy under \( \mathbb{P}^* \) with Lévy density \( \nu^* (z) = e^z \nu (z) \);

2 What about a jump-diffusion case \( \nu (x, z) = \nu (z) \)?

- The expansion for \( \mathbb{P} [X_t \geq \kappa] \) was obtained in F-L, Luo, & Ouyang (2014);
- Is \( X_t \) a jump-diffusion process under \( \mathbb{P}^* \)? No...
- Under \( \mathbb{P}^* \), \( X_t \) has generator \( L^* f(x) \) of the form

\[
\begin{align*}
  b^*(x)f'(x) + \frac{\sigma^2(x)}{2} f''(x) + \int \{f(x + \gamma(x, z)) - f(x) - 1_{\{|z| \leq 1\}} \gamma(x, z) f'(x)\} \nu^*(x, z) dz
\end{align*}
\]

where

\[
\nu^*(x, z) = e^{\gamma(x, z)} \nu(z), \quad b^*(x) = b(x) + \sigma^2(x) + \int_{|z| \leq 1} \left( e^{\gamma(x, z)} - 1 \right) \gamma(x, z) \nu(z) dz.
\]

3 For a general state-dependent jump intensity \( \nu (x, z), \) \( X \) is of the same type, under \( \mathbb{P}^* \), with \( \nu^* (x, z) = e^{\gamma(x, z)} \nu (x, z) \);

So, we only need to find an expansion for \( \Pi_t^{Digital} = \mathbb{P}[X_t \geq \kappa] \).
Motivation and Idea

1. In the Lévy case \((\sigma(x) = \sigma, b(x) = b, \text{ and } \nu(x, z) = \nu(z))\),
   - \(\Pi_t^{\text{Digital}} = \mathbb{P}[X_t \geq \kappa] \iff \Pi_t^{\text{Call}} = \mathbb{E}(e^{X_t} - \kappa)_+ \quad (\text{F-L & Forde, 2012});\)
   - \(\mathbb{E}(e^{X_t} - e^\kappa)_+ = \mathbb{E}(e^{X_t} - e^\kappa)1\{X_t \geq \kappa\} = \mathbb{P}^* [X_t \geq \kappa] - e^\kappa \mathbb{P} [X_t \geq \kappa] \)
     \(\leftarrow \text{ with } \mathbb{E}^*[\chi] := \mathbb{E} \left[ \chi e^{X^*_t} \right];\)
   - \(\{X_t\}_{t \geq 0} \text{ is Lévy under } \mathbb{P}^* \text{ with Lévy density } \nu^*(z) = e^z \nu(z);\)

2. What about a jump-diffusion case \((\nu(x, z) = \nu(z))\)?
   - The expansion for \(\mathbb{P} [X_t \geq \kappa] \) was obtained in \(\text{F-L, Luo, & Ouyang (2014)};\)
   - Is \(X_t\) a jump-diffusion process under \(\mathbb{P}^*\)? No...
   - Under \(\mathbb{P}^*\), \(X_t\) has generator \(L^* f(x)\) of the form
     \[ b^*(x)f'(x) + \frac{\sigma^2(x)}{2} f''(x) + \int \left\{ f(x + \gamma(x, z)) - f(x) - 1\{|z| \leq 1\} \gamma(x, z)f'(x) \right\} \nu^*(x, z)dz \]
     where
     \[ \nu^*(x, z) = e^{\gamma(x, z)} \nu(z), \quad b^*(x) = b(x) + \sigma^2(x) + \int_{|z| \leq 1} \left( e^{\gamma(x, z)} - 1 \right) \gamma(x, z) \nu(z)dz. \]

3. For a general state-dependent jump intensity \(\nu(x, z), X\) is of the same type, under \(\mathbb{P}^*\), with \(\nu^*(x, z) = e^{\gamma(x, z)} \nu(x, z);\)
   So, we only need to find an expansion for \(\Pi_t^{\text{Digital}} = \mathbb{P}[X_t \geq \kappa].\)
In the Lévy case \((\sigma(x) = \sigma, b(x) = b, \text{ and } \nu(x, z) = \nu(z))\),

1. \(\Pi_t^{Digital} = \mathbb{P}[X_t \geq \kappa] \implies \Pi_t^{Call} = \mathbb{E} \left( e^{X_t} - e^{\kappa} \right)_+ \) (F-L & Forde, 2012);
2. \(\mathbb{E} \left( e^{X_t} - e^{\kappa} \right)_+ = \mathbb{E} \left( e^{X_t} - e^{\kappa} \right) 1_{\{X_t \geq \kappa\}} = \mathbb{P}^* \left[ X_t \geq \kappa \right] - e^{\kappa} \mathbb{P} \left[ X_t \geq \kappa \right] \)
   \(\downarrow\) with \(\mathbb{E}^* [\chi] := \mathbb{E} \left[ \chi e^{X_t} \right] \);
3. \(\{X_t\}_{t \geq 0} \) is Lévy under \(\mathbb{P}^*\) with Lévy density \(\nu^*(z) = e^z \nu(z)\);

2. What about a jump-diffusion case \((\nu(x, z) = \nu(z))\)?
   - The expansion for \(\mathbb{P} \left[ X_t \geq \kappa \right]\) was obtained in F-L, Luo, & Ouyang (2014);
   - Is \(X_t\) a jump-diffusion process under \(\mathbb{P}^*\)? No...
   - Under \(\mathbb{P}^*\), \(X_t\) has generator \(L^* f(x)\) of the form
     \[
     b^*(x) f'(x) + \frac{\sigma^2(x)}{2} f''(x) + \int_{|z| \leq 1} \left( f(x + \gamma(x, z)) - f(x) - 1_{|z| \leq 1} \gamma(x, z) f'(x) \right) \nu^*(x, z)dz
     \]
     where
     \[
     \nu^*(x, z) = e^{\gamma(x, z)} \nu(z), \quad b^*(x) = b(x) + \sigma^2(x) + \int_{|z| \leq 1} \left( e^{\gamma(x, z)} - 1 \right) \gamma(x, z) \nu(z)dz.
     \]

3. For a general state-dependent jump intensity \(\nu(x, z)\), \(X\) is of the same type, under \(\mathbb{P}^*\), with \(\nu^*(x, z) = e^{\gamma(x, z)} \nu(x, z)\);
   So, we only need to find an expansion for \(\Pi_t^{Digital} = \mathbb{P}[X_t \geq \kappa]\).
Equivalent formulation as a jump-diffusion model

1 Can we find \( \tilde{\gamma}(x, z) \) and \( \tilde{\nu}(z) \) such that

\[
\int \left\{ f(x + \gamma(x, z)) - f(x) - 1_{|z| \leq 1} \gamma(x, z)f'(x) \right\} \nu(x, z)dz = \int \left\{ f(x + \tilde{\gamma}(x, z)) - f(x) - 1_{|z| \leq 1} \tilde{\gamma}(x, z)f'(x) \right\} \tilde{\nu}(z)dz ?
\]

2 Equivalently,

\[
\int F(\gamma(x, z)) \nu(x, z)dz = \int F(\tilde{\gamma}(x, z)) \tilde{\nu}(z)dz, \quad (\ast)
\]

3 Proposition: (F-L & Luo, 2015) Suppose that \( 0 < \nu(x, z) \leq h(z) \) for a Lévy density \( h \). Let

\[
\bar{h}(w) = \begin{cases} 
\int_{-\infty}^{w} h(r)dr, & w < 0, \\
- \int_{w}^{\infty} h(r)dr, & w > 0,
\end{cases}
\]

\[
\tilde{\nu}(x, w) := \begin{cases} 
\int_{-\infty}^{w} \nu(x, r)dr, & w < 0, \\
- \int_{w}^{\infty} \nu(x, r)dr, & w > 0.
\end{cases}
\]

Then, (\ast) holds with

\[
\tilde{\gamma}(x, z) = \gamma \left( x, \tilde{\nu}^{-1}(x, \bar{h}(z)) \right), \quad \tilde{\nu}(z) = h(z);
\]
Equivalent formulation as a jump-diffusion model

1. Can we find \( \tilde{\gamma}(x, z) \) and \( \tilde{\nu}(z) \) such that

\[
\int \{ f(x + \gamma(x, z)) - f(x) - 1_{|z| \leq 1} \gamma(x, z)f'(x) \} \nu(x, z)dz = \int \{ f(x + \tilde{\gamma}(x, z)) - f(x) - 1_{|z| \leq 1} \tilde{\gamma}(x, z)f'(x) \} \tilde{\nu}(z)dz
\]

2. Equivalently,

\[
\int F(\gamma(x, z))\nu(x, z)dz = \int F(\tilde{\gamma}(x, z))\tilde{\nu}(z)dz, \quad (\star)
\]

3. Proposition: (F-L & Luo, 2015) Suppose that \( 0 < \nu(x, z) \leq h(z) \) for a Lévy density \( h \). Let

\[
\bar{h}(w) = \begin{cases} 
\int_{-\infty}^{w} h(r)dr, & w < 0, \\
-\int_{w}^{\infty} h(r)dr, & w > 0,
\end{cases} \quad \tilde{\nu}(x, w) := \begin{cases} 
\int_{-\infty}^{w} \nu(x, r)dr, & w < 0, \\
-\int_{w}^{\infty} \nu(x, r)dr, & w > 0.
\end{cases}
\]

Then, \( (\star) \) holds with

\[
\tilde{\gamma}(x, z) = \gamma \left( x, \tilde{\nu}^{-1}(x, \bar{h}(z)) \right), \quad \tilde{\nu}(z) = h(z);
\]
Equivalent formulation as a jump-diffusion model

1. Can we find \( \tilde{\gamma}(x, z) \) and \( \tilde{\nu}(z) \) such that
   \[
   \int \{ f(x + \gamma(x, z)) - f(x) - 1_{|z| \leq 1} \gamma(x, z)f'(x) \} \nu(x, z)dz
   = \int \{ f(x + \tilde{\gamma}(x, z)) - f(x) - 1_{|z| \leq 1} \tilde{\gamma}(x, z)f'(x) \} \tilde{\nu}(z)dz
   \]

2. Equivalently,
   \[
   \int F(\gamma(x, z))\nu(x, z)dz = \int F(\tilde{\gamma}(x, z))\tilde{\nu}(z)dz, \quad (\star)
   \]

3. Proposition: (F-L & Luo, 2015) Suppose that \( 0 < \nu(x, z) \leq h(z) \) for a Lévy density \( h \). Let
   \[
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   \int_{-\infty}^{w} h(r)dr, & w < 0, \\
   -\int_{w}^{\infty} h(r)dr, & w > 0,
   \end{cases}
   \]
   \[
   \bar{\nu}(x, w) := \begin{cases} 
   \int_{-\infty}^{w} \nu(x, r)dr, & w < 0, \\
   -\int_{w}^{\infty} \nu(x, r)dr, & w > 0.
   \end{cases}
   \]
   Then, \((\star)\) holds with
   \[
   \tilde{\gamma}(x, z) = \gamma\left(x, \bar{\nu}^{-1}(x, \bar{h}(z))\right), \quad \tilde{\nu}(z) = h(z);
   \]
Regularity of $\gamma(x, \tilde{\nu}^{-1}(x, \tilde{h}(z)))$

1. We need $\tilde{\gamma}(x, z)$ to be $C^3_b(\mathbb{R} \times \mathbb{R}_0)$;

2. If $\tilde{\nu} \in C^2_b(\mathbb{R} \times \mathbb{R}_0)$, then it suffices that $F(x, z) := \tilde{\nu}^{-1}(x, \tilde{h}(z)) \in C^3_b(\mathbb{R} \times \mathbb{R}_0)$.

3. It turns out that
   
   $$
   \partial_z F(x, z) = \frac{h(z)}{\nu(x, F(x, z))}, \quad \partial_x F(x, z) = -\frac{\partial_x \tilde{\nu}(x, w)}{\nu(x, F(x, z))}.
   $$

   The regularity of $F$ at $z = 0$ and $z = \infty$ are not trivial.

4. Theorem: (F-L & Luo, 2015) Suppose that $0 < \nu(x, z) \leq h(z)$ with $h(z) = g(z)|z|^{-\alpha-1}$, $\alpha \in (1, 2)$, and bounded twice differentiable $g : \mathbb{R} \to [0, \infty)$ such that $g(0) > 0$; Then, $F(x, z) \in C^3_b(\mathbb{R} \times \mathbb{R}_0)$.

5. A necessary and sufficient condition for $\nu$ to satisfy the desired condition is that $\tilde{\nu}(z) := \sup_x \nu(x, z)$ is a Lévy density bounded outside any neighborhood of the origin and
   
   $$
   \limsup_{z \to 0} |z|^\alpha \tilde{\nu}(z) < \infty;
   $$
Regularity of $\gamma(x, \tilde{\nu}^{-1}(x, \bar{h}(z)))$

1. We need $\tilde{\gamma}(x, z)$ to be $C_b^{3,2}(\mathbb{R} \times \mathbb{R}_0)$;

2. If $\tilde{\nu} \in C_b^{3,2}(\mathbb{R} \times \mathbb{R}_0)$, then it suffices that $F(x, z) := \tilde{\nu}^{-1}(x, \bar{h}(z)) \in C_b^{3,2}(\mathbb{R} \times \mathbb{R}_0)$.

3. It turns out that

$$\partial_z F(x, z) = \frac{h(z)}{\nu(x, F(x, z))}, \quad \partial_x F(x, z) = -\frac{\partial_x \tilde{\nu}(x, w)}{\nu(x, F(x, z))};$$

The regularity of $F$ at $z = 0$ and $z = \infty$ are not trivial.

4. Theorem: (F-L & Luo, 2015) Suppose that $0 < \nu(x, z) \leq h(z)$ with $h(z) = g(z)|z|^{-\alpha - 1}$, $\alpha \in (1, 2)$, and bounded twice differentiable $g : \mathbb{R} \to [0, \infty)$ such that $g(0) > 0$; Then, $F(x, z) \in C_b^{3,2}(\mathbb{R} \times \mathbb{R}_0)$.

5. A necessary and sufficient condition for $\nu$ to satisfy the desired condition is that $\tilde{\nu}(z) := \sup_x \nu(x, z)$ is a Lévy density bounded outside any neighborhood of the origin and

$$\limsup_{z \to 0} |z|^\alpha \tilde{\nu}(z) < \infty;$$
Regularity of $\gamma(x, \tilde{\nu}^{-1}(x, \tilde{h}(z)))$

1. We need $\tilde{\gamma}(x, z)$ to be $C_b^{3,2}(\mathbb{R} \times \mathbb{R}_0)$;

2. If $\tilde{\epsilon} \in C_b^{3,2}(\mathbb{R} \times \mathbb{R}_0)$, then it suffices that $F(x, z) := \tilde{\nu}^{-1}(x, \tilde{h}(z)) \in C_b^{3,2}(\mathbb{R} \times \mathbb{R}_0)$.

3. It turns out that

$$ \partial_z F(x, z) = \frac{h(z)}{\nu(x, F(x, z))}, \quad \partial_x F(x, z) = -\frac{\partial_x \tilde{\nu}(x, w)}{\nu(x, F(x, z))}; $$

The regularity of $F$ at $z = 0$ and $z = \infty$ are not trivial.

4. Theorem: (F-L & Luo, 2015) Suppose that $0 < \nu(x, z) \leq h(z)$ with $h(z) = g(z)|z|^{-\alpha-1}$, $\alpha \in (1, 2)$, and bounded twice differentiable $g : \mathbb{R} \rightarrow [0, \infty)$ such that $g(0) > 0$; Then, $F(x, z) \in C_b^{3,2}(\mathbb{R} \times \mathbb{R}_0)$.

5. A necessary and sufficient condition for $\nu$ to satisfy the desired condition is that $\tilde{\nu}(z) := \sup_x \nu(x, z)$ is a Lévy density bounded outside any neighborhood of the origin and

$$ \limsup_{z \to 0} |z|^\alpha \tilde{\nu}(z) < \infty; $$
Regularity of $\gamma(x, \bar{\nu}^{-1}(x, \bar{h}(z)))$

1. We need $\tilde{\gamma}(x, z)$ to be $C^{3,2}_b(\mathbb{R} \times \mathbb{R}_0)$;

2. If $\tilde{\nu} \in C^{3,2}_b(\mathbb{R} \times \mathbb{R}_0)$, then it suffices that $F(x, z) := \bar{\nu}^{-1}(x, \bar{h}(z)) \in C^{3,2}_b(\mathbb{R} \times \mathbb{R}_0)$.

3. It turns out that

$$
\partial_z F(x, z) = \frac{h(z)}{\nu(x, F(x, z))}, \quad \partial_x F(x, z) = -\frac{\partial_x \tilde{\nu}(x, w)}{\nu(x, F(x, z))}.
$$

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5. A necessary and sufficient condition for $\nu$ to satisfy the desired condition is that $\tilde{\nu}(z) := \sup_x \nu(x, z)$ is a Lévy density bounded outside any neighborhood of the origin and

$$
\limsup_{z \to 0} |z|^{\alpha+1} \tilde{\nu}(z) < \infty;
$$
Regularity of $\gamma(x, \bar{\nu}^{-1}(x, \bar{h}(z)))$

1. We need $\tilde{\gamma}(x, z)$ to be $C^{3,2}_b(\mathbb{R} \times \mathbb{R}_0)$;

2. If $\tilde{\nu} \in C^{3,2}_b(\mathbb{R} \times \mathbb{R}_0)$, then it suffices that $F(x, z) := \bar{\nu}^{-1}(x, \bar{h}(z)) \in C^{3,2}_b(\mathbb{R} \times \mathbb{R}_0)$.

3. It turns out that

$$\partial_z F(x, z) = \frac{h(z)}{\nu(x, F(x, z))}, \quad \partial_x F(x, z) = -\frac{\partial_x \tilde{\nu}(x, w)}{\nu(x, F(x, z))};$$

The regularity of $F$ at $z = 0$ and $z = \infty$ are not trivial.

4. Theorem: (F-L & Luo, 2015) Suppose that $0 < \nu(x, z) \leq h(z)$ with $h(z) = g(z)|z|^{-\alpha - 1}$, $\alpha \in (1, 2)$, and bounded twice differentiable $g : \mathbb{R} \to [0, \infty)$ such that $g(0) > 0$; Then, $F(x, z) \in C^{3,2}_b(\mathbb{R} \times \mathbb{R}_0)$.

5. A necessary and sufficient condition for $\nu$ to satisfy the desired condition is that $\tilde{\nu}(z) := \sup_x \nu(x, z)$ is a Lévy density bounded outside any neighborhood of the origin and

$$\limsup_{z \to 0} |z|^{\alpha + 1} \tilde{\nu}(z) < \infty;$$
In addition to the already mentioned assumptions, suppose that

$$|\partial_z \gamma(x, z)| > \eta, \quad |1 + \partial_x \gamma(x, z)| > \eta,$$

for some $\eta > 0$.

1st Order Term: For any $y > 0$,

$$\lim_{t \to 0} \frac{1}{t} \mathbb{P} \left[ X_t^{(x)} \geq x + y \right] = \int_{\{\gamma(x, z) \geq y\}} \nu(x, z) dz. $$

The leading order term is only determined by the jump component.

Intuition:

Suppose $\lambda(x) := \int \nu(x, z) dz < \infty$ and let $N_t = \#\{s > 0 : \Delta X_s \neq 0\}$;

$$\mathbb{P} \left[ X_t^{(x)} \geq x + y \right] \approx \mathbb{P} \left[ X_t^{(x)} \geq x + y \mid N_t = 1 \right] \lambda(x) t \approx \mathbb{P} \left[ \gamma(x, J_i) \geq y \mid N_t = 1 \right] \lambda(x) t,$$

$$\mathbb{P}[N_t = 1] \approx \lambda(x) t + o(t) \quad \left\langle J_i \sim \nu(x, \cdot)/\lambda(x) \right\rangle.$$
The Main Result

1st Order Term

1. In addition to the already mentioned assumptions, suppose that

\[ |\partial_z \gamma(x, z)| > \eta, \quad |1 + \partial_x \gamma(x, z)| > \eta, \quad \text{for some} \quad \eta > 0. \]

2. 1st Order Term: For any \( y > 0 \),

\[
\lim_{t \to 0} \frac{1}{t} \mathbb{P} \left[ X_t^{(x)} \geq x + y \right] = \int_{\{\gamma(x, z) \geq y\}} \nu(x, z) dz.
\]

The leading order term is only determined by the jump component.

3. Intuition:

Suppose \( \lambda(x) := \int \nu(x, z) dz < \infty \) and let \( N_t = \#\{s > 0 : \Delta X_s \neq 0\} \);

\[
\mathbb{P} \left[ X_t^{(x)} \geq x + y \right] \approx \mathbb{P} \left[ X_t^{(x)} \geq x + y | N_t = 1\right] \lambda(x)t \approx \mathbb{P} \left[ \gamma(x, J_i) \geq y | N_t = 1\right] \lambda(x)t,
\]

\[
\mathbb{P}[N_t = 1] \approx \lambda(x)t + o(t)
\]

\( J_i \sim \nu(x, \cdot)/\lambda(x) \).
In addition to the already mentioned assumptions, suppose that

\[ |\partial_z \gamma(x, z)| > \eta, \quad |1 + \partial_x \gamma(x, z)| > \eta, \quad \text{for some } \eta > 0. \]

1st Order Term: For any \( y > 0 \),

\[
\lim_{t \to 0} \frac{1}{t} \mathbb{P} \left[ X_t^{(x)} \geq x + y \right] = \int_{\{\gamma(x, z) \geq y\}} \nu(x, z) dz.
\]

The leading order term is only determined by the jump component.

Intuition:

Suppose \( \lambda(x) := \int \nu(x, z) dz < \infty \) and let \( N_t = \#\{s > 0 : \Delta X_s \neq 0\} \);

\[
\mathbb{P} \left[ X_t^{(x)} \geq x + y \right] \approx \mathbb{P} \left[ X_t^{(x)} \geq x + y | N_t = 1 \right] \lambda(x) t \approx \mathbb{P} \left[ \gamma(x, J_t) \geq y | N_t = 1 \right] \lambda(x) t,
\]

\[
\mathbb{P}[N_t = 1] \approx \lambda(x) t + o(t) \quad \quad \quad J_t \sim \nu(x, \cdot) / \lambda(x).
\]
The Main Result

2nd Order Term

1. **2nd Order Term**: For some (explicit) $\mathcal{D}(x, y)$ and $\mathcal{I}(x, y)$,

$$
\lim_{t \to 0} \frac{1}{t} \left( \frac{1}{t} \text{IP} \left[ X_t^{(x)} \geq x + y \right] - \int_{\gamma(x,z) \geq y} \nu(x, z) dz \right) = \mathcal{D}(x, y) + \mathcal{I}(x, y);
$$

2. Suppose $\gamma(x, z) = z$ (so that $\nu(x, z)$ has the interpretation of intensity of jumps near $z$ when the state is at $x$). Then, for $\varepsilon \in (0, 1)$ small enough,

$$
\mathcal{D}(x, y) = \frac{b_{\varepsilon}(x)}{2} \left[ \nu(x, y) + \int_y^\infty \partial_x \nu(x, r) dr \right] + \frac{\sigma^2(x)}{4} \left[ - \partial_y \nu(x, y) + \int_y^\infty \partial_y^2 \nu(x, r) dr \right] + \frac{1}{2} \left( b_{\varepsilon}(x + y) - \sigma(x + y) \sigma'(x + y) \right) \nu(x, y) - \frac{\sigma^2(x + y)}{4} \partial_y \nu(x, y)
$$

$$
\mathcal{I}(x, y) = \frac{1}{2} \int \left[ \int_y^\infty \nu(x + r, r_1) dr_1 - \int_y^\infty \nu(x, r_1) dr_1 - r \left( \nu(x, y) + \int_y^\infty \partial_1 \nu(x, r_1) dr_1 \right) \right] \bar{\nu}_{\varepsilon}(x, r) dr
$$

$$
+ \frac{1}{2} \int \left[ \int_y^r \nu_{\varepsilon}(x, r_1) \bar{\nu}_{\varepsilon}(x + r_1, r) dr_1 - \nu_{\varepsilon}(x, y) r \bar{\nu}_{\varepsilon}(x + y, r) \right] dr
$$

$$
+ \frac{1}{2} \int \nu_{\varepsilon}(x, r_1) \int_{r_1}^\infty \nu_{\varepsilon}(x, r_2) dr_2 dr_1
$$

$$
- \frac{1}{2} \int_y^\infty \nu(x, r_1) \int \nu_{\varepsilon}(x + r_1, r_2) dr_2 dr_1 - \int_y^\infty \nu(x, r) dr \int \nu_{\varepsilon}(x, r) dr.
$$

where $b_{\varepsilon}(x) = b(x) - \int_{r < |r| \leq 1} r \nu(x, r) dr$, $\nu_{\varepsilon}(x, r) = \nu(x, r) \mathbf{1}_{|r| \geq \varepsilon}$, $\bar{\nu}_{\varepsilon}(x, r) := \nu(x, r) \mathbf{1}_{|r| < \varepsilon}$.
2nd Order Term: For some (explicit) $D(x, y)$ and $I(x, y)$,
\[
\lim_{t \to 0} \frac{1}{t} \left( \frac{1}{t} \mathbb{P} \left[ X_t^{(x)} \geq x + y \right] - \int_{\gamma(x, z) \geq y} \nu(x, z) dz \right) = D(x, y) + I(x, y);
\]

Suppose $\gamma(x, z) = z$ (so that $\nu(x, z)$ has the interpretation of intensity of jumps near $z$ when the state is at $x$). Then, for $\varepsilon \in (0, 1)$ small enough,
\[
D(x, y) = \frac{b_{\varepsilon}(x)}{2} \left[ \nu(x, y) + \int_y^\infty \partial_x \nu(x, r) dr \right] + \frac{\sigma^2(x)}{4} \left[ - \partial_y \nu(x, y) + \int_y^\infty \partial_x^2 \nu(x, r) dr \right] \\
+ \frac{1}{2} \left( \frac{b_{\varepsilon}(x + y) - \sigma(x + y) \sigma'(x + y)}{\sigma^2(x + y)} \right) \nu(x, y) - \frac{\sigma^2(x + y)}{4} \partial_y \nu(x, y)
\]

\[
I(x, y) = \frac{1}{2} \int \left[ \int_{y-r}^\infty \nu(x + r, r_1) dr_1 - \int_y^\infty \nu(x, r_1) dr_1 - r \left( \nu(x, y) + \int_y^\infty \partial_1 \nu(x, r_1) dr_1 \right) \right] \tilde{\nu}_\varepsilon(x, r) dr
\]
\[
+ \frac{1}{2} \int \left[ \int_{y-r}^y \nu_\varepsilon(x, r_1) \tilde{\nu}_\varepsilon(x + r_1, r) dr_1 - \nu_\varepsilon(x, y) r \tilde{\nu}_\varepsilon(x + y, r) \right] dr
\]
\[
+ \frac{1}{2} \int \nu_\varepsilon(x, r_1) \int_{y-r_1}^\infty \nu_\varepsilon(x, r_2) dr_2 dr_1
\]
\[
- \frac{1}{2} \int_y^\infty \nu(x, r_1) \int \nu_\varepsilon(x + r_1, r_2) dr_2 dr_1 - \int_y^\infty \nu(x, r) dr \int \nu_\varepsilon(x, r) dr.
\]

where $b_{\varepsilon}(x) = b(x) - \int_{|r| \leq 1} r \nu(x, r) dr$, $\nu_\varepsilon(x, r) = \nu(x, r) 1_{|r| \geq \varepsilon}$, $\tilde{\nu}_\varepsilon(x, r) := \nu(x, r) 1_{|r| < \varepsilon}$.
The Main Result

2nd Order Term

1. **2nd Order Term:** For some (explicit) $D(x, y)$ and $I(x, y)$,

$$
\lim_{t \to 0} \frac{1}{t} \left( \frac{1}{t} \mathbb{P} \left[ X_t^{(x)} \geq x + y \right] - \int_{\{\gamma(x, z) \geq y\}} \nu(x, z) \, dz \right) = D(x, y) + I(x, y);
$$

2. Suppose $\gamma(x, z) = z$ (so that $\nu(x, z)$ has the interpretation of intensity of jumps near $z$ when the state is at $x$). Then, for $\varepsilon \in (0, 1)$ small enough,

$$
D(x, y) = \frac{b_\varepsilon(x)}{2} \left[ \nu(x, y) + \int_y^\infty \partial_x \nu(x, r) \, dr \right] + \frac{\sigma^2(x)}{4} \left[ - \partial_y \nu(x, y) + \int_y^\infty \partial_x^2 \nu(x, r) \, dr \right] \\
+ \frac{1}{2} \left( b_\varepsilon(x + y) - \sigma(x + y) \sigma'(x + y) \right) \nu(x, y) - \frac{\sigma^2(x + y)}{4} \partial_y \nu(x, y)
$$

$$
I(x, y) = \frac{1}{2} \int \left[ \int_y^\infty \nu(x + r, r_1) \, dr_1 - \int_y^\infty \nu(x, r_1) \, dr_1 - \left( \nu(x, y) + \int_y^\infty \partial_1 \nu(x, r_1) \, dr_1 \right) \right] \bar{\nu}_\varepsilon(x, r) \, dr
$$

$$
+ \frac{1}{2} \int \left[ \int_{y-r}^y \nu_\varepsilon(x + r_1, r) \bar{\nu}_\varepsilon(x + r_1, r) \, dr_1 - \nu_\varepsilon(x, y) \, r \bar{\nu}_\varepsilon(x + y, r) \right] \, dr
$$

$$
+ \frac{1}{2} \int \nu_\varepsilon(x, r_1) \int_{y-r_1}^\infty \nu_\varepsilon(x, r_2) \, dr_2 \, dr_1
$$

$$
- \frac{1}{2} \int_y^\infty \nu(x, r_1) \int \nu_\varepsilon(x + r_1, r_2) \, dr_2 \, dr_1 - \int_y^\infty \nu(x, r) \, dr \int \nu_\varepsilon(x, r) \, dr
$$

where $b_\varepsilon(x) = b(x) - \int_{\varepsilon < |r| \leq 1} r \nu(x, r) \, dr$, $\nu_\varepsilon(x, r) = \nu(x, r) 1_{|r| \geq \varepsilon}$, $\bar{\nu}_\varepsilon(x, r) := \nu(x, r) 1_{|r| < \varepsilon}$.
The Main Result

2nd Order Term

1. 2nd Order Term: For some (explicit) \( D(x, y) \) and \( I(x, y) \),

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\lim_{t \to 0} \frac{1}{t} \left( \frac{1}{t} \mathbb{P} \left[ X_t^{(x)} \geq x + y \right] - \int_{\{\gamma(x, z) \geq y\}} \nu(x, z) dz \right) = D(x, y) + I(x, y);
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2. Suppose \( \gamma(x, z) = z \) (so that \( \nu(x, z) \) has the interpretation of intensity of jumps near \( z \) when the state is at \( x \)). Then, for \( \varepsilon \in (0, 1) \) small enough,

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D(x, y) = \frac{b_\varepsilon(x)}{2} \left[ \nu(x, y) + \int_y^\infty \partial_x \nu(x, r) dr \right] + \frac{\sigma^2(x)}{4} \left[ - \partial_y \nu(x, y) + \int_y^\infty \partial_x^2 \nu(x, r) dr \right] + \frac{1}{2} \left( b_\varepsilon(x + y) - \sigma(x + y) \sigma'(x + y) \right) \nu(x, y) - \frac{\sigma^2(x + y)}{4} \partial_y \nu(x, y)
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I(x, y) = \frac{1}{2} \int \left[ \int_{y-r}^\infty \nu(x + r, r_1) dr_1 - \int_y^\infty \nu(x, r_1) dr_1 - r \left( \nu(x, y) + \int_y^\infty \partial_1 \nu(x, r_1) dr_1 \right) \right] \bar{\nu}_\varepsilon(x, r) dr
\]

where \( b_\varepsilon(x) = b(x) - \int_{|r| \leq 1} r \nu(x, r) dr \), \( \nu_\varepsilon(x, r) = \nu(x, r) \mathbf{1}_{|r| \geq \varepsilon} \), \( \bar{\nu}_\varepsilon(x, r) := \nu(x, r) \mathbf{1}_{|r| < \varepsilon} \).
Some Interesting Consequences \( (\gamma(x, z) = z) \)

1. A constant positive “drift” \( b \) would increase the probability of a “large” move of more than \( y \) by

\[
\frac{t^2}{2} b \left( 2\nu(x, y) + \int_y^\infty \frac{\partial}{\partial x} \nu(x, r) dr \right) (1 + o(1)).
\]

2. A constant nonzero volatility \( \sigma \) will change the probability of a “large” move of more than \( y \) by

\[
\frac{t^2}{2} \sigma^2 \left( -\frac{\partial}{\partial y} \nu(x, y) + \frac{1}{2} \int_y^\infty \frac{\partial^2}{\partial x^2} \nu(x, r) dr \right) (1 + o(1)),
\]

3. The leading order term of \( C(\kappa, t) = \mathbb{E} \left( e^{X_t} - e^\kappa \right)_+ \) (assuming \( X_0 = 0 \) and \( \kappa > 0 \)) is

\[
t \int \left( e^r - e^\kappa \right)_+ \nu(0, r) dr.
\]

4. The curvature of the OTM call premium \( K \rightarrow C(t, K) \) (where \( K = e^\kappa \)) is strongly determined by jump intensity \( \nu(0, \kappa) \):

\[
\frac{\partial^2 C(t, K)}{\partial K^2} \approx te^{-\kappa} \nu(0, \kappa) \iff \nu(0, \kappa) \approx \frac{1}{t} e^\kappa \frac{\partial^2 C(t, K)}{\partial K^2};
\]

5. The effect of a nonzero constant volatility \( \sigma(x) \equiv \sigma \) in the price of an OTM call option is of order

\[
\frac{t^2 \sigma^2}{2} \left[ e^k \nu(0, k) + \int_k^\infty \frac{e^r + e^k}{2} \partial_x \nu(0, r) dr + \int_k^\infty \frac{e^r - e^k}{2} \partial^2_x \nu(0, r) dr \right] (1 + o(1)),
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Some Interesting Consequences \((\gamma(x, z) = z)\)

1. A constant positive “drift” \(b\) would increase the probability of a “large” move of more than \(y\) by
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   \]

2. A constant nonzero volatility \(\sigma\) will change the probability of a “large” move of more than \(y\) by
   \[
   \frac{t^2}{2} \sigma^2 \left(-\frac{\partial}{\partial y} \nu(x, y) + \frac{1}{2} \int_y^\infty \frac{\partial^2}{\partial x^2} \nu(x, r) dr \right) (1 + o(1)),
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   \]
Some Interesting Consequences ($\gamma(x, z) = z$)

1. A constant positive "drift" $b$ would increase the probability of a "large" move of more than $y$ by
   \[ \frac{t^2}{2} b \left( 2\nu(x, y) + \int_y^\infty \frac{\partial}{\partial x} \nu(x, r) dr \right) (1 + o(1)). \]

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For Further Reading I

Figueroa-López, J.E., & Luo, Y.
Small-time expansions for state-dependent local jump-diffusion models with infinite jump activity. *Arxiv 2015*.

Figueroa-López, J.E., Luo, Y., & Ouyang, C.
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