Modeling and estimation of the dynamics of asset prices

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Outline

1. Modeling of asset prices
   - Asset price evolution
   - The Black-Scholes model
   - Jump-based models

2. Nonparametric estimation of the Lévy density
   - The problem and the method
   - An example
   - Some results
   - Conclusions
1 Objectives:
   - Define a parsimonious stochastic model that accounts for the “stylized” features observed in historical asset prices.
   - Estimate the parameters of the model to the historical prices.

2 Types of assets: Stocks, financial indexes, bond prices, etc.

3 Why?
   - Adequate allocation of money in a portfolio of assets.
   - Risk management.

4 What are these stylized features?
   - Heavy-tails of short-term returns
   - “Sudden big” changes in price (Jumps)
   - “Volatility” clustering (intermittency)
   - Leverage phenomenon
What has been done?

1. Black-Scholes model
2. Lévy based models
3. Stochastic volatility models
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2. Lévy based models $\implies$ Allows jumps
3. Stochastic volatility models
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1. Black-Scholes model
2. Lévy based models $\Rightarrow$ Allows jumps
3. Stochastic volatility models $\Rightarrow$ Allows volatility intermittency
Black-Scholes Model

1 Principles:
   - The (log) returns during disjoint time periods are independent from one another.
   - The log returns on equal size time periods have the same statistical properties.
   - The process evolves continuously in time.

2 The model:

\[
\log \left\{ \frac{S_{t+\Delta t}}{S_t} \right\} \sim \mathcal{N} \left( m \Delta t, \sigma^2 \Delta t \right)
\]

- \( m = \text{Mean rate of return} \)
- \( \sigma = \text{Volatility or measure of variability} \)
The Brownian motion

1. Equivalent formulation:

\[ \log \left( \frac{S_{t+\Delta t}}{S_t} \right) = m \Delta t + \sigma \{ W_{t+\Delta t} - W_t \} \]

where \( W \) is the standard Brownian Motion:

- \( W_{t+\Delta t} - W_t \sim \mathcal{N}(0, \Delta t) \)
- Independent increments
- Continuous paths


- Distributions with heavier tails
- "Sudden big" changes (Jumps)
Need for price jumps

1. Introduce jumps via a Compound Poisson Process:
   - The "arrival" of jumps are independent from one another
   - Jumps don’t occur simultaneously
   - Jumps arrive "homogeneously" across time at an average expected rate of $\lambda$ jumps per unit time.
   - The size of the jumps has the same distribution with density $f$.

   \[ N_t = \text{Number of jumps by time } t; \]
   \[ Y_i = \text{Size of the } i\text{th jump} \]
   \[ Z_t = \sum_{i=1}^{N_t} Y_i, \]

2. Jump-Diffusion model with finite-jump activity:

   \[ \log \left\{ \frac{S_{t+\Delta t}}{S_t} \right\} = m \Delta t + \sigma \{ W_{t+\Delta t} - W_t \} + \{ Z_{t+\Delta t} - Z_t \} \]
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   \]
Other alternatives based on Poisson jumps

Time-changed jump-diffusion: [Carr, Madan, Geman, Yor etc.]

\[
\log \frac{S_t}{S_0} = X_{T_t},
\]

\[
X(t) = mt + \sigma W_t + Z_t, \quad T(t) = \text{Random Clock}.
\]

Stochastic volatility driven by a jump-diffusion: [B-N and Shephard]

\[
\log \frac{S_t}{S_0} = mt + \int_0^t \sigma_s dW_s.
\]

\[
\sigma_t^2 = \sigma_0^2 + \int_0^t \alpha \sigma_s^2 ds + X_{\alpha t}.
\]

Shot-noise jump diffusion model:

\[
\log \frac{S_t}{S_0} = \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s + \sum_{s \leq t} h(s, \Delta Z_s),
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Nonparametric estimation

1 Set-up:
   • \( \{X_t\}_{t \geq 0} \) is a jump-diffusion process with density of jumps \( f \) and intensity of jumps \( \lambda \).
   • The process is discretely sampled at \( 0 = t_0 < \cdots < t_n = T \).

2 \( s(x) = \lambda f(x) \) is called the Lévy density of the process.

3 Problem: Estimate \( s \) in a non-parametric fashion.

4 Why is estimation hard?
   The times and sizes of the jumps are latent unobservable variables.

5 Intuition:
   • \( \max_i \{ t_i - t_{i-1} \} \downarrow 0 \implies \text{Recover jumps from } \{X_{t_i} - X_{t_{i-1}}\}_i \).
   • \( t_n \uparrow \infty \implies \text{Increase of relevant sample size.} \)
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Histogram type estimators

1. Building blocks:

\[ \hat{\beta}(\varphi) := \frac{1}{tn} \sum_{k=1}^{n} \varphi \left( X_{tk} - X_{tk-1} \right) \]

Realized \( \varphi \)-variation per unit time.

2. Histogram estimators:

- Fix an estimation window \([a, b]\).
- Divide the window on \(m\) classes: \((x_0, x_1], \ldots, (x_{m-1}, x_m]\).
- Construct the function estimator:

\[ \hat{s}(x) = \hat{\beta}(\varphi_1)\varphi_1(x) + \cdots + \hat{\beta}(\varphi_m)\varphi_m(x), \]

where \( \varphi_i(x) = \frac{1}{\sqrt{x_i-x_{i-1}}}1_{(x_{i-1}, x_i]} \).
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An example: Gamma Lévy process

1. Model: Pure-jump process 
   \[ s(x) = \frac{\alpha}{x} e^{-x/\beta} 1_{\{x > \varepsilon\}}. \]

2. Histogram estimators:
Performance

- **Maximum-Likelihood estimators:** \( \hat{\alpha}_{MLE} = 1.01 \) and \( \hat{\beta}_{MLE} = 0.94 \).

- **Non-parametric least-squares estimators:** \( \hat{\alpha}_{LSE} = 0.93 \) and \( \hat{\beta}_{LSE} = 1.055 \)
  
  Obtained from fitting the model \( \frac{\alpha}{x} e^{-x/\beta} \) (using least-squares) to the histogram estimator.

- **Sampling mean and standard errors** based on 1000 repetitions.

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Example: One-sided tempered stable distribution

1 Model: Pure-jump Lévy process with Lévy density

\[ s(x) = \frac{a}{x^{\alpha+1}} e^{-x/b} 1_{\{x > \varepsilon\}}. \]

2 Sampling means and standard errors based on 100 repetitions.

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Example: One-sided tempered stable distribution

1. **Model:** Pure-jump Lévy process with Lévy density
   \[ s(x) = \frac{a}{x^{\alpha+1}} e^{-x/b} 1\{x \geq \varepsilon\}. \]

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Some asymptotic results

1. Set-up:

- \( \pi : 0 = t_0 < \cdots < t_n = T \)
  - \( T \) = time horizon and \( \bar{\pi} = \sup_i \{ t_i - t_{i-1} \} \) = mesh.
- \( \hat{s}_m^\pi \) = Histogram estimator on \([a, b]\) based on \( m \) classes and on observation \( X_{t_0}, \ldots, X_{t_n} \).
- \( s_m^\perp \) be the histogram function based on \( m \)-classes that is “closest” to \( s \).
- \( \| p - q \|^2 := \int_a^b (p(x) - q(x))^2 dx \) = distance between \( p \) and \( q \).

2. Results: As \( \bar{\pi} \downarrow 0 \) and \( T \uparrow \infty \),

- \( \| \hat{s}_m - s_m^\perp \| \to 0 \).
- \( \sqrt{T} (\hat{s}_m^\pi(x) - s_m^\perp(x)) \xrightarrow{D} \bar{\sigma}(x) \mathcal{N}(0, 1) \).
- \( \| s_m^\perp - s \| = O(m^{-\alpha}) \), as \( m \to \infty \) if for all \( x, y \in [a, b] \)

\[
|s(x) - s(y)| \leq L|x - y|^\alpha, \quad 0 < \alpha \leq 1.
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Some more asymptotic results

1. How good could an estimator do?
   For any estimators $\hat{s}_T$ of $s$, based on the whole path of the process up to time $T$, there exists a "smooth" $s$ and a constant $\varepsilon > 0$ such that:

   $$T^{\frac{2\alpha}{2\alpha+1}} \mathbb{E} \| \hat{s}_T - s \|^2 > \varepsilon, \quad \text{for all } T.$$

2. The fastest rate of convergence that one can hope for is $T^{-2\alpha/(2\alpha+1)}$.

3. How good can we do?
   There exists a "critical" mesh $\delta_T > 0$ such that if $\bar{\pi} < \delta_T$ and $m_T := \left\lfloor T^{1/(2\alpha+1)} \right\rfloor$, then for a constant $B < \infty$,

   $$T^{2\alpha/(2\alpha+1)} \mathbb{E} \| s - \hat{s}_{m_T}^\pi \|^2 < B, \quad \text{for all } s \text{ and } T.$$
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1. In the finite-jump activity case, $\delta(T) = o\left(\frac{1}{T}\right)$.
2. In the general case, $\delta(T) < T^{-\frac{1}{a}}$, if $D = [a, b]$ with $a > 0$. 
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**Theorem**

1. *In the finite-jump activity case*, $\delta(T) = o\left(\frac{1}{T}\right)$.
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Conclusions I

1. Exponential Lévy models are some of the simplest and most practical alternatives to the shortfalls of the Black-Scholes model.
2. Capture several stylized empirical features of historical returns.
3. Limitations: Lack of stochastic volatility, leverage, quasi-long-memory, etc.
4. Lévy processes are important in constructing more robust asset price models.
Conclusions II

1. We develop a feasible estimation scheme for the Lévy density based on discrete-sampling.
2. The method is flexible (histograms, splines, wavelets), model free, and (in principle) work well for high-frequency data.
3. We are able to estimate the rate of convergence of the method under smoothness assumptions.
4. The rate of convergence in the long-run match the optimal (minimax) rate even under continuous sampling.
For Further Reading I

Figueroa-López.
Small-time asymptotics for generalized moments of Lévy processes and some applications.
Preprint available online, 2007.

Woerner.
Variational Sums and Power Variation: a unifying approach to model selection and estimation in semimartingale models.

Jacod.
Asymptotic properties of power variations of Lévy processes.
Figueroa-Lopez and Houdré.
Empirical distribution of returns

Return during a given time period = $\log \frac{\text{Final price}}{\text{Initial price}}$.

Figure 1: Empirical density of one-hour returns (Bayer) vs. density of fitted hyperbolic (blue) and fitted normal distribution (red).
FIGURE 1.2: Evolution of SLM (NYSE), January-March 1993, compared with a scenario simulated from a Black-Scholes model with same annualized return and volatility.
Times series of returns

Five-minute log-return for Yen/Deutsehemark exchange rate, 1992-1995

BMW daily log-returns
Lévy-Itô decomposition

\[ X_t = bt + \sigma W_t + \text{Pure jump process} \]

Examples of Lévy processes: compound Poisson process (left) and Lévy jump-diffusion
Figure 1.5: Densities and log-densities of high frequency data.
Empirical performance of the CGMY Lévy model
Madan, Carr, Geman, Yor, and others
Lévy processes with jumps

Compound Poisson Process

Examples of Lévy processes: compound Poisson process (left) and Lévy jump-diffusion