Short-time Asymptotics for
Stochastic Differential Equations with Jumps

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Joint work with C. Houdré, Y. Luo, S. Ólafsson, and C. Ouyang.
Outline

1 Motivation and Background

2 The Problem

3 Tools and Some Results

4 Conclusions
Many phenomena are best understood as dynamical systems operating in continuous time.

A Stochastic Process is a collection of random variables $X = \{X_t\}_{t \geq 0}$, which are indexed by time $t$ and are generated by the same random experiment $\mathcal{E}$.

Constructions follow a bottom-up approach from simple to more complex models.

The most basic processes are:

- Brownian Motion $W = \{W_t\}_{t \geq 0}$ with volatility $\sigma \in (0, \infty)$:
  - The paths $t \rightarrow W_t(\omega)$ are continuous for any outcome $\omega$ of the experiment $\mathcal{E}$;
  - $t_1 < \cdots < t_n : W_{t_2} - W_{t_1}, \ldots, W_{t_n} - W_{t_{n-1}}$ indep. $\mathcal{N}(0, \sigma(t_n - t_{n-1}))$.
- Poisson Process $N = \{N_t\}_{t \geq 0}$ with intensity $\lambda \in (0, \infty)$ of jumps:
  - The paths $t \rightarrow N_t(\omega)$ are nondecreasing piece-wise constant jumping by ones for any outcome $\omega$ of $\mathcal{E}$;
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An overview of common SDEs

1 Continuous Diffusion Process:

\[ dX_t = b(X_t) \, dt + \sigma(X_t) \, dW_t \]

\[ b : \mathbb{R} \to \mathbb{R}, \quad \sigma : \mathbb{R} \to [0, \infty), \]

\[ W \sim \text{standard B.M.} \]

\[ X_{t+dt} - X_t \approx b(X_t) \, dt + \sigma(X_t) (W_{t+dt} - W_t) \]

2 Simple Jump-Diffusion Process:

\[ dX_t = b(X_t) \, dt + \sigma(X_t) \, dW_t + \gamma(X_t, J_t) \, dN_t \]

\[ \{J_t\}_{t \geq 0} \overset{\text{i.i.d.}}{\sim} g, \quad N \sim \text{Poisson}(\lambda), \]

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3 Jump-Diffusion Process with STATE-Dependent Jump Intensity:

\[ dX_t = b(X_t) \, dt + \sigma(X_t) \, dW_t + \gamma(X_t, J_t) \, dM_t \]

\[ t \to M_t \text{ is a counting process}^1 \text{ such that } \mathbb{E}[M_{t+dt} - M_t] \approx \mathbb{E} [\lambda(X_t) \, dt], \text{ for} \]

\[ \lambda : \mathbb{R} \to [0, \infty); \]

\[ J_t \sim g(\cdot; X_t), \text{ where, for each } x \in \mathbb{R}, g(\cdot; x) \text{ is a pdf (i.e., } \int g(z; x)dz = 1) \]

---

1 Nondecreasing piece-wise constant paths jumping by ones
An overview of common SDEs

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   \[
   dX_t = b(X_t) \, dt + \sigma(X_t) \, dW_t \quad b : \mathbb{R} \to \mathbb{R}, \quad \sigma : \mathbb{R} \to [0, \infty),
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Simulation of State-Dependent Jump-Diffusions

Goal: (Approximately) generate a fine discrete “skeleton” $X_{t_0}, X_{t_1}, \ldots, X_{t_n}$ of the process where $0 = t_0 < t_1 < \cdots < t_n$ are frequent enough sampling times;

1. Set $X_{t_0} = x$ (given starting point) and $\Delta_i = t_{i+1} - t_i$;
2. Iteratively generate, for $i = 0, \ldots, n - 1$,
   - Standard normal r.v. $Z_i$
   - A r.v. $J_i$ with density $g(\cdot; X_{t_i})$
   - A Bernoulli r.v. $I_i$ with $\mathbb{P}(I_i = 1) = 1 - \mathbb{P}(I_i = 0) = \lambda(X_{t_i})\Delta_i$
   - $X_{t_{i+1}} = X_{t_i} + b(X_{t_i})\Delta_i + \sigma(X_{t_i})\sqrt{\Delta_i} Z_i + \gamma(X_{t_i}, J_i) I_i$, 

The Problem

- Let \( \{X_t\}_{t \geq 0} \) be a state-dependent jump diffusion;
- **Goal**: Given a suitable function \( f : \mathbb{R} \to \mathbb{R} \), we wish to study the asymptotic behavior, as \( t \to 0 \), of the generalized moment functional

\[
\beta_t(f) := \mathbb{E}[f(X_t)].
\]

- **Common Examples**:

\[
\begin{align*}
  f(x) &= x^k, \ (k \in \mathbb{N}), \\
  f(x) &= e^{iux}, \ (u \in \mathbb{R}), \\
  f(x) &= 1_{a < x < b}, \ (-\infty \leq a < b \leq \infty),
\end{align*}
\]

- If, say, \( f \) is continuous in a neighborhood of the starting point \( x \),

\[
\lim_{t \to 0} \beta_t(f) = f(x),
\]

(provided that \( f \) is appropriately bounded);

- Rate of convergence in \( \beta_t(f) \xrightarrow{t \to 0} f(x) \)?
Applications

1 Statistical Inference based on high-frequency data

- A great deal of inference methods are based on high-frequency realized variations;
- Given a discrete record of consecutive sampling observations \(X_{t_1}, \ldots, X_{t_n}\) regular in time \((t_i = iT/n)\),
  \[
  \hat{\beta}_{n,t}(f) := \sum_{i=1}^{n} f \left( X_{t_i} - X_{t_{i-1}} \right)
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  is called the realized \(f\)-variation of \(X\);
- The statistical properties of \(\hat{\beta}_n(f)\), as \(n \to \infty\), boil down to short-time asymptotics for certain moment functionals

2 Mathematical Finance

- In finance, \(\beta_t(f) := \mathbb{E}[f(X_t)]\) is interpreted as the price of a claim that pays the amount \(f(X_t)\) at time \(t\) when the price of a stock security is \(e^{X_t}\);
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Smooth moment functions (Dynkin’s formula)

- **Infinitesimal Generator:** For \( f \in C_b^2(\mathbb{R}) \), let
  \[
  Lf(x) = b(x)f'(x) + \frac{\sigma^2(x)}{2}f''(x) + \int_\mathbb{R} \{f(x + \gamma(x, z)) - f(x)\} \lambda(x)g(z; x) \, dz;
  \]

- **Key Property:**
  \[
  \lim_{t \to 0} \frac{\mathbb{E}[f(X_t)] - f(x)}{t} = Lf(x) \iff \beta_t(f) := \mathbb{E}[f(X_t)] = f(x) + tLf(x) + o(t).
  \]

- **Dynkin’s formula:**
  \[
  \mathbb{E}[f(X_t)] = f(x) + \int_0^t \mathbb{E}[Lf(X_r)] \, dr = f(x) + tLf(x) + o(t) \quad \text{(since } Lf \in C_b(\mathbb{R}));
  \]

- **Implication:** If \( Lf \in C_b^2 \),
  \[
  \mathbb{E}[f(X_t)] = f(x) + \int_0^t \mathbb{E}[Lf(X_r)] \, dr
  = f(x) + \int_0^t \left\{ Lf(x) + \int_0^r \mathbb{E}\left( (L^2 f)(X_s) \right) \, ds \right\} \, dr
  = f(x) + tLf(x) + \frac{t^2}{2}L^2 f(x) + o(t^2);
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  \lim_{t \to 0} \frac{\mathbb{E}[f(X_t)] - f(x)}{t} = Lf(x) \iff \beta_t(f) := \mathbb{E}[f(X_t)] = f(x) + tLf(x) + o(t).
  \]

- **Dynkin’s formula:**
  \[
  \mathbb{E}[f(X_t)] = f(x) + \int_0^t \mathbb{E}[Lf(X_r)]\,dr = f(x) + tLf(x) + o(t) \quad \text{ (since }Lf \in C^1_b(\mathbb{R}));
  \]

- **Implication:** If \( Lf \in C^2_b \),
  \[
  \mathbb{E}[f(X_t)] = f(x) + \int_0^t \mathbb{E}[Lf(X_r)]\,dr = f(x) + \int_0^t \left\{Lf(x) + \int_0^r \mathbb{E}\left[(L^2f)(X_s)\right]ds\right\}\,dr = f(x) + tLf(x) + \frac{t^2}{2}L^2f(x) + o(t^2);
  \]
Smooth moment functions (Dynkin’s formula)

- **Infinitesimal Generator:** For $f \in C^2_b(\mathbb{R})$, let
  \[
  Lf(x) = b(x)f'(x) + \frac{\sigma^2(x)}{2}f''(x) + \int_{\mathbb{R}} \{f(x + \gamma(x, z)) - f(x)\} \lambda(x)g(z; x) \, dz;
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  \mathbb{E}[f(X_t)] = f(x) + \int_0^t \mathbb{E}[Lf(X_r)] \, dr = f(x) + tLf(x) + o(t^2).
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Smooth moment functions (Dynkin’s formula)

- **Infinitesimal Generator**: For $f \in C_0^2(\mathbb{R})$, let
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  \]

- **Implication**: If $Lf \in C_b^2$,
  \[
  \mathbb{E}[f(X_t)] = f(x) + \int_0^t \mathbb{E}[Lf(X_r)] \, dr
  = f(x) + \int_0^t \left\{ Lf(x) + \int_0^r \mathbb{E}\left[ (L^2f)(X_s) \right] \, ds \right\} \, dr
  = f(x) + tLf(x) + \frac{t^2}{2} L^2f(x) + o(t^2);
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Is $f \in C_0^2(\mathbb{R})$ necessary? No... if $f$ vanishes around $x$ and is bounded;

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- Consider a simple jump-diffusion $\gamma(x, z) = z$:

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dX_t = b(X_t) \, dt + \sigma(X_t) \, dW_t + J_t \, dN_t \implies X_t = x + \int_0^t b(X_s) \, ds + \int_0^t \sigma(X_s) \, dW_s + \sum_{i=1}^{N_t} J_i,
\]

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\[
\mathbb{E} [f(X_t)] = e^{-\lambda t} \mathbb{E} [f(X_t) | N_t = 0] + e^{-\lambda t} \mathbb{E} [f(X_t) | N_t = 1] \lambda t + e^{-\lambda t} \mathbb{E} [f(X_t) | N_t = 2] \frac{(\lambda t)^2}{2}
\]

- Let $d\bar{X}_t = b(\bar{X}_t) \, dt + \sigma(\bar{X}_t) \, dW_t$, $\bar{X}_0 = x$;

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\mathbb{E} [f(X_t) | N_t = 0] = \mathbb{E} [f(\bar{X}_t)] + o(t^k), \text{ for any } k \geq 0;
\]

\[
\mathbb{E} [f(X_t) | N_t = 1] = \mathbb{E} [f(\bar{X}_t + J_1)] = \mathbb{E} [\mathbb{E} [f(\bar{X}_t + J_1) | \bar{X}_t]] = \mathbb{E} [F(\bar{X}_t)],
\]

\[ F(x) := \mathbb{E} [f(\bar{X}_t + J_1) | \bar{X}_t = x] = \mathbb{E} [f(x + J_1)] = f(x)g(x)dx, \]

which equals $f(x)g(x-x)dx$ and, thus, is bounded and smooth whenever $g$ is smooth enough;

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\mathbb{E} [f(X_t) | N_t = 2] = \mathbb{E} [f(\bar{X}_t + J_1 + J_2)] = \mathbb{E} [f(x + J_1 + J_2)] + O(t)
\]

\[
\mathbb{E} [f(X_t) | N_t = 3] = \mathbb{E} [f(\bar{X}_t + J_1 + J_2 + J_3)] = \mathbb{E} [f(x + J_1 + J_2 + J_3)] + O(t^2)
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Is \( f \in C^2_0(\mathbb{R}) \) necessary? No... if \( f \) vanishes around \( x \) and is bounded;

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- Let \( d\bar{X}_t = b(\bar{X}_t) \, dt + \sigma(\bar{X}_t) \, dW_t, \quad \bar{X}_0 = x; \)

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where \( F(x) := \mathbb{E}[f(\bar{X}_t + J_1) \mid \bar{X}_t = x] = \mathbb{E}[f(x + J_1)] = \int f(x + z)g(z) \, dz \), which equals \( f(t)g(t,x) \) \( dx \) and, thus, is bounded and smooth whenever \( g \) is smooth enough;

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Regularization by smooth jump densities

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     \]
   - Let \( d\tilde{X}_t = b(\tilde{X}_t) \, dt + \sigma(\tilde{X}_t) \, dW_t, \quad \tilde{X}_0 = x \);
   - \( \mathbb{E}[f(X_t) | N_t = 0] = \mathbb{E}[f(\tilde{X}_t)] = o(t^k) \), for any \( k \geq 0 \);
   - \( \mathbb{E}[f(X_t) | N_t = 1] = \mathbb{E}[f(\tilde{X}_t + J_1)] = \mathbb{E}[\mathbb{E}[f(\tilde{X}_t + J_1) | \tilde{X}_t]] = \mathbb{E}[F(\tilde{X}_t)] \), where 
     \[
     F(x) := \mathbb{E}[f(\tilde{X}_t + J_1) | \tilde{X}_t = x] = \mathbb{E}[f(x + J_1)] = \int f(x + z) g(z) \, dz,
     \]
     which equals \( \int f(u) g(u - x) \, du \) and, thus, is bounded and smooth whenever \( g \) is smooth enough;
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1. Consider a simple jump-diffusion \( \gamma(x, z) = z \):
   \[
   dX_t = b(X_t) \, dt + \sigma(X_t) \, dW_t + J_t \, dN_t \quad \Rightarrow \quad X_t = x + \int_0^t b(X_s) \, ds + \int_0^t \sigma(X_s) \, dW_s + \sum_{i=1}^{N_t} J_i,
   \]

2. Conditioning on \( N_t \sim \text{Poisson}(\lambda t) \):
   \[
   \mathbb{E}[f(X_t)] = e^{-\lambda t} \mathbb{E}[f(X_t) | N_t = 0] + e^{-\lambda t} \mathbb{E}[f(X_t) | N_t = 1] \lambda t + e^{-\lambda t} \mathbb{E}[f(X_t) | N_t = 2] \frac{(\lambda t)^2}{2}
   \]

3. Let \( d\bar{X}_t = b(\bar{X}_t) \, dt + \sigma(\bar{X}_t) \, dW_t, \quad \bar{X}_0 = x \);

4. \( \mathbb{E}[f(X_t) | N_t = 0] = \mathbb{E}[f(\bar{X}_t)] = o(t^k) \), for any \( k \geq 0 \);

5. \( \mathbb{E}[f(X_t) | N_t = 1] = \mathbb{E}[f(\bar{X}_t + J_1)] = \mathbb{E}[\mathbb{E}[f(\bar{X}_t + J_1) | \bar{X}_t]] = \mathbb{E}[F(\bar{X}_t)] \), where

   \[
   F(x) := \mathbb{E}[f(\bar{X}_t + J_1) | \bar{X}_t = x] = \mathbb{E}[f(x + J_1)] = \int f(x + z)g(z)dz,
   \]

   which equals \( \int f(u)g(u - x)du \) and, thus, is bounded and smooth whenever \( g \) is smooth enough;

6. In particular, \( \mathbb{E}[f(X_t) | N_t = 1] = F(x) + (LF)(x)t + O(t^2) \)

7. \( \mathbb{E}[f(X_t) | N_t = 2] = \mathbb{E}[f(\bar{X}_t + J_1 + J_2)] = \mathbb{E}[f(x + J_1 + J_2)] + O(t) \)
Regularization by smooth jump densities

1. Is \( f \in C^2_b(\mathbb{R}) \) necessary? No... if \( f \) vanishes around \( x \) and is bounded;

2. Idea:

   - Consider a simple jump-diffusion \( \gamma(x, z) = z \):
     \[
     dX_t = b(X_t) \, dt + \sigma(X_t) \, dW_t + J_t \, dN_t \quad \implies \quad X_t = x + \int_0^t b(X_s) \, ds + \int_0^t \sigma(X_s) \, dW_s + \sum_{i=1}^{N_t} J_i,
     \]

   - Conditioning on \( N_t \sim \text{Poisson}(\lambda t) \):
     \[
     \mathbb{E} [f(X_t)] = e^{-\lambda t} \mathbb{E} [f(X_t) \mid N_t = 0] + e^{-\lambda t} \mathbb{E} [f(X_t) \mid N_t = 1] \lambda t + e^{-\lambda t} \mathbb{E} [f(X_t) \mid N_t = 2] \frac{(\lambda t)^2}{2}
     \]

   - Let \( d\bar{X}_t = b(\bar{X}_t) \, dt + \sigma(\bar{X}_t) \, dW_t \), \( \bar{X}_0 = x \):

   - \( \mathbb{E} [f(X_t) \mid N_t = 0] = \mathbb{E} [f(\bar{X}_t)] = o(t^k) \), for any \( k \geq 0 \);

   - \( \mathbb{E} [f(X_t) \mid N_t = 1] = \mathbb{E} [f(\bar{X}_t + J_1)] = \mathbb{E} \left[ \mathbb{E} \left[ f(\bar{X}_t + J_1) \mid \bar{X}_t \right] \right] = \mathbb{E} [F(\bar{X}_t)], \) where
     \[
     F(x) := \mathbb{E} \left[ f(\bar{X}_t + J_1) \mid \bar{X}_t = x \right] = \mathbb{E} [f(x + J_1)] = \int f(x + z) g(z) \, dz,
     \]
     which equals \( \int f(u) g(u - x) \, du \) and, thus, is bounded and smooth whenever \( g \) is smooth enough;

   - In particular, \( \mathbb{E} [f(X_t) \mid N_t = 1] = F(x) + (LF)(x) t + O(t^2) \)

   - \( \mathbb{E} [f(X_t) \mid N_t = 2] = \mathbb{E} [f(\bar{X}_t + J_1 + J_2)] = \mathbb{E} [f(x + J_1 + J_2)] + O(t) \)
The Result

1 2nd Order Expansion (F-L & Houdré, 2009):

\[
\mathbb{E}[f(X_t)] = \lambda t \mathbb{E}[f(x + J_1)] + \frac{t^2}{2} \left( \lambda^2 \mathbb{E}[f(x + J_1 + J_2)] - 2\lambda^2 \mathbb{E}[f(x + J_1)] + 2\lambda LF(x) \right) + \ldots
\]

2 Consequences:

- The leading order term depends only on the jump component;
- What is the contribution of the drift \( b \) to the moment?
  \[
t^2 \lambda b(x) F'(x) = -t^2 \lambda b(x) \int f(u) g'(u - x) du.
\]
- What is the contribution of \( \sigma \) to the moment?
  \[
t^2 \frac{\lambda \sigma^2(x)}{2} F''(x) = \frac{t^2}{2} \lambda \sigma^2(x) \int f(u) g''(u - x) du.
\]

3 Is the smoothness of \( g \) necessary? Yes... There are counterexamples for \( g \)'s that are not differentiable.

4 Does \( f \) have to vanish in a neighborhood of \( x \)? No... it suffices that \( f \) is \( C^2 \) in a neighborhood of \( x \):

\[
\mathbb{E}[f(X_t) | N_t = 0] = \mathbb{E}[f(\bar{X}_t)] = f(x) + t(\bar{L}f)(x) + \frac{t^2}{2} (\bar{L}^2 f)(x) + \ldots
\]

where \( \bar{L}f(x) = b(x)f'(x) + \frac{\sigma^2(x)}{2} f''(x) \).
The Result

1. **2nd Order Expansion (F-L & Houdré, 2009):**

\[
\mathbb{E}[f(X_t)] = \lambda t \mathbb{E}[f(x + J_1)] + \frac{t^2}{2} (\lambda^2 \mathbb{E}[f(x + J_1 + J_2)] - 2\lambda^2 \mathbb{E}[f(x + J_1)] + 2\lambda LF(x)) + \ldots
\]

2. **Consequences:**

   - The leading order term depends only on the jump component;
   - What is the contribution of the drift \(b\) to the moment?
     \[
t^2 \lambda b(x) F'(x) = -t^2 \lambda b(x) \int f(u)g'(u - x)du.
     \]
   - What is the contribution of \(\sigma\) to the moment?
     \[
     \frac{t^2}{2} \lambda \sigma^2(x) F''(x) = \frac{t^2}{2} \lambda \sigma^2(x) \int f(u)g''(u - x)du.
     \]

3. Is the smoothness of \(g\) necessary? Yes... There are counterexamples for \(g\)'s that are not differentiable.

4. Does \(f\) have to vanish in a neighborhood of \(x\)? No... it suffices that \(f\) is \(C^2\) in a neighborhood of \(x\):

\[
\mathbb{E}[f(X_t) \mid N_t = 0] = \mathbb{E}[f(\bar{X}_t)] = f(x) + t(\bar{L}f)(x) + \frac{t^2}{2}(\bar{L}^2 f)(x) + \ldots
\]

where \(\bar{L}f(x) = b(x)f'(x) + \frac{\sigma^2(x)}{2}f''(x)\).
The Result

1. 2nd Order Expansion (F-L & Houdré, 2009):

   \[ \mathbb{E} [f(X_t)] = \lambda t \mathbb{E} [f(x + J_1)] + \frac{t^2}{2} (\lambda^2 \mathbb{E} [f(x + J_1 + J_2)] - 2\lambda^2 \mathbb{E} [f(x + J_1)] + 2\lambda LF(x)) + \ldots \]

2. Consequences:

   - The leading order term depends only on the jump component;
   - What is the contribution of the drift \( b \) to the moment?
     \[ t^2 \lambda b(x) F'(x) = -t^2 \lambda b(x) \int f(u)g'(u - x)du. \]
   - What is the contribution of \( \sigma \) to the moment?
     \[ \frac{t^2}{2} \lambda \sigma^2(x) F''(x) = \frac{t^2}{2} \lambda \sigma^2(x) \int f(u)g''(u - x)du. \]

3. Is the smoothness of \( g \) necessary? Yes... There are counterexamples for \( g \)'s that are not differentiable.

4. Does \( f \) have to vanish in a neighborhood of \( x \)? No... it suffices that \( f \) is \( C^2 \) in a neighborhood of \( x \):

   \[ \mathbb{E} [f(X_t) | N_t = 0] = \mathbb{E} [f(\bar{X}_t)] = f(x) + t(\bar{L}f)(x) + \frac{t^2}{2} (\bar{L}^2 f)(x) + \ldots \]

   where \( \bar{L}f(x) = b(x)f'(x) + \frac{\sigma^2(x)}{2} f''(x). \)
The Result

1. 2nd Order Expansion (F-L & Houdré, 2009):

\[
\mathbb{E}[f(X_t)] = \lambda t \mathbb{E}[f(x + J_1)] + \frac{t^2}{2} (\lambda^2 \mathbb{E}[f(x + J_1 + J_2)] - 2\lambda^2 \mathbb{E}[f(x + J_1)] + 2\lambda LF(x)) + \ldots
\]

2. Consequences:

- The leading order term depends only on the jump component;
- What is the contribution of the drift \( b \) to the moment?
  
  \[
t^2 \lambda b(x) F'(x) = -t^2 \lambda b(x) \int f(u) g'(u - x) du.
  \]
- What is the contribution of \( \sigma \) to the moment?
  
  \[
  \frac{t^2}{2} \lambda \sigma^2(x) F''(x) = \frac{t^2}{2} \lambda \sigma^2(x) \int f(u) g''(u - x) du.
  \]

3. Is the smoothness of \( g \) necessary? Yes... There are counterexamples for \( g \)'s that are not differentiable.

4. Does \( f \) have to vanish in a neighborhood of \( x \)? No... it suffices that \( f \) is \( C^2 \) in a neighborhood of \( x \):

\[
\mathbb{E}[f(X_t) | N_t = 0] = \mathbb{E}[f(\bar{X}_t)] = f(x) + t(\bar{L}f)(x) + \frac{t^2}{2}(\bar{L}^2 f)(x) + \ldots
\]

where \( \bar{L}f(x) = b(x)f'(x) + \frac{\sigma^2(x)}{2} f''(x) \).
The Result

1 **2nd Order Expansion (F-L & Houdré, 2009):**

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\mathbb{E}[f(X_t)] = \lambda t \mathbb{E}[f(x + J_1)] + \frac{t^2}{2} \left( \lambda^2 \mathbb{E}[f(x + J_1 + J_2)] - 2\lambda^2 \mathbb{E}[f(x + J_1)] + 2\lambda LF(x) \right) + \ldots
\]

2 **Consequences:**

- The leading order term depends only on the jump component;
- What is the contribution of the drift \(b\) to the moment?
  \[
t^2 \lambda b(x) F'(x) = -t^2 \lambda b(x) \int f(u)g'(u-x)du.
\]
- What is the contribution of \(\sigma\) to the moment?
  \[
  \frac{t^2}{2} \lambda \sigma^2(x) F''(x) = \frac{t^2}{2} \lambda \sigma^2(x) \int f(u)g''(u-x)du.
  \]

3 **Is the smoothness of \(g\) necessary?** Yes... There are counterexamples for \(g\)'s that are not differentiable.

4 **Does \(f\) have to vanish in a neighborhood of \(x\)?** No... it suffices that \(f\) is \(C^2\) in a neighborhood of \(x\):

\[
\mathbb{E}[f(X_t) | N_t = 0] = \mathbb{E}[f(\bar{X}_t)] = f(x) + t(\bar{L}f)(x) + \frac{t^2}{2}(\bar{L}^2 f)(x) + \ldots
\]

where \(\bar{L}f(x) = b(x)f'(x) + \frac{\sigma^2(x)}{2}f''(x)\).
The Result

1. 2nd Order Expansion (F-L & Houdré, 2009):

\[
\mathbb{E}[f(X_t)] = \lambda t \mathbb{E}[f(x + J)] + \frac{t^2}{2} \left( \lambda^2 \mathbb{E}[f(x + J_1 + J_2)] - 2\lambda^2 \mathbb{E}[f(x + J_1)] + 2\lambda LF(x) \right) + \ldots
\]

2. Consequences:
   - The leading order term depends only on the jump component;
   - What is the contribution of the drift \( b \) to the moment?
     \[
t^2 \lambda b(x) F'(x) = -t^2 \lambda b(x) \int f(u) g'(u - x) du.
\]
   - What is the contribution of \( \sigma \) to the moment?
     \[
     \frac{t^2}{2} \lambda \sigma^2(x) F''(x) = \frac{t^2}{2} \lambda \sigma^2(x) \int f(u) g''(u - x) du.
     \]

3. Is the smoothness of \( g \) necessary? Yes... There are counterexamples for \( g \)'s that are not differentiable.

4. Does \( f \) have to vanish in a neighborhood of \( x \)? No... it suffices that \( f \) is \( C^2 \) in a neighborhood of \( x \):

\[
\mathbb{E}[f(X_t) | N_t = 0] = \mathbb{E}[f(\bar{X}_t)] = f(x) + t(\bar{L}f)(x) + \frac{t^2}{2}(\bar{L}^2 f)(x) + \ldots
\]

where \( \bar{L}f(x) = b(x)f'(x) + \frac{\sigma^2(x)}{2}f''(x) \).
• Denoting the jump times of $N$ by $\tau_1 < \tau_2 < \ldots$,

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s + \sum_{i=1}^{N_t} \gamma(X_{\tau_i}, J_i).$$

• Conditioning on $N_t$,

$$\mathbb{E}[f(X_t)] = e^{-\lambda t}\mathbb{E}[f(X_t)|N_t = 0] + e^{-\lambda t}\mathbb{E}[f(X_t)|N_t = 1] + e^{-\lambda t}\mathbb{E}[f(X_t)|N_t = 2] \frac{(\lambda t)^2}{2} + \ldots$$

• Let $\bar{X}(y) = \{\bar{X}_t(y)\}_{t \geq 0}$ be defined as $\bar{X}_t(y) = y + \int_0^t b(\bar{X}_s)ds + \int_0^t \sigma(\bar{X}_s)d\bar{W}_s$

• One-Jump Term:

$$\mathbb{E}[f(X_t)|N_t = 1] = \mathbb{E}\left[f\left(x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s + \gamma(X_{\tau_1}, J_1)\right)|N_t = 1\right]$$

$$= \frac{1}{t} \int_0^t \mathbb{E}\left[f\left(x + \int_0^s b(X_s)ds + \int_0^s \sigma(X_s)dW_s + \gamma(X_{\tau_1}, J_1)\right)|N_t = 1\right] ds$$

$$= \frac{1}{t} \int_0^t \mathbb{E}[F_{t-s}(\bar{X}_{s-}(x))] ds,$$

where $F_u(z) = \mathbb{E}[f(\bar{X}_u(z + \gamma(z, J_1)))]$

• As it turns out, when the density of $J_1$ and $\gamma$ are smooth, $\gamma(z, J_1)$ regularized $f(\bar{X}_u(z + \gamma(z, J_1)))$ and

$$F_u(z) = F^{(0)}(z) + uF^{(1)}(z) + u^2 R(z).$$
Denoting the jump times of $N$ by $\tau_1 < \tau_2 < \ldots$,

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s + \sum_{i=1}^{N_t} \gamma(X_{\tau_i}, J_i).$$

Conditioning on $N_t$,

$$\mathbb{E}[f(X_t)] = e^{-\lambda t}\mathbb{E}[f(X_t)|N_t=0] + e^{-\lambda t}\mathbb{E}[f(X_t)|N_t=1] \lambda t + e^{-\lambda t}\mathbb{E}[f(X_t)|N_t=2] \frac{(\lambda t)^2}{2} + \ldots$$

Let $\bar{X}(y) = \{\bar{X}_t(y)\}_{t \geq 0}$ be defined as $\bar{X}_t(y) = y + \int_0^t b(\bar{X}_s)ds + \int_0^t \sigma(\bar{X}_s)d\bar{W}_s$

One-Jump Term:

$$\mathbb{E}[f(X_t)|N_t=1] = \mathbb{E}\left[f\left(x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s + \gamma(X_{\tau_1}, J_1)\right)|N_t=1\right]$$

$$= \frac{1}{t} \int_0^t \mathbb{E}\left[f\left(x + \int_0^s b(X_s)ds + \int_0^s \sigma(X_s)dW_s + \gamma(X_{\tau_1}, J_1)\right)|N_t=1\right] ds$$

$$= \frac{1}{t} \int_0^t \mathbb{E}\left[F_{\bar{X}_s}(\bar{X}_s(x))\right] ds,$$

where $F_u(z) = \mathbb{E}\left[f(\bar{X}_u(z + \gamma(z, J_1))))\right]$

As it turns out, when the density of $J_1$ and $\gamma$ are smooth, $\gamma(z, J_1)$ regularized

$$f(\bar{X}_u(z + \gamma(z, J_1)))$$

and

$$F_u(z) = F^{(0)}(z) + uF^{(1)}(z) + u^2 R(z).$$
General Jump-Diffusion Process (F-L, Luo, and Ouyang, 2014)

- Denoting the jump times of $N$ by $\tau_1 < \tau_2 < \ldots$,
  \[ X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s + \sum_{i=1}^{N_t} \gamma \left( X_{\tau_i}, J_i \right) . \]

- Conditioning on $N_t$,
  \[ \mathbb{E} [ f(X_t) ] = e^{-\lambda t} \mathbb{E} [ f(X_t) | N_t = 0 ] + e^{-\lambda t} \mathbb{E} [ f(X_t) | N_t = 1 ] \lambda t + e^{-\lambda t} \mathbb{E} [ f(X_t) | N_t = 2 ] \frac{(\lambda t)^2}{2} + \ldots \]

- Let $\bar{X}(y) = \{ \bar{X}_t(y) \}_{t \geq 0}$ be defined as $\bar{X}_t(y) = y + \int_0^t b(\bar{X}_s)ds + \int_0^t \sigma(\bar{X}_s)d\bar{W}_s$

- One-Jump Term:
  \[ \mathbb{E} [ f(X_t) | N_t = 1 ] = \mathbb{E} \left[ f \left( x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s + \gamma \left( X_{\tau_1}, J_1 \right) \right) | N_t = 1 \right] \]
  \[ = \frac{1}{t} \int_0^t \mathbb{E} \left[ f \left( x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s + \gamma \left( X_{\tau_1}, J_1 \right) \right) | N_t = 1 \right] du \]
  \[ = \frac{1}{t} \int_0^t \mathbb{E} \left[ F_{t-s} \left( \bar{X}_{t-s}(x) \right) \right] du, \]
  where $F_u(z) = \mathbb{E} [ f \left( \bar{X}_u(z + \gamma(z, J_1)) \right) ]$

- As it turns out, when the density of $J_1$ and $\gamma$ are smooth, $\gamma(z, J_1)$ regularized $f \left( \bar{X}_u(z + \gamma(z, J_1)) \right)$ and
  \[ F_u(z) = F^{(0)}(z) + uF^{(1)}(z) + u^2 R(z). \]
General Jump-Diffusion Process  (F-L, Luo, and Ouyang, 2014)

- Denoting the jump times of $N$ by $\tau_1 < \tau_2 < \ldots$,
  \[
  X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s + \sum_{i=1}^{N_t} \gamma \left( X_{\tau_i}^-, J_i \right).
  \]

- Conditioning on $N_t$,
  \[
  \mathbb{E} [f(X_t)] = e^{-\lambda t} \mathbb{E} [f(X_t) | N_t = 0] + e^{-\lambda t} \mathbb{E} [f(X_t) | N_t = 1] \lambda t + e^{-\lambda t} \mathbb{E} [f(X_t) | N_t = 2] \frac{(\lambda t)^2}{2} + \ldots
  \]

- Let $\bar{X}(y) = \{ \bar{X}_t(y) \}_{t \geq 0}$ be defined as $\bar{X}_t(y) = y + \int_0^t b(\bar{X}_s)ds + \int_0^t \sigma(\bar{X}_s)d\bar{W}_s$

- One-Jump Term:
  \[
  \mathbb{E} [f(X_t) | N_t = 1] = \mathbb{E} \left[ f \left( x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s + \gamma(X_{\tau_1}^-, J_1) \right) | N_t = 1 \right]
  \]
  \[
  = \frac{1}{t} \int_0^t \mathbb{E} \left[ f \left( x + \int_0^u b(X_s)ds + \int_0^u \sigma(X_s)dW_s + \gamma(X_{u}^-, J_1) \right) | N_t = 1 \right] du
  \]
  \[
  = \frac{1}{t} \int_0^t \mathbb{E} \left[ F_{t-u} (\bar{X}_u^- (x)) \right] du,
  \]
  where $F_u(z) = \mathbb{E} [f \left( \bar{X}_u (z + \gamma(z, J_1)) \right)]$

- As it turns out, when the density of $J_1$ and $\gamma$ are smooth, $\gamma(z, J_1)$ regularized $f \left( \bar{X}_u (z + \gamma(z, J_1)) \right)$ and
  \[
  F_u(z) = F^{(0)}(z) + uF^{(1)}(z) + u^2 R(z).
  \]
General Jump-Diffusion Process (F-L, Luo, and Ouyang, 2014)

- Denoting the jump times of $N$ by $\tau_1 < \tau_2 < \ldots$, 
  $$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s + \sum_{i=1}^{N_t} \gamma \left( X_{\tau_i^-}, J_i \right).$$

- Conditioning on $N_t$,
  $$\mathbb{E} [f(X_t)] = e^{-\lambda t} \mathbb{E} [f(X_t) \mid N_t = 0] + e^{-\lambda t} \mathbb{E} [f(X_t) \mid N_t = 1] \lambda t + e^{-\lambda t} \mathbb{E} [f(X_t) \mid N_t = 2] \frac{(\lambda t)^2}{2} + \ldots$$

- Let $\tilde{X}(y) = \{\tilde{X}_t(y)\}_{t \geq 0}$ be defined as $\tilde{X}_t(y) = y + \int_0^t b(\tilde{X}_s)ds + \int_0^t \sigma(\tilde{X}_s)d\tilde{W}_s$

- One-Jump Term:
  $$\mathbb{E} [f(X_t) \mid N_t = 1] = \mathbb{E} \left[ f \left( x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s + \gamma(X_{\tau_1^-}, J_1) \right) \mid N_t = 1 \right]$$
  $$= \frac{1}{t} \int_0^t \mathbb{E} \left[ f \left( x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s + \gamma(X_{u^-}, J_1) \right) \mid N_t = 1 \right] du$$
  $$= \frac{1}{t} \int_0^t \mathbb{E} \left[ F_{t-s} (\tilde{X}_{s^-}(x)) \right] du,$$
  where $F_{u}(z) = \mathbb{E} [f(\tilde{X}_{u}(z + \gamma(z, J_1)))].$

- As it turns out, when the density of $J_1$ and $\gamma$ are smooth, $\gamma(z, J_1)$ regularized $f(\tilde{X}_{u}(z + \gamma(z, J_1)))$ and
  $$F_u(z) = F^{(0)}(z) + uF^{(1)}(z) + u^2 R(z).$$
State-Dependent Jump-Diffusion Process (F-L and Luo, 2015)

- **Idea:** Reduce the problem to that for a suitably defined "Jump-Diffusion Process".
- Infinitesimal Generator of a State-Dependent Jump-Diffusion:
  \[
  Lf(x) = b(x)f'(x) + \frac{\sigma^2(x)}{2}f''(x) + \int_{\mathbb{R}} \{f(x + \gamma(x, z)) - f(x)\} \, dz;
  \]
- Infinitesimal Generator of a Jump-Diffusion:
  \[
  \tilde{L}f(x) = b(x)f'(x) + \frac{\sigma^2(x)}{2}f''(x) + \int_{\mathbb{R}} \{f(x + \tilde{\gamma}(x, z)) - f(x)\} \, dz;
  \]
- Assumption: For each \(x, z \to \nu(x, z)\) is positive and continuous s.t. \(\int_{-\infty}^{0} \nu(x, z) \, dz\) and \(\int_{0}^{\infty} \nu(x, z) \, dz\) remain constant;
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  \tilde{\nu}(w) = \begin{cases} 
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State-Dependent Jump-Diffusion Process (F-L and Luo, 2015)

- **Idea:** Reduce the problem to that for a suitably defined "Jump-Diffusion Process".

- **Infinitesimal Generator of a State-Dependent Jump-Diffusion:**
  
  $Lf(x) = b(x)f'(x) + \frac{\sigma^2(x)}{2} f''(x) + \int_{\mathbb{R}} \{f(x + \gamma(x, z)) - f(x)\} \lambda(x)g(z; x)dz$;

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  Let
  
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Future and Ongoing Work

1. Extensions to path-dependent functionals:
   e.g., asymptotic behavior of \( \mathbb{E} [f(X_s, X_t)] \) or \( \mathbb{E} [f(\sup_{u \leq t} X_u)] \) as \( s, t \to 0 \);

2. Extensions to \( \mathbb{R}^k \)-dimensional processes \( X \);

3. Extensions to time-dependent moment functions:
   \( \mathbb{E} [f_t(X_t)] \), where \( f_t(x) \to f_0(x) \) as \( t \to 0 \).

4. Probabilistic characterizations of the expansion coefficients amicable to Monte Carlo valuations of the expansions.
For Further Reading I

Figueroa-López, J.E. & Houdré, C.
Small-time expansions for the transition distributions of Lévy processes.

Figueroa-López, J.E., Luo, Y., & Ouyang, C.
Small-time expansions for local jump-diffusion models with infinite jump activity.
*Bernoulli* 20(3) 1165-1209, 2014.

Figueroa-López, J.E., & Luo, Y.
Small-time expansions for state-dependent local jump-diffusion models with infinite jump activity.
*Arxiv* 2015.