Short-time asymptotics for Lévy models with a view towards financial mathematics

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Outline

1. Lévy processes
   Some Financial models driven by Lévy processes
2. The general problems
3. High-order asymptotic behavior for OTM Option Prices
   Jump-Diffusion Lévy Model
   General Lévy Model
4. High-order asymptotic behavior for ATM Option Prices
   Tempered Stable-Like Processes
   Main Results
5. ATM implied volatility slope
6. Conclusions
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Lévy processes \( \{X_t\}_{t \geq 0} \)

1. **Key features:**
   - The process starts at 0 and its increments, \( X_\delta, X_{2\delta} - X_\delta, \ldots, X_{n\delta} - X_{(n-1)\delta} \), are independent identically distributed;
   - Paths \( t \mapsto X_t \) are right-continuous with left-limits everywhere: time-\( t \) jump \( \Delta X_t := X_t - \lim_{s \nearrow t} X_s = X_t - X_{t-} \) is well-defined;
   - The distribution of \( \{X_t\}_{t \geq 0} \) is uniquely determined by the distribution of \( X_1 \).

2. **Fundamental examples:**
   - Wiener process \( \{W_t\}_{t \geq 0} \):
     \[ W_1 \sim \mathcal{N}(0, 1); \quad W_{i\delta} - W_{(i-1)\delta} \sim \mathcal{N}(0, \delta); \quad \text{Continuous paths}; \]
   - Poisson process \( \{N_t\}_{t \geq 0} \):
     \[ N_1 \sim \text{Pois}(\lambda); \quad N_{i\delta} - N_{(i-1)\delta} \sim \text{Pois}(\lambda \delta); \]
     Piece-wise constant paths with jumps of size \( 1 \);
   - Compound Poisson:
     \[ X_{t\alpha}^{cp} = \sum_{i=1}^{N_t} \xi_i, \text{ where } \xi_i \sim p(\cdot) \text{ and } (\xi_i)_{i \perp} (N_t); \]
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Parameters and construction of Lévy processes

1. Parameters: \((b_1, \sigma^2, s)\) (called Lévy triplet)
   - \(b_1 \in \mathbb{R}\) \underline{drift}
   - \(\sigma \in \mathbb{R}_+\) \underline{Volatility}
   - \(s : \mathbb{R} \setminus \{0\} \to \mathbb{R}_+\) \underline{Lévy density}

2. Lévy-Itô decomposition: For each \(\varepsilon \in (0, \infty)\),

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X_t = b_\varepsilon t + \sigma W_t + X_t^{(\varepsilon, \infty)} + \lim_{\delta \downarrow 0} \left( X_t^{(\delta, \varepsilon)} - \mathbb{E}X_t^{(\delta, \varepsilon)} \right)
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where \(X_t^{(\delta, \varepsilon)} := \sum_{s \leq t} \Delta X_s 1_{\delta \leq |\Delta X_s| < \varepsilon}, \quad 0 < \delta < \varepsilon \leq \infty\).

3. \(X^{(\delta, \varepsilon)}\) is Compound Poisson with intensity of jumps \(\lambda^{(\delta, \varepsilon)} := \int_\delta^\varepsilon s(x)dx\)
   and jump density \(1_{\delta \leq x < \varepsilon} s(x)/\lambda^{(\delta, \varepsilon)}\).

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Exponential Lévy model

1. The model: The time-$t$ price of a financial asset is given by
\[ S_t = S_0 e^{X_t}, \text{ for a Lévy process } \{X_t\}_{t \geq 0}. \]

2. Underlying postulate:
Log returns $R_i^\delta = \log \left\{ \frac{S_i^\delta}{S_{(i-1)\delta}} \right\} = X_i^\delta - X_{(i-1)^\delta}$ on disjoint periods of equal length $\delta$ (say, hourly, daily, etc.) are independent with common distribution;

3. Advantages:
Log return distributions can exhibit heavy-tails, high kurtosis, asymmetry.

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Time-Changed Lévy models (Carr et al.)

1. The model: \( S_t = S_0 e^{X_t} = S_0 e^{Z_{\tau(t)}} \), where
   - \( Z \) is a Lévy process; (Controls the jump activity and volatility)
   - \( \tau(t) = \int_0^t r_u du \), with speed \( r \geq 0 \); (Random clock of business activity)
     \[ r(t) \text{ value} \implies \text{clock} \implies \text{volatility and jump intensity} \]
   - Prototypical example: CIR process \( dr_t = \alpha(m - r_t)dt + \nu \sqrt{r_t} dW_t \).

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Small-time behavior of generalized Lévy moments

Goal: Given a function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi(0) = 0$, determine the asymptotic behavior of the $\varphi$-moment $\mathbb{E}\{\varphi(X_t)\}$ when $t \to 0$;

Regularity Conditions

1. $\varphi$ is almost everywhere continuous.
2. As $x \to 0$, $\varphi(x) = o(x^2)$;
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(e.g., $g(x) = |x|^p$, $p > 0$, or $g(x) = e^{cx}$, $c > 0$).

Theorem (Jacod 06, F-L 08)

$$\lim_{t \to 0} \frac{1}{t} \mathbb{E}\{\varphi(X_t)\} = \int \varphi(x)s(x)dx$$
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   (e.g., $g(x) = |x|^p$, $p > 0$, or $g(x) = e^{cx}$, $c > 0$).

**Theorem (Jacod 06, F-L 08)**

$$\lim_{t \rightarrow 0} \frac{1}{t} \mathbb{E} \{ \varphi(X_t) \} = \int \varphi(x)s(x)dx + \sigma^2$$
Small-time behavior of generalized Lévy moments

Goal: Given a function \( \varphi : \mathbb{R} \to \mathbb{R} \) such that \( \varphi(0) = 0 \), determine the asymptotic behavior of the \( \varphi \)-moment \( \mathbb{E} \{ \varphi(X_t) \} \) when \( t \to 0 \);

Regularity Conditions

1. \( \varphi \) is almost everywhere continuous.
2. As \( x \to 0 \), \( \varphi(x) = o(x^2) \); or \( \varphi(x) \sim x^2 \); or \( \varphi(x) \sim |x|, \sigma = 0 \), and \( X \) is of bounded variation
3. \( \varphi(x) = O(g(x)) \), as \( |x| \to \infty \), where \( g \) is submultiplicative or subadditive such that \( \int_{|x|>1} g(x)s(x)dx < \infty \)

(e.g., \( g(x) = |x|^p, p > 0 \), or \( g(x) = e^{cx}, c > 0 \)).

Theorem (Jacod 06, F-L 08)

\[
\lim_{t \to 0} \frac{1}{t} \mathbb{E} \{ \varphi(X_t) \} = \int \varphi(x)s(x)dx + \sigma^2 + |\bar{b}|, \text{ where } \bar{b} = \text{drift of } X.
\]
Applications to Mathematical Finance

1. **Set-up:** The asset price follows an exponential Lévy model: $S_t = S_0 e^{X_t}$;

2. **Out-of-The-Money (OTM) Digital Options:** For $\kappa > 0$,
   \[
   \mathbb{P}(S_t \geq S_0 e^{\kappa}) = \mathbb{P}(X_t \geq \kappa) \sim t \int_{\kappa}^{\infty} s(x) \, dx.
   \]

3. **Out-of-The-Money (OTM) Call Options:** For $\kappa > 0$,
   \[
   \frac{1}{S_0} \mathbb{E}(S_t - S_0 e^{\kappa})_+ = \mathbb{E}(e^{X_t} - e^{\kappa})_+ \sim t \int (e^x - e^{\kappa})_+ s(x) \, dx.
   \]
General Problems

1. Higher order approximations:

\[
\frac{1}{t} \mathbb{P}(X_t \geq \kappa) - \int_\kappa^\infty s(x) dx \sim d t^\nu, \quad \text{for some } \nu > 0, d \neq 0?
\]

2. At-the-money or close-to-the-money asymptotic behavior

Asymptotic behavior of \( \mathbb{E} (e^{X_t} - e^\kappa)_+ \) when \( \kappa = 0 \) or \( \kappa = \kappa_t \to 0 \)?

3. Incorporation of a stochastic volatility component

Asymptotics for \( \mathbb{E} (e^{X_t + V_t} - e^\kappa)_+ \) when \( V_t \) is a stoch. volatility factor?

4. Translation into implied volatility asymptotics
Why short-time/high-order approximation schemes?

1. Option traders could experience large losses near expiry due to the marked discrepancy of the anticipated behavior of option prices under a purely continuous model, a pure-jump model, or a mixed model.

2. Lack of tractable formulas for the marginal densities, distributions, or moments of Lévy models.

3. Existing numerical methods could be unstable and computational expensive for short-expiry options.

4. First-order approximation are in general inaccurate for practical data sets and parameter values.

5. Allows for a qualitative model assessment or model testing based on the behavior of market option prices in short-time.

6. Helpful for calibration or fitting purposes based on observed option prices.
Outline

1 Lévy processes
   Some Financial models driven by Lévy processes

2 The general problems

3 High-order asymptotic behavior for OTM Option Prices
   Jump-Diffusion Lévy Model
   General Lévy Model

4 High-order asymptotic behavior for ATM Option Prices
   Tempered Stable-Like Processes
   Main Results

5 ATM implied volatility slope

6 Conclusions
Higher-order expansions:

Compound Poisson Process $X_{t}^{cp} = \sum_{i=1}^{N_{t}} \xi_{i}$

Conditioning on the number of jumps $N_{t}$:

$$
E \phi (X_{t}^{cp}) = \sum_{k=0}^{\infty} P (N_{t} = k) E \phi \left( \sum_{i=1}^{k} \xi_{i} \right)
$$

$$
= \sum_{k=1}^{n} e^{-\lambda t} \frac{(\lambda t)^{k}}{k!} E \phi \left( \sum_{i=1}^{k} \xi_{i} \right) + O(t^{n+1})
$$

$$
= \sum_{k=1}^{n} d_{k}^{cp} t^{k} + O(t^{n+1}),
$$

where, in terms of the Lévy density $s$ of the process,

$$
d_{1}^{cp} = \lambda E \phi (\xi_{1}) = \int \varphi(x)s(x)dx
$$

$$
d_{2}^{cp} = \frac{\lambda^{2}}{2} E \phi (\xi_{1} + \xi_{2}) - \lambda^{2} E \phi (\xi_{1}), \text{ etc.}
$$
Higher-order expansions:

Jump-Diffusion Lévy Model $X_t = bt + \sigma W_t + \sum_{i=1}^{N_t} \xi_i$

1. Conditioning on the number of jumps $N_t$:

\[
\mathbb{E}\varphi(X_t) = \sum_{k=0}^{n} e^{-\lambda t} \frac{e^{\lambda t}^k}{k!} \mathbb{E}\varphi\left(bt + \sigma W_t + \sum_{i=1}^{k} \xi_i\right) + O(t^{n+1})
\]

2. Estimating the contribution of the no-jump term:

\[
\mathbb{E}\varphi(bt + \sigma W_t) \leq \mathbb{E}\left(e^{bt+\sigma W_t} \mathbf{1}_{\{bt+\sigma W_t \geq \kappa\}}\right) = O(t^{n+1}),
\]

if, e.g., $\varphi(x) \leq e^x \mathbf{1}_{x \geq \kappa}$, as in the Call Option payoff.

3. Estimating the effect of the drift $b$ and volatility $\sigma$:

\[
\mathbb{E}\varphi\left(bt + \sigma W_t + \sum_{i=1}^{k} \xi_i\right) - \mathbb{E}\varphi\left(\sum_{i=1}^{k} \xi_i\right) = ?
\]
I. Dynkin’s Formula

1. Let $X$ be a Lévy process with triplet $(b_1, \sigma^2, s)$. 

2. For a function $f : \mathbb{R} \to \mathbb{R}$ in $C_b^2$,

$$
\mathbb{E}f(X_t) = f(0) + t \int_0^1 \mathbb{E}(Lf)(X_{\alpha t}) d\alpha,
$$

where $L$ is the so-called infinitesimal generator of $X$ defined by

$$(Lf)(x) := \frac{\sigma^2}{2}f''(x) + b_1 f'(x) + \int (f(x + z) - f(x) - zf'(x)1_{|z| \leq 1}) s(z) dz.$$

3. Furthermore, for $f \in C_b^{2n+2}$,

$$
\mathbb{E}f(X_t) = f(0) + \sum_{k=1}^{n} \frac{t^k}{k!} (L^k f)(0) + \frac{t^{n+1}}{n!} \int_0^1 (1 - \alpha)^n \mathbb{E}(L^{n+1} f)(X_{\alpha t}) d\alpha.
$$
II. Regularization

1 Goal: Determine the asymptotic behavior in small $t$ of the quantity

$$
\mathbb{E}\phi \left( bt + \sigma W_t + \sum_{i=1}^{k} \xi_i \right) - \mathbb{E}\phi \left( \sum_{i=1}^{k} \xi_i \right),
$$

2 Note that

$$
\mathbb{E}\phi \left( bt + \sigma W_t + \sum_{i=1}^{k} \xi_i \right) - \mathbb{E}\phi \left( \sum_{i=1}^{k} \xi_i \right) = \mathbb{E}f_k \left( bt + \sigma W_t \right) - f_k(0),
$$

where, in terms of the density of $\xi_i$’s, say $p(\cdot)$,

$$
f_k(x) := \mathbb{E}\phi \left( x + \sum_{i=1}^{k} \xi_i \right) = \int \phi(x + z)p^*k(z)dz
$$

3 Key observation: If the density $p$ is smooth, then $f_k$ is smooth (even if $\phi$ is not) and, furthermore,

$$
f_k^{(j)}(x) = (-1)^j \int \phi(u) \frac{d^j p^*k}{dz^j}(u - x)du.
$$
II. Regularization

1. **Goal:** Determine the asymptotic behavior in small $t$ of the quantity

$$
\mathbb{E}\varphi \left( bt + \sigma W_t + \sum_{i=1}^{k} \xi_i \right) - \mathbb{E}\varphi \left( \sum_{i=1}^{k} \xi_i \right),
$$

2. Note that

$$
\mathbb{E}\varphi \left( bt + \sigma W_t + \sum_{i=1}^{k} \xi_i \right) - \mathbb{E}\varphi \left( \sum_{i=1}^{k} \xi_i \right) = \mathbb{E}f_k (bt + \sigma W_t) - f_k(0),
$$

where, in terms of the density of $\xi_i$’s, say $p(\cdot)$,

$$
f_k(x) := \mathbb{E}\varphi \left( x + \sum_{i=1}^{k} \xi_i \right) = \int \varphi(x + z)p^*k(z)dz
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f_k^{(j)}(x) = (-1)^j \int \varphi(u)\frac{d^j p^*k}{dz^j}(u - x)du.
$$
II. Regularization

1. **Goal:** Determine the asymptotic behavior in small $t$ of the quantity

$$\mathbb{E} \phi \left( bt + \sigma W_t + \sum_{i=1}^{k} \xi_i \right) - \mathbb{E} \phi \left( \sum_{i=1}^{k} \xi_i \right),$$

2. Note that

$$\mathbb{E} \phi \left( bt + \sigma W_t + \sum_{i=1}^{k} \xi_i \right) - \mathbb{E} \phi \left( \sum_{i=1}^{k} \xi_i \right) = \mathbb{E} f_k (bt + \sigma W_t) - f_k (0),$$

where, in terms of the density of $\xi_i$’s, say $p(\cdot)$,

$$f_k(x) := \mathbb{E} \phi \left( x + \sum_{i=1}^{k} \xi_i \right) = \int \phi(x + z) p^k(z) dz = \int \phi(u) p^k(u - x) du.$$

3. **Key observation:** If the density $p$ is smooth, then $f_k$ is smooth (even if $\phi$ is not) and, furthermore,

$$f_k^{(j)}(x) = (-1)^j \int \phi(u) \frac{d^j p^k}{dz^j}(u - x) du.$$
Second-order asymptotic expansion for general $\varphi$

1. $\mathbb{E}\varphi(bt + \sigma W_t + \xi_1) - \mathbb{E}\varphi(\xi_1) = \mathbb{E}f_1(bt + \sigma W_t) - f_1(0)$ admits the expansion

$$t(Lf_1)(0) + O(t^2) = t \left( \frac{\sigma^2}{2} f_1''(0) + bf_1'(0) \right) + O(t^2)$$

$$= t \left( \frac{\sigma^2}{2} \int \varphi(u)p''(u)du - b \int \varphi(u)p'(u)du \right) + O(t^2).$$

2. $\mathbb{E}\varphi(bt + \sigma W_t + \xi_1 + \xi_2) - \mathbb{E}\varphi(\xi_1 + \xi_2) = O(t)$;

3. Putting together everything:

$$\mathbb{E}\varphi(X_t) - \mathbb{E}\varphi(X_t^{qp}) = \lambda t^2 \left( \frac{\sigma^2}{2} \int \varphi(u)p''(u)du - b \int \varphi(u)p'(u)du \right)$$

$$+ O(t^3).$$
Second-order asymptotic expansion for general $\varphi$

1. $E\varphi (bt + \sigma W_t + \xi_1) - E\varphi (\xi_1) = E f_1 (bt + \sigma W_t) - f_1 (0)$ admits the expansion

   $$t(Lf_1)(0) + O(t^2) = t \left( \frac{\sigma^2}{2} f_1''(0) + b f_1'(0) \right) + O(t^2)$$

   $$= t \left( \frac{\sigma^2}{2} \int \varphi(u)p''(u)du - b \int \varphi(u)p'(u)du \right) + O(t^2).$$

2. $E\varphi (bt + \sigma W_t + \xi_1 + \xi_2) - E\varphi (\xi_1 + \xi_2) = O(t)$;

3. Putting together everything:

   $$E\varphi (X_t) - E\varphi (X_t^{cp}) = t^2 \left( \frac{\sigma^2}{2} \int \varphi(u)s''(u)du - b \int \varphi(u)s'(u)du \right)$$

   $$+ O(t^3).$$

   in terms of the Lévy density $s(x) := \lambda p(x)$ of the process.
Some consequences

OTM Digital Options [F-L & Houdré, SPA 2009]

\[ P \left( bt + \sigma W_t + X_t^{cp} \geq \kappa \right) - P \left( X_t^{cp} \geq \kappa \right) = t^2 \left( -\frac{\sigma^2}{2} s'(\kappa) + bs(\kappa) \right) + O(t^2) \]

- If \( b > 0 \) (\( b < 0 \)) the probability increases (decreases) by \( t^2 |b| s(\kappa) \)
- If \( \sigma \neq 0 \), the probability increases by \( t^2 \frac{\sigma^2}{2} |s'(\kappa)| \) (\( s'(\kappa) < 0, \kappa > 0 \)).

OTM Call Options [F-L & Forde, SIFIN 2012]

\[ E \left( e^{bt+\sigma W_t} + X_t^{cp} - e^\kappa \right) + E \left( e^{X_t^{cp}} - e^\kappa \right) = t^2 \left( \frac{\sigma^2}{2} \int_{\kappa}^{\infty} (e^u - e^\kappa) s''(u)du - b \int_{\kappa}^{\infty} (e^u - e^\kappa) s'(u)du \right) + O(t^2) \]

- By the martingale condition \( b = -\sigma^2/2 - \int (e^x - 1 - x1_{|x|\leq1}) s(x)dx \)

the price would increase by \( t^2 \frac{\sigma^2}{2} e^\kappa s(\kappa) \), if \( \sigma \neq 0 \).
Some consequences

OTM Digital Options [F-L & Houdré, SPA 2009]

\[ \mathbb{P}(bt + \sigma W_t + X_{t}^{cp} \geq \kappa) - \mathbb{P}(X_{t}^{cp} \geq \kappa) = t^2 \left( -\frac{\sigma^2}{2} s'(\kappa) + bs(\kappa) \right) + O(t^2) \]

- If \( b > 0 \) (\( b < 0 \)) the probability increases (decreases) by \( t^2 |b| s(\kappa) \)

- If \( \sigma \neq 0 \), the probability increases by \( t^2 \frac{\sigma^2}{2} |s'(\kappa)| \) (\( s'(\kappa) < 0, \kappa > 0 \)).

OTM Call Options [F-L & Forde, SIFIN 2012]

\[ \mathbb{E}(e^{bt + \sigma W_t + X_{t}^{cp}} - e^{\kappa}) + \mathbb{E}(e^{X_{t}^{cp}} - e^{\kappa}) = t^2 \left( \frac{\sigma^2}{2} \int_{\kappa}^{\infty} (e^u - e^{\kappa}) s''(u)du - b \int_{\kappa}^{\infty} (e^u - e^{\kappa}) s'(u)du \right) + O(t^2) \]

- By the martingale condition \( b = -\frac{\sigma^2}{2} - \int (e^x - 1 - x1_{|x|\leq1}) s(x)dx \)

the price would increase by \( t^2 \frac{\sigma^2}{2} e^{\kappa} s(\kappa) \), if \( \sigma \neq 0 \).
General Lévy Model

Small-time asymptotic behavior of tail distributions

1 Key Assumption: For all $\varepsilon > 0$ and $k \geq 0$,

$$\sup_{|x| \geq \varepsilon} |s^{(k)}(x)| < \infty;$$

2 Theorem: [F-L & Houdré, SPA 2009]

For any $x_0 > 0$ & $n \geq 0$, there exists a $t_0 > 0$ s.t. $\forall 0 < t < t_0$ & $\forall x \geq x_0$,

$$\mathbb{P}(X_t \geq x) = d_1(x)t + d_2(x)t^2 + \cdots + d_n(x)t^n + O_{x_0,n}(t^{n+1}).$$

3 The coefficients:

- $d_1(x) = \int_x^\infty s(u)du$;
- $d_2(x) = -\int_{|v| \geq \varepsilon} s(v)dv \int_x^\infty s(u)du + \frac{1}{2} \int_{|u| \geq \varepsilon} \int_{|v| \geq \varepsilon} 1_{u+v \geq x} s(u)s(v)du dv$;
- $d_n(x) = \int_{|u| \leq \varepsilon} \int_{|v| \leq \varepsilon} \int_{|w| \leq \varepsilon} 1_{w+u+v \geq x} s(u)s(v)s(w)du dv dw$. 
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- $d_2(x) = -\int_{|v| \geq \varepsilon} s(v)dv \int_x^\infty s(u)du + \frac{1}{2} \int_{|u| \geq \varepsilon} \int_{|v| \geq \varepsilon} 1_{(u+v \geq x)} s(u)s(v)dudv$;
- $d_3(x) = \int_{|u| \leq \varepsilon} \int_{|v| \leq \varepsilon} s(x - u)(1 - \beta u)dudv s(\varepsilon)v^2$. 

High-order asymptotic behavior for OTM Option Prices
General Lévy Model

Small-time asymptotic behavior of tail distributions

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For any $x_0 > 0$ & $n \geq 0$, there exists a $t_0 > 0$ s.t. $\forall 0 < t < t_0$ & $\forall x \geq x_0$,

$$\mathbb{P}(X_t \geq x) = d_1(x)t + d_2(x)t^2 + \cdots + d_n(x)t^n + O_{x_0,n}(t^{n+1}).$$

3. The coefficients:

- $d_1(x) = \int_x^\infty s(u)du$;
- $d_2(x) = -\int_{|v| \geq \varepsilon} s(v)dv \int_x^\infty s(u)du + \frac{1}{2} \int_{|u| \geq \varepsilon} \int_{|v| \geq \varepsilon} 1(u+v\geq x)s(u)s(v)dudv$;
General Lévy Model

Small-time asymptotic behavior of tail distributions

1. **Key Assumption:** For all $\varepsilon > 0$ and $k \geq 0$,

   $\sup_{|x| \geq \varepsilon} |s^{(k)}(x)| < \infty$;

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   For any $x_0 > 0$ & $n \geq 0$, there exists a $t_0 > 0$ s.t. $\forall 0 < t < t_0$ & $\forall x \geq x_0$,

   $\mathbb{P}(X_t \geq x) = d_1(x)t + d_2(x)t^2 + \cdots + d_n(x)t^n + O_{x_0,n}(t^{n+1})$.

3. **The coefficients:**
   - $d_1(x) = \int_x^{\infty} s(u)du$;
   - $d_2(x) = -\int_{|v| \geq \varepsilon} s(v)dv \int_x^{\infty} s(u)du + \frac{1}{2} \int_{|u| \geq \varepsilon} \int_{|v| \geq \varepsilon} 1_{\{u+v \geq x\}} s(u)s(v)du dv$
     $- \frac{\sigma^2}{2} s'(x) + b \cdot s(x) - \int_{|u| \leq \varepsilon} \int_0^{1} s'(x - \beta u)(1 - \beta)d\beta u^2 s(u)du$. 


General Lévy Model

Small-time asymptotic behavior of tail distributions

1. **Key Assumption:** For all $\epsilon > 0$ and $k \geq 0$,

$$
\sup_{|x| \geq \epsilon} |s^{(k)}(x)| < \infty;
$$

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For any $x_0 > 0$ & $n \geq 0$, there exists a $t_0 > 0$ s.t. $\forall \ 0 < t < t_0$ & $\forall \ x \geq x_0$,

$$
P(X_t \geq x) = d_1(x)t + d_2(x)t^2 + \cdots + d_n(x)t^n + O_{x_0,n}(t^{n+1}).
$$

3. **The coefficients:**

- $d_1(x) = \int_x^\infty s(u)du$;
- $d_2(x) = -\int_{|v| \geq \epsilon} s(v)dv \int_x^\infty s(u)du + \frac{1}{2} \int_{|u| \geq \epsilon} \int_{|v| \geq \epsilon} 1_{\{u+v \geq x\}} s(u)s(v)dudv$
  
  \[= \frac{\sigma^2}{2} s'(x) + b \epsilon s(x) - \int_{|u| \leq \epsilon} \int_0^1 s'(x - \beta u)(1 - \beta) d\beta u^2 s(u)du.\]
General Lévy Model

Small-time asymptotic behavior of tail distributions

1 Key Assumption: For all $\varepsilon > 0$ and $k \geq 0$,

$$\sup_{|x| \geq \varepsilon} |s^{(k)}(x)| < \infty;$$

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For any $x_0 > 0$ & $n \geq 0$, there exists a $t_0 > 0$ s.t. $\forall 0 < t < t_0$ & $\forall x \geq x_0$,

$$\mathbb{P}(X_t \geq x) = d_1(x)t + d_2(x)t^2 + \cdots + d_n(x)t^n + O_{x_0,n}(t^{n+1}).$$

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High-order asymptotic behavior for ATM Option Prices

Outline

1. Lévy processes
   - Some Financial models driven by Lévy processes

2. The general problems

3. High-order asymptotic behavior for OTM Option Prices
   - Jump-Diffusion Lévy Model
   - General Lévy Model

4. High-order asymptotic behavior for ATM Option Prices
   - Tempered Stable-Like Processes
   - Main Results

5. ATM implied volatility slope

6. Conclusions
The problem

1 Consider a price process of the form $S_t = S_0 e^{X_t}$ such that $S_t$ is a Martingale; in particular,

$$\mathbb{E}(S_t) = \mathbb{E}(S_0 e^{X_t}) = S_0;$$

2 Then, by the Dominated Convergence,

$$\Pi_t = \mathbb{E}\left[ (e^{X_t} - 1)^+ \right] \rightarrow 0, \quad (t \rightarrow 0)$$

3 Aims:
   - Characterize the asymptotic behavior of $\Pi_t$ as $t \rightarrow 0$;
   - Determine the effect of the different model’s parameters in the short-time asymptotic behavior of $\Pi_t$. 
Outlook

1. The problem can be rephrased in terms of absolute values:

$$E |e^{X_t} - 1| = E [2 (e^{X_t} - 1)^+ - e^{X_t} + 1] = 2E (e^{X_t} - 1)^+$$

2. Since $\varphi(x) := |e^x - 1| \sim |x|$, for a bounded variation Lévy process $X$,

$$E |e^{X_t} - 1| \sim t \left( \int |e^x - 1| s(x) dx + |\bar{b}| \right), \text{ where } \bar{b} = \text{drift of } X.$$

3. In the case that $E |X_t| < \infty$, one would expect that

$$E (e^{X_t} - 1)^+ = E (e^{X_t^+} - 1) \sim E X_t^+, \quad t \to 0.$$

4. This suggests to consider a Lévy process with a local self-similar behavior:

$$t^{-1/Y} X_t \overset{D}{\to} Z,$$

for some $Y > 0$ and $Z \neq 0$. 
Tempered stable-like Lévy process

1. A pure-jump Lévy process \{X_t\}_{t \geq 0} whose Lévy density \( s \) is of the form:

\[
s(x) = C \left( \frac{x}{|x|} \right) \bar{q}(x)|x|^{-Y-1}, \quad \text{with}
\]

\[
C(1), C(-1) \in (0, \infty), \quad Y \in (0, 2), \quad \bar{q}(x) \xrightarrow{x \to 0} 1; \quad \sup_{x \neq 0} \bar{q}(x) < \infty.
\]

2. \( Y \) is referred to as the index of jump activity.

3. For \( Y \in (0, 1) \), the process is of bounded variation.

4. The case \( Y \in (1, 2) \) is more relevant for financial applications based on empirical evidence.
Connection to other processes

1. If \( \bar{q}(x) \equiv 1 \), the resulting Lévy process is a Stable Lévy Process \( \{Z_t\}_{t \geq 0} \).

2. For certain \( c \), the centered process \( \bar{Z}_t := Z_t - ct \) is self-similar:
   \[
   \left\{ h^{-1/Y} \bar{Z}_{ht} \right\}_{t \geq 0} \cong \left\{ \bar{Z}_t \right\}_{t \geq 0} \quad (h > 0).
   \]
   If \( c = 0 \), we say \( Z \) is strictly \( Y \)-stable.

3. The tails of the distribution are often too heavy for applications:
   \[
   \mathbb{E}(|Z_t|^p) = \infty, \text{ for any } p > Y.
   \]

4. A condition for the exponential moment to exist is given by:
   \[
   \mathbb{E}(e^{X_t}) < \infty \iff \int_{x \geq 1} e^x x^{-Y-1} \bar{q}(x) dx < \infty.
   \]

5. When \( \bar{q}(x) = e^{-Mx}1_{x>0} + e^{-G|x|}1_{x<0} \), for some \( M, G > 0 \), the process is called Tempered Stable Process (TSP). If, in addition, \( C(1) = C(-1) \), the process is called CGMY process.
Short and long time behavior TSP

1. In short-time or locally, \( \{X_t\}_{t \geq 0} \) behaves like a stable process:
   - \( 1 < Y < 2 \):
     \[
     \{ h^{-1/Y} X_{ht} \} \xrightarrow{\mathbb{D}} \{ \tilde{Z}_t \}_{t \geq 0}, \quad (h \to 0),
     \]
     for a strictly \( Y \)-stable process \( \{ \tilde{Z}_t \}_{t \geq 0} \);
   - \( 0 < Y < 1 \):
     \[
     \{ h^{-1/Y} (X_{ht} - cht) \} \xrightarrow{\mathbb{D}} \{ \tilde{Z}_t \}_{t \geq 0}, \quad (h \to 0),
     \]
     for a suitable drift \( c \) and strictly \( Y \)-stable process \( \{ \tilde{Z}_t \}_{t \geq 0} \);

2. In long-time, \( \{X_t\}_{t \geq 0} \) behaves like a Brownian Motion:
   \[
   \{ h^{-1/2} X_{ht} \} \xrightarrow{\mathbb{D}} \{ B_t \}_{t \geq 0}, \quad (h \to \infty),
   \]
   where \( \{ B_t \}_{t \geq 0} \) is a suitable Brownian motion.
Relevant literature: Exponential Lévy processes


\[
\mathbb{E} \left( e^{X_t} - 1 \right)^+ = t^{1/Y} \mathbb{E} \left( \bar{Z}_1^+ \right) + o \left( t^{1/Y} \right), \quad (t \to 0),
\]

where \( \{\bar{Z}_t\}_{t \geq 0} \) is a centered \( Y \)-stable process.

2. Tankov (2011): If \( L = \sigma W + X \) with non-zero \( \sigma \),

\[
\mathbb{E} \left( e^{\sigma W_t + X_t} - 1 \right)^+ = t^{1/2} \frac{\sigma}{\sqrt{2\pi}} + o(t^{1/2}), \quad (t \to 0).
\]
Extensions

1. What is the accuracy of the previous asymptotics?

2. Incorporate a nonhomogeneous component

3. Include “close-to-the-money” options
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1. What is the accuracy of the previous asymptotics?

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3. Include "close-to-the-money" options
Extensions: Nonhomogeneous component

1. Consider $S_t = S_0 e^{L_t}$ with

$$dL_t = \underbrace{dX_t}_{\text{Jump Component}} + \underbrace{dV_t}_{\text{Continuous Component}}$$

$$= dX_t + \mu(Y_t)dt + \sigma(Y_t)dW_t;$$

2. $Y = \{Y_t\}_{t \geq 0}$ is a latent (unobservable) risk factor with dynamics

$$dY_t = \alpha(Y_t)dt + \gamma(Y_t)dW'_t, \quad Y_0 = y_0;$$

3. $X = \{X_t\}_{t \geq 0}$ is a pure-jump tempered stable-like process

4. $W$, and $W'$ have correlation coefficient $\rho \in (-1, 1);$
Extensions: Nonhomogeneous component

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$$dL_t = \left\{ \begin{array}{ll}
\text{Jump Component} & dX_t \\
\text{Continuous Component} & dV_t
\end{array} \right\}$$

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4. $W$ and $W'$ have correlation coefficient $\rho \in (-1, 1)$;
Extensions: Nonhomogeneous component

1. Consider \( S_t = S_0 e^{L_t} \) with

\[
dL_t = dX_t + dV_t
\]

Jump Component + Continuous Component

\[
= dX_t + \mu(Y_t)dt + \sigma(Y_t)dW_t;
\]

2. \( Y = \{Y_t\}_{t \geq 0} \) is a latent (unobservable) risk factor with dynamics

\[
dY_t = \alpha(Y_t)dt + \gamma(Y_t)dW'_t, \quad Y_0 = y_0;
\]

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4. \( W, \) and \( W' \) have correlation coefficient \( \rho \in (-1, 1); \)
Extensions: “Close-to-the-money” options

- The previous asymptotics deal with ATM options $\kappa = 0$

- The most liquid options in practice are “close-to-the-money”:

$$C(t, \kappa) := \mathbb{E}\left[ (e^{Lt} - e^{\kappa})^+ \right], \quad \text{with} \quad \kappa \approx 0;$$

Consider another asymptotic regime of the form $C(t, \kappa_t)$ with $\kappa_t \to 0$;
Pure-jump Case

Theorem (F-L & Ólafsson, F&S 2015-)

Suppose the Lévy density \( s(x) := C \left( \frac{x}{|x|} \right) \tilde{q}(x) |x|^{-Y-1} \) is such that

(i) \( Y \in (1, 2) \),
(ii) \( C(\pm 1) \in (0, \infty) \);
(iii) \( \tilde{q}(x) \xrightarrow{x \to 0} 1 \);
(iv) \( \sup_{x \neq 0} \tilde{q}(x) < \infty \);
(v) \( \int_{x \geq 1} e^x x^{-Y-1} \tilde{q}(x) dx < \infty \),
(vi) \( \int_{|x| \leq 1} |x|^{-Y} |1 - \tilde{q}(x)| \, dx < \infty \).
Pure-jump Case

Theorem (F-L & Ólafsson, F&S 2015-)

Suppose the Lévy density \( s(x) := C \left( \frac{x}{|x|} \right) \bar{\varphi}(x)|x|^{-y-1} \) is such that

(i) \( Y \in (1, 2) \),  
(ii) \( C(\pm 1) \in (0, \infty) \);  
(iii) \( \bar{\varphi}(x) \xrightarrow{x\to 0} 1 \);  
(iv) \( \sup_{x \neq 0} \bar{\varphi}(x) < \infty \);  
(v) \( \int_{x \geq 1} e^x x^{-y-1} \bar{\varphi}(x) \, dx < \infty \),  
(vi) \( \int_{|x| \leq 1} |x|^{-y} |1 - \bar{\varphi}(x)| \, dx < \infty \).
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(vi) \( \int_{|x| \leq 1} |x|^{-y} |1 - \tilde{q}(x)| \, dx < \infty \).
Pure-jump Case. Cont...

Theorem (F-L & Ólafsson, F&S 2015-)

Then, for $\kappa_t := \theta t + o(t)$, as $t \to 0$, with $\theta \in \mathbb{R}$,

$$C(t, \kappa_t) = \mathbb{E} \left( e^{X_t} - e^{\kappa_t} \right)^+ = d_1 t^\frac{1}{Y} + d_2 t + o(t), \quad (t \to 0),$$

where $d_1 = \mathbb{E}(\tilde{Z}_1^+)$ and

\[
d_2 := C(1) \mathbb{P}(\tilde{Z}_1 < 0) \int_0^\infty (e^x \bar{q}(x) - \bar{q}(x) - x) x^{-Y-1} dx \\
- C(-1) \mathbb{P}(\tilde{Z}_1 \geq 0) \int_{-\infty}^0 (e^x \bar{q}(x) - \bar{q}(x) - x) |x|^{-Y-1} dx - \theta \mathbb{P}(\tilde{Z}_1 \geq 0)
\]

Remark: The condition (vi) $\int_{|x| \leq 1} |x|^{-Y} |1 - \bar{q}(x)| \ dx < \infty$ is a necessary conditions for such a second-order expansion to exist.
Nonzero Diffusion Component

Theorem (F-L & Ólafsson, F&S 2015-)

Let

\[ dV_t = \mu(Y_t)dt + \sigma(Y_t)dW_t, \quad dY_t = \alpha(Y_t)dt + \gamma(Y_t)dW_t', \quad Y_0 = y_0; \]

independent of \( X \) such that \( \sigma(y_0) > 0 \) and \( \sigma(\cdot) \) is Lipschitz continuous at \( y_0 \).

Then, for \( \kappa_t := \theta t^{3-\gamma} + o(t^{3-\gamma}) \), as \( t \to 0 \), for some \( \theta \in \mathbb{R} \),

\[
C(t, \kappa_t) = \mathbb{E} \left( e^{X_t+V_t} - e^{\kappa_t} \right)^+ = d_1 t^{\frac{1}{2}} + d_2 t^{\frac{3-\gamma}{2}} + o \left( t^{\frac{3-\gamma}{2}} \right), \quad (t \to 0),
\]

where

\[
d_1 := \frac{\sigma(y_0)}{\sqrt{2\pi}}, \quad d_2 := \frac{C(1) + C(-1)}{2Y(Y - 1)} \sigma(y_0)^{1-\gamma} \mathbb{E} \left( |W_1|^{1-\gamma} \right) + \frac{\theta}{2}.
\]
Nonzero Diffusion Component

Theorem (F-L & Ólafsson, F&S 2015-)

Let

\[ dV_t = \mu(Y_t)dt + \sigma(Y_t)dW_t, \quad dY_t = \alpha(Y_t)dt + \gamma(Y_t)dW'_t, \quad Y_0 = y_0; \]

independent of \( X \) such that \( \sigma(y_0) > 0 \) and \( \sigma(\cdot) \) is Lipschitz continuous at \( y_0 \).

Then, for \( \kappa_t := \theta t^{\frac{3-Y}{2}} + o(t^{\frac{3-Y}{2}}) \), as \( t \to 0 \), for some \( \theta \in \mathbb{R} \),

\[ C(t, \kappa_t) = \mathbb{E} \left( e^{X_t+V_t} - e^{\kappa_t} \right)^+ = d_1 t^{\frac{1}{2}} + d_2 t^{\frac{3-Y}{2}} + o \left( t^{\frac{3-Y}{2}} \right), \quad (t \to 0), \]

where

\[ d_1 := \frac{\sigma(y_0)}{\sqrt{2\pi}}, \quad d_2 := \frac{C(1) + C(-1)}{2Y(Y-1)} \sigma(y_0)^{1-Y} \mathbb{E} (|W_1|^{1-Y}) + \frac{\theta}{2}. \]
Remarks

• The first-order term $d_1$ is the same as in the Black-Scholes model with volatility $\sigma(y_0)$; in particular,

$$
\mathbb{E} \left( e^{X_t + V_t} - e^{\kappa t} \right)^+ - \mathbb{E} \left( e^{\sigma(y_0) W_t - \frac{\sigma^2(y_0)}{2} t} - e^{\kappa t} \right)^+ = O \left( t^{\frac{3}{2} - \frac{Y}{2}} \right),
$$

• The second-order term $d_2$ also incorporates information on the degree of jump activity $Y$, and the intensity of small jumps as measured by the quantity $C(1) + C(-1)$; in particular, suppose that $X_{CGMY}$ is a CGMY process such that $G = M = 1$ and $C = \frac{C(1) + C(2)}{2}$. Then, we have that

$$
\mathbb{E} \left( e^{X_t + V_t} - e^{\kappa t} \right)^+ - \mathbb{E} \left( e^{X_{CGMY}^t + \sigma(y_0) W_t + \mu t} - e^{\kappa t} \right)^+ = o \left( t^{\frac{3}{2} - \frac{Y}{2}} \right),
$$

where $\mu$ is chosen so that $e^{X_{CGMY}^t + \sigma(y_0) W_t + \mu t}$ is a martingale.

• The second order term $d_2$ is of order $\frac{3-Y}{2} \in \left( \frac{1}{2}, 1 \right)$; i.e., the convergence of the option price to zero is slower with increased jump activity.
CGMY Model: \[ s(x) = C|x|^{-Y-1} \left( e^{-M|x|} 1_{x>0} + e^{-G|x|} 1_{x<0} \right) \]

Figure: Comparisons of ATM call option prices (computed by Monte-Carlo simulation) and the first- and second-order approximations. Left panel: Pure-jump CGMY. Right panel: CGMY with a Brownian component. Parameter values taken from Andersen et al. (2012)
Outline

1. Lévy processes
   Some Financial models driven by Lévy processes

2. The general problems

3. High-order asymptotic behavior for OTM Option Prices
   Jump-Diffusion Lévy Model
   General Lévy Model

4. High-order asymptotic behavior for ATM Option Prices
   Tempered Stable-Like Processes
   Main Results

5. ATM implied volatility slope

6. Conclusions
ATM implied volatility slope

- Option prices are usually quoted in terms of implied volatilities: The unique value of $\hat{\sigma}(\kappa, t)$ such that

$$C_{BS}(\kappa, t, \hat{\sigma}(\kappa, t)) = C(\kappa, t) \leftarrow \text{Observed Price}$$

where $C_{BS}(\kappa, t, \sigma) = \Phi(d_+) - e^{\kappa} \Phi(d_-)$, with $d_\pm = (-\kappa \pm \frac{1}{2} \sigma^2 t) / (\sigma \sqrt{t})$

- The close-to-the-money option price expansions can be translated into short-time approximations for the close-to-the-money implied volatility level $\hat{\sigma}(\kappa t, t)$.

- Related quantities of interest are the ATM slope ($\frac{\partial \hat{\sigma}(\kappa, t)}{\partial \kappa}$) and convexity ($\frac{\partial^2 \hat{\sigma}(\kappa, t)}{\partial \kappa^2}$) of the implied volatility curve.
ATM implied volatility slope

An Important Formula:

\[
\left. \frac{\partial \hat{\sigma} (\kappa, t)}{\partial \kappa} \right|_{\kappa = 0} = \sqrt{\frac{2\pi}{t}} \left( \frac{1}{2} - \mathbb{P}(S_t \geq 1) - \frac{\hat{\sigma}(t) \sqrt{t}}{2\sqrt{2\pi}} + O \left( \left( \frac{\hat{\sigma}(t) \sqrt{t}}{2\sqrt{2\pi}} \right)^3 \right) \right)
\]

- \( \hat{\sigma}(t) := \hat{\sigma}(0, t) \) is the ATM implied volatility, whose small-time behavior is known (F-L, Gong, & Houdré, MF 2014):

\[
\hat{\sigma}(t) = \sqrt{2\pi} d_1 t^{\frac{1}{2}} + \sqrt{2\pi} d_2 t^{\frac{1}{2}} + o(t^{\frac{1}{2}})
\]

\[
\hat{\sigma}(t) = \sigma_0 + \frac{(C_+ + C_-) 2^{-Y/2}}{Y(Y - 1)} \Gamma \left( 1 - \frac{Y}{2} \right) \sigma_0^{1-Y} t^{1-\frac{Y}{2}} + o(t^{1-\frac{Y}{2}})
\]

- Need to study the small time behavior of ATM digital call options:

\[
\mathbb{P}(S_t \geq 1) = \mathbb{P}(X_t + V_t \geq 0)
\]
ATM implied volatility slope

An Important Formula:

$$\left. \frac{\partial \hat{\sigma}(\kappa, t)}{\partial \kappa} \right|_{\kappa = 0} = \sqrt{\frac{2\pi}{t}} \left( \frac{1}{2} - \mathbb{P}(S_t \geq 1) - \frac{\hat{\sigma}(t)\sqrt{t}}{2\sqrt{2\pi}} + O\left( \left( \hat{\sigma}(t)\sqrt{t} \right)^3 \right) \right)$$

- $\hat{\sigma}(t) := \hat{\sigma}(0, t)$ is the ATM implied volatility, whose small-time behavior is known (F-L, Gong, & Houdré, MF 2014):

$$\hat{\sigma}(t) = \sqrt{2\pi}d_1t^{\frac{Y}{2} - 1} + \sqrt{2\pi}d_2t^{\frac{1}{2}} + o(t^{\frac{1}{2}})$$

$$\hat{\sigma}(t) = \sigma_0 + \frac{(C_+ + C_-)2^{-Y/2}}{Y(Y - 1)}\Gamma\left(1 - \frac{Y}{2}\right)\sigma_0^{1-Y}t^{1-\frac{Y}{2}} + o(t^{1-\frac{Y}{2}})$$

- Need to study the small time behavior of ATM digital call options:

$$\mathbb{P}(S_t \geq 1) = \mathbb{P}(X_t + V_t \geq 0)$$
ATM implied volatility slope

An Important Formula:

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\left. \frac{\partial \hat{\sigma}(\kappa, t)}{\partial \kappa} \right|_{\kappa=0} = \sqrt{\frac{2\pi}{t}} \left( \frac{1}{2} - \mathbb{P}(S_t \geq 1) - \frac{\hat{\sigma}(t)\sqrt{t}}{2\sqrt{2\pi}} + O\left(\left(\hat{\sigma}(t)\sqrt{t}\right)^3\right) \right)
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• \( \hat{\sigma}(t) := \hat{\sigma}(0, t) \) is the ATM implied volatility, whose small-time behavior is known (F-L, Gong, & Houdré, MF 2014):

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\hat{\sigma}(t) = \sqrt{2\pi} d_1 t^{\frac{1}{2}} - \frac{1}{2} + \sqrt{2\pi} d_2 t^{\frac{1}{2}} + o(t^{\frac{1}{2}})
\]

\[
\hat{\sigma}(t) = \sigma_0 + \frac{(C_+ + C_-)2^{-Y/2}}{Y(Y-1)} \Gamma\left(1 - \frac{Y}{2}\right) \sigma_0^{1-Y} t^{1-\frac{Y}{2}} + o(t^{1-\frac{Y}{2}})
\]

• Need to study the small time behavior of ATM digital call options:

\[
\mathbb{P}(S_t \geq 1) = \mathbb{P}(X_t + V_t \geq 0)
\]
Existing literature

1. Tankov & Rosenbaum (2011): Leading order term for a pure-jump Lévy process with stable-like jump behavior:

\[ \mathbb{P}(X_t \geq 0) \rightarrow \mathbb{P}(\bar{Z}_1 \geq 0), \quad t \rightarrow 0, \]

where \( \bar{Z}_1 \) is a strictly \( Y \)-stable random variable.

2. For \( L = X + \sigma W \) with nonzero \( \sigma \), there is a small-time CLT:

\[ \mathbb{P}(X_t + \sigma W_t \geq 0) \rightarrow \frac{1}{2}, \quad t \rightarrow 0. \]
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Pure-jump Case

**Theorem (F-L & Ólafsson, 2015-)**

Suppose the Lévy density $s(x) := C \left(\frac{x}{|x|}\right) \bar{q}(x)|x|^{-Y-1}$ is such that

(i) $Y \in (1, 2)$,  (ii) $C(\pm 1) \in (0, \infty)$;  (iii) $\bar{q}(x) \xrightarrow{x \to 0} 1$;  (iv) $\sup_{x \neq 0} \bar{q}(x) < \infty$;

(v) $\int_{|x| \leq 1} \left| \bar{q}(x) - 1 - \alpha \left(\frac{x}{|x|}\right) x \right| |x|^{-Y-1} dx < \infty$, $\alpha(1), \alpha(-1) \in \mathbb{R}$
Pure-jump Case

Theorem (F-L & Ólafsson, 2015-)

Suppose the Lévy density \( s(x) := C \left( \frac{x}{|x|} \right) \bar{q}(x)|x|^{-Y-1} \) is such that

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Pure-jump Case. Cont...

Theorem (F-L & Ólafsson, 2014)

Then,

$$
\mathbb{P}(X_t \geq 0) - \mathbb{P}(\bar{Z}_1 \geq 0) = \sum_{k=1}^{n} d_k t^{k\left(1 - \frac{1}{Y}\right)} + e t^{\frac{1}{Y}} + f t + o(t), \quad t \to 0,
$$

where $n := \sup\{k \geq 3 : k \left(1 - 1/Y\right) \leq 1\}$. The $d_k$'s are given by

$$
d_k := \left(\frac{-1}{k!}\right)^{k+1} \tilde{\gamma}^k f^{(k-1)}(0), \quad 1 \leq k \leq n,
$$

where

$$
\tilde{\gamma} := b + \frac{C(1) - C(-1)}{Y - 1} + \int_{|x| \leq 1} C \left(\frac{x}{|x|}\right) x |x|^{-Y-1} (1 - \bar{q}(x)) \, dx,
$$

while, the coefficients $e$ and $f$ are given by...
Pure-jump Case. Cont...

\[ e := \alpha(1) \mathbb{E} \left( \bar{Z}_1^{(p)} 1_{\{\bar{Z}_1^{(p)} + \bar{Z}_1^{(n)} \geq 0\}} \right) + \alpha(-1) \mathbb{E} \left( \bar{Z}_1^{(n)} 1_{\{\bar{Z}_1^{(p)} + \bar{Z}_1^{(n)} \geq 0\}} \right), \]

\[ f := \tilde{\gamma} \left\{ \alpha(1) \mathbb{E} \left( \bar{Z}_1^{(p)} f_{\bar{Z}_1^{(n)}} (-\bar{Z}_1^{(p)}) \right) + \alpha(-1) \mathbb{E} \left( \bar{Z}_1^{(n)} f_{\bar{Z}_1^{(p)}} (-\bar{Z}_1^{(n)}) \right) \right\} \]

\[ + \tilde{\mathbb{P}} (\tilde{Z}_1 \leq 0) \ C(1) \int_{0}^{\infty} (\bar{q}(x) - 1 - \alpha(1)x) x^{-Y-1} dx \]

\[ - \tilde{\mathbb{P}} (\tilde{Z}_1 > 0) \ C(-1) \int_{-\infty}^{0} (\bar{q}(x) - 1 - \alpha(-1)x) |x|^{-Y-1} dx , \]

where \( \bar{Z}_1^{(p)} \) and \( \bar{Z}_1^{(n)} \) are strictly \( Y \)-stable r.v.'s under \( \tilde{\mathbb{P}} \), and \( \tilde{Z}_1 := \bar{Z}_1^{(p)} + \bar{Z}_1^{(n)} \).

**Remark:** The condition

\[ (v) \ \int_{|x|\leq 1} \left| \bar{q}(x) - 1 - \alpha \left( \frac{x}{|x|} \right) x \right| |x|^{-Y-1} dx < \infty, \]

is a necessary conditions for \( d_{n+2} \) to be well defined.
Remarks

- The $d_k'$ terms depend only on $C(1)$, $C(-1)$, and $\tilde{\gamma}$. In particular, if $X_{\text{stbl}}$ denotes a $Y$-stable process with the same $C(1)$, $C(-1)$, and $\tilde{\gamma}$ as $X$, then

$$\mathbb{P}(X_t \geq 0) - \mathbb{P}(X_{\text{stbl}}^t \geq 0) = O\left(t^{1/Y}\right).$$

- The coefficient $e$ depends, in addition, on $\alpha(1)$ and $\alpha(-1)$. Therefore, if $X_{\text{CGMY}}$ represents a CGMY process with the same $C(\pm 1)$, $\alpha(\pm 1)$, and $\tilde{\gamma}$ as $X$, then

$$\mathbb{P}(X_t \geq 0) - \mathbb{P}(X_{\text{CGMY}}^t \geq 0) = O(t).$$

- The tempering function $\bar{q}$ appears only up to a term of order $O(t)$, via the parameters $\int_0^\infty (\bar{q}(x) - 1 - \alpha(1)x) x^{-Y-1} dx$ and $\int_{-\infty}^0 (\bar{q}(x) - 1 - \alpha(-1)x) |x|^{-Y-1} dx$. 
Corollary (F-L & Ólafsson, 2014)

Let $X$ be a tempered stable-like Lévy process as in the previous theorem, and assume that $S_t := e^{X_t}$ is a martingale. Then,

$$\left. \frac{\partial \hat{\sigma} (\kappa, t)}{\partial \kappa} \right|_{\kappa = 0} = \sqrt{\frac{2\pi}{t}} \left( \frac{1}{2} - \mathbb{P} (\bar{Z}_1 \geq 0) \right) - \sum_{k=1}^{n} d_k t^k (1 - \frac{1}{\nu})$$

$$- \left( e + \frac{\sigma_1}{2} \right) t^{\frac{1}{\nu}} - \left( f + \frac{\sigma_2}{2} \right) t + o(t), \quad t \to 0,$$

where $\sigma_1$ and $\sigma_2$ are the coefficients of the first and second order terms for the ATM implied volatility $\hat{\sigma} (t) := \hat{\sigma} (0, t)$ expansion.
Nonzero Diffusion Component

Theorem (F-L & Ólafsson, 2014)

Let $X$ be a pure-jump tempered stable process, and $V$ such that

$$
dV_t = \mu(Y_t) dt + \sigma(Y_t) dW_t, \quad V_0 = 0,
$$

$$
dY_t = \alpha(Y_t) dt + \gamma(Y_t) dW'_t, \quad Y_0 = y_0;
$$

and $\sigma(y_0) > 0$. Then,

$$
P(X_t + V_t \geq 0) = \frac{1}{2} + \sum_{k=1}^{n} d_k t^{k(1 - Y/2)} + e t^{1/2} + f t^{3-Y/2} + o(t^{3-Y/2}), \quad t \to 0,
$$

where $n := \sup \{k \geq 3 : k (1 - Y/2) \leq (3 - Y)/2\}$, and $d_k, e, and, f$, are given by:
Nonzero Diffusion Component, cont.

d_k := \frac{(\sqrt{1 - \rho^2 \sigma(y_0)})^{-k Y}}{k!} L_Z^k \Psi^{(\delta_0)}(0), \quad 1 \leq k \leq n,

e := \left( \tilde{\gamma} + \mu(y_0) - \rho \sigma'(y_0) \gamma(y_0) \left( 1 - \frac{\rho^2}{2} \right) \right) \phi_{\sigma_0}(0),

f := \left[ \frac{\alpha(1) C(1) + \alpha(-1) C(-1)}{Y - 1} - \frac{C(1) + C(-1)}{\sigma^2(y_0) Y} \right] \left( \tilde{\gamma} + \mu(y_0) - \rho \sigma'(y_0) \gamma(y_0) \left( 1 - \frac{\rho^2}{2} \right) (1 + Y) \right) \int_0^\infty \phi_{\sigma_0}(x)x^{1-Y} \, dx,

where \( L_Z \) denotes the infinitesimal generator of the strictly stable process \( \tilde{Z} \), and \( \Psi^{(\delta)} = \mathbb{E} \left( \int_{\omega = 0}^{\omega + \delta W_1} \phi(x) \, dx \right) \), with \( \delta_0 := \rho \sigma(y_0) (\sqrt{1 - \rho^2 \sigma(y_0)})^{-1} \).
Remarks

• The $d_k$’s and $e$ depend on $X$ only through $C(1)$ and $C(-1)$. In particular, if $	ilde{Z}$ denotes the strictly $Y$-stable process with parameters $C(1)$ and $C(-1)$, then

$$\mathbb{P}(X_t + V_t \geq 0) - \mathbb{P}(\tilde{Z}_t + V_t \geq 0) = O \left( t^{(3-Y)/2} \right).$$

• The coefficient $f$ depends, in addition, on $\alpha(1)$ and $\alpha(-1)$. Therefore, if $X^{CGMY}$ represents a CGMY process with the same $C(\pm 1)$, $\alpha(\pm 1)$, and $\tilde{\gamma}$ as $X$, then

$$\mathbb{P}(X_t + V_t \geq 0) - \mathbb{P}(X_t^{CGMY} + V_t \geq 0) = o \left( t^{(3-Y)/2} \right).$$
Corollary (F-L & Ólafsson, 2014)

Let \( X \) and \( V \) be as in the previous theorem, with \( b \) and \( \mu(\cdot) = -\frac{1}{2} \sigma(\cdot) \), such that \( e^{Vt} \) is true a martingale. Then,

\[
- \left. \frac{\partial \hat{\sigma}(\kappa, t)}{\partial \kappa} \right|_{\kappa=0} = \sqrt{2\pi} \sum_{k=1}^{n} d_k t^{(1-\frac{\gamma}{2})k-\frac{1}{2}} + \frac{1}{\sigma_0} \left( \tilde{\gamma} - \rho \sigma'(y_0) \gamma(y_0) \left( 1 - \frac{\rho^2}{2} \right) \right) \\
+ \left( \sqrt{2\pi} f + \frac{1}{2} \bar{\sigma}_1 \right) t^{1-\frac{\gamma}{2}} + o(t^{1-\frac{\gamma}{2}}), \quad t \to 0,
\]

where \( \bar{\sigma}_1 \) is the coefficient of the second order term for the ATM implied volatility expansion.
CGMY Model: \( s(x) = C|x|^{-y-1} \left( e^{-M|x|}1_{x>0} + e^{-G|x|}1_{x<0} \right) \)

Figure: Comparisons of \( \log_{10}(\mathbb{P}(X_t + \sigma W_t > 0)) \) (computed by Monte-Carlo simulation) and the first- and second-order approximations. Left panel: Pure-jump CGMY. Right panel: CGMY with a Brownian component. Parameter values taken from Andersen et al. (2012)
**CGMY Model:** \( s(x) = c \left( \frac{x}{|x|} \right) |x|^{-\gamma - 1} \left( e^{-M|x|} 1_{x>0} + e^{-G|x|} 1_{x<0} \right) \)

**Figure:** Solid line: The volatility smile with maturity \( t=0.1 \), as a function of log-strike. Dashed line: The ATM slope approximation. Left panel: Pure-jump CGMY. Right panel: CGMY with a Brownian component. Parameter values taken from Andersen et al. (2012)
Outline

1. Lévy processes
   - Some Financial models driven by Lévy processes
2. The general problems
3. High-order asymptotic behavior for OTM Option Prices
   - Jump-Diffusion Lévy Model
   - General Lévy Model
4. High-order asymptotic behavior for ATM Option Prices
   - Tempered Stable-Like Processes
   - Main Results
5. ATM implied volatility slope
6. Conclusions
Future Work

1 Obtaining higher order terms would improve the approximation
   - The second order term $d_2$ does not incorporate information on the tempering function $\bar{q}$ in the case of a continuous component.
   - Ultimately, what information about the underlying process can be extracted from the asymptotic behavior of option prices in small time?

2 Calibration of the implied volatility smile
   - The second order approximation gives an expression for the small-time behavior of the implied volatility:
     \[
     \hat{\sigma}(t) = \sigma(y_0) + \frac{(C(1) + C(-1)) 2^{-Y/2}}{Y(Y - 1)} \Gamma \left(1 - \frac{Y}{2}\right) \sigma(y_0)^{1-Y} t^{1-\frac{Y}{2}} + o(t^{1-\frac{Y}{2}}).
     \]
   - Issue: The expression involves the unobserved spot volatility $\sigma(y_0)$; but, there is a limited number of outstanding short-term options at any given time.

3 Short-time asymptotics for the convexity of the implied volatility ATM.
Conclusions

1. Obtained high-order short-time expansions for close-to-the-money European call option prices under stable-like small jumps and a possible nonzero independent diffusion component.

2. Obtained high-order short-time expansion for ATM digital call options prices, and the ATM implied volatility slope.

3. Characterized explicitly the effect of the model parameters on the behavior of ATM option prices near expiration.
Further Reading

J.E. Figueroa-López, R. Gong, and C. Houdré,

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M. Rosenbaum and P. Tankov.
Asymptotic results for time-changed Lévy processes sampled at hitting times. *Stochastic processes and their applications, 121, 2011*.

P. Tankov.
Times series of returns

Graph taken from “Financial Modeling with Jump Processes” by Cont and Tankov, 2004

Five-minute log-return for Yen/Deutschemark exchange rate, 1992-1995

BMW daily log-returns
Outline of the proof (pure-jump case)

Step 2: For a closer look at \( \int_0^\infty e^{-t^{\frac{1}{Y}}} u \mathbb{P}^*(t^{\frac{1}{Y}} X_t > u) du \to \widetilde{\mathbb{E}} \left( \tilde{Z}_1^+ \right) \), consider \( \mathbb{P}^* \to \tilde{\mathbb{P}} \) such that, under \( \tilde{\mathbb{P}} \), \( X \) is a stable process:

\[
\int_0^\infty e^{-t^{\frac{1}{Y}}} u \mathbb{P}^*(t^{\frac{1}{Y}} X_t > u) du = \int_0^\infty e^{-t^{\frac{1}{Y}}} u \tilde{\mathbb{E}} \left( e^{-U_t} 1_{\{t^{\frac{1}{Y}} X_t > u\}} \right) du
\]

where \( d\tilde{\mathbb{P}} \big|_{\mathcal{F}_t} = e^{U_t} d\mathbb{P}^* \big|_{\mathcal{F}_t} \); Eventually...

\[
t^{\frac{1}{Y} - 1} \left( \int_0^\infty e^{-t^{\frac{1}{Y}}} u \mathbb{P}^*(t^{\frac{1}{Y}} X_t > u) du - \tilde{\mathbb{E}} \left( Z_t^+ \right) \right)
\]

\[
= \int_0^\infty (e^{-v} - 1) \frac{1}{t} \tilde{\mathbb{P}}(Z_t^+ + U_t \geq v) dv
\]

\[
- \int_0^\infty (e^v - 1) \frac{1}{t} \tilde{\mathbb{P}}(Z_t^+ + U_t \leq -v) dv \to d_2, \quad (t \to 0).
\]
Outline of the proof (pure-jump case)

Step 2: For a closer look at \( \int_0^\infty e^{-t^{1/\gamma}u}P^*(t^{-1/\gamma}X_t > u)du \rightarrow \tilde{E}(\tilde{Z}_1^+) \), consider \( P^* \rightarrow \tilde{P} \) such that, under \( \tilde{P} \), \( X \) is a stable process:

\[
\int_0^\infty e^{-t^{1/\gamma}u}P^*(t^{-1/\gamma}X_t > u)du = \int_0^\infty e^{-t^{1/\gamma}u}\tilde{E}(e^{-U_t}1_{\{t^{-1/\gamma}X_t > u\}}) \, du
\]

where \( d\tilde{P} \bigg|_{\mathcal{F}_t} = e^{U_t}dP^* \bigg|_{\mathcal{F}_t} \); Eventually…

\[
t^{1/\gamma - 1} \left( \int_0^\infty e^{-t^{1/\gamma}u}P^*(t^{-1/\gamma}X_t > u)du - \tilde{E}(Z_1^+) \right)
\]

\[
= \int_0^\infty (e^{-v} - 1) \frac{1}{t} \tilde{P}(Z_t^+ + U_t \geq v) \, dv
\]

\[
- \int_0^\infty (e^{v} - 1) \frac{1}{t} \tilde{P}(Z_t^+ + U_t \leq -v) \, dv \rightarrow d_2, \quad (t \rightarrow 0).
\]
Outline of the proof (pure-jump case)

Step 3: The previous approach allow us to prove the result under the conditions:

\[ |1 - \bar{q}(x)| = O(x) \quad \& \quad \limsup_{|x| \to \infty} \frac{|\ln \bar{q}(x)|}{|x|} < \infty. \]

Then, the idea is to approximate \( X \) by a process \( X^{(\delta)} \) such that

- \( X^{(\delta)} \) satisfies those conditions
- \( X^{(\delta)} \overset{D}{\to} X \), as \( \delta \to 0 \)

It turns out that the condition \( \int_{|x| \leq 1} |x|^{-\gamma} |1 - \bar{q}(x)| \, dx < \infty \) is exactly what is needed to do that.

\[
\begin{align*}
\mathbb{E} \left( e^{X_t^{(\delta)}} - e^{\kappa_t} \right)^+ & - \mathbb{E} \left| e^{X_t} - e^{X_t^{(\delta)}} \right| \\
& \leq \mathbb{E} \left( e^{X_t} - e^{\kappa_t} \right)^+ \\
& \leq \mathbb{E} \left( e^{X_t^{(\delta)}} - e^{\kappa_t} \right)^+ + \mathbb{E} \left| e^{X_t} - e^{X_t^{(\delta)}} \right|. \\
\end{align*}
\]
Outline of the proof (pure-jump case)

Step 3: The previous approach allow us to prove the result under the conditions:

\[ |1 - \bar{q}(x)| = O(x) \quad \& \quad \limsup_{|x| \to \infty} \frac{\ln \bar{q}(x)}{|x|} < \infty. \]

Then, the idea is to approximate \( X \) by a process \( X^{(\delta)} \) such that

- \( X^{(\delta)} \) satisfies those conditions
- \( X^{(\delta)} \overset{D}{\to} X \), as \( \delta \to 0 \)

It turns out that the condition \( \int_{|x| \leq 1} |x|^{-Y} |1 - \bar{q}(x)| \, dx < \infty \) is exactly what is needed to do that.

\[
E\left( e^{X^{(\delta)}_t} - e^{\kappa t} \right)^+ - E\left| e^{X_t} - e^{X^{(\delta)}_t} \right| \\
\leq E\left( e^{X_t} - e^{\kappa t} \right)^+ \\
\leq E\left( e^{X^{(\delta)}_t} - e^{\kappa t} \right)^+ + E\left| e^{X_t} - e^{X^{(\delta)}_t} \right|.
\]
Outline of the proof (pure-jump case)

Step 3: The previous approach allow us to prove the result under the conditions:

\[ |1 - \tilde{q}(x)| = O(x) \quad \& \quad \limsup_{|x| \to \infty} \frac{|\ln \tilde{q}(x)|}{|x|} < \infty. \]

Then, the idea is to approximate \( X \) by a process \( X^{(\delta)} \) such that

- \( X^{(\delta)} \) satisfies those conditions
- \( X^{(\delta)} \xrightarrow{D} X, \) as \( \delta \to 0 \)

It turns out that the condition \( \int_{|x| \leq 1} |x|^{-Y} |1 - \tilde{q}(x)| \, dx < \infty \) is exactly what is needed to do that.

\[
\mathbb{E} \left( e^{X^{(\delta)}_t} - e^{\kappa_t} \right)^+ - \mathbb{E} \left| e^{X_t} - e^{X^{(\delta)}_t} \right| \\
\leq \mathbb{E} \left( e^{X_t} - e^{\kappa_t} \right)^+ \\
\leq \mathbb{E} \left( e^{X^{(\delta)}_t} - e^{\kappa_t} \right)^+ + \mathbb{E} \left| e^{X_t} - e^{X^{(\delta)}_t} \right|. \]
Outline of the proof (pure-jump case)

Step 3: $X_t - X_t^{(\delta)}$ is of finite variation so

$$\limsup_{t \to 0} \frac{1}{t} \left| e^{X_t} - e^{X_t^{(\delta)}} \right| \leq K \int_{|x| \leq \delta} |x|^{-\gamma} |1 - \bar{q}(x)| \, dx,$$

for some $K < \infty$.

$$t^{1/\gamma - 1} \left( t^{-1/\gamma} \mathbb{E} \left( e^{X_t^{(\delta)}} - e^{\kappa t} \right)^+ - d_1 \right) - \frac{1}{t} \mathbb{E} \left| e^{X_t} - e^{X_t^{(\delta)}} \right|$$

$$\leq t^{1/\gamma - 1} \left( t^{-1/\gamma} \mathbb{E} \left( e^{X_t} - e^{\kappa t} \right)^+ - d_1 \right)$$

$$\leq t^{1/\gamma - 1} \left( t^{-1/\gamma} \mathbb{E} \left( e^{X_t^{(\delta)}} - e^{\kappa t} \right)^+ - d_1 \right) + \frac{1}{t} \mathbb{E} \left| e^{X_t} - e^{X_t^{(\delta)}} \right|$$

First, let $t \to 0 \ldots$
Outline of the proof (pure-jump case)

Step 3: $X_t - X_t^{(\delta)}$ is of finite variation so

$$\limsup_{t \to 0} \frac{1}{t} \mathbb{E} \left| e^{X_t} - e^{X_t^{(\delta)}} \right| \leq K \int_{|x| \leq \delta} |x|^{-Y} |1 - \bar{q}(x)| \, dx,$$

for some $K < \infty$.

$$d_2^{(\delta)} - K \int_{|x| \leq \delta} |x|^{-Y} |1 - \bar{q}(x)| \, dx$$

$$\leq \liminf_{t \to 0} t^{\frac{1}{Y} - 1} \left( t^{-\frac{1}{Y}} \mathbb{E} (e^{X_t} - e^{\kappa_t})^+ - d_1 \right)$$

$$\leq \limsup_{t \to 0} t^{\frac{1}{Y} - 1} \left( t^{-\frac{1}{Y}} \mathbb{E} (e^{X_t} - e^{\kappa_t})^+ - d_1 \right)$$

$$\leq d_2^{(\delta)} + K \int_{|x| \leq \delta} |x|^{-Y} |1 - \bar{q}(x)| \, dx$$

Then, let $\delta \to 0$, and notice that $d_2^{(\delta)} \to d_2$. 