The Arzelà-Ascoli Theorem

The Arzelà-Ascoli Theorem gives sufficient conditions for compactness in certain function spaces. Among other things, it helps provide some additional perspective on what compactness means.

Let $C([0,1])$ denote the set of continuous functions $f : [0,1] \to \mathbb{R}$. Because the domain is compact, one can show (I leave this as an exercise) that any $f \in C([0,1])$ is uniformly continuous: for any $\varepsilon > 0$ there is a $\delta > 0$ such that if $|x - \hat{x}| < \delta$ then $|f(x) - f(\hat{x})| < \varepsilon$. An example of a function that is continuous but not uniformly continuous is $f : (0,1] \to \mathbb{R}$ given by $f(x) = 1/x$.

A set $F \subseteq C([0,1])$ is (uniformly) equicontinuous iff for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $f \in F$, if $|x - \hat{x}| < \delta$ then $|f(x) - f(\hat{x})| < \varepsilon$. That is, if $F$ is (uniformly) equicontinuous then every $f \in F$ is uniformly continuous and for every $\varepsilon > 0$ and every $f \in F$, I can use the same $\delta > 0$.

Example 3 below gives an example of a set of (uniformly) continuous functions that is not equicontinuous. A trivial example of an equicontinuous set of functions is a set of functions such that any pair of functions differ from each other by an additive constant: for any $f$ and $g$ in the set, there is an $a$ such that for all $x \in [0,1]$, $f(x) = \hat{f}(x) + a$. A more interesting example is given by a set of differentiable functions for which the derivative is uniformly bounded: there is a $W > 0$ such that for all $x, \hat{x} \in [0,1]$ and all $\theta \in (x, \hat{x})$, $|Df(x)| < W$. In this case, for any $x, \hat{x} \in [0,1], \hat{x} > x$, there is, by the Mean Value Theorem, a $\theta \in (x, \hat{x})$ such that

$$(f(x) - f(\hat{x})) = Df(\theta)(x - \hat{x}),$$

hence

$$|f(x) - f(\hat{x})| < W|x - \hat{x}|,$$

which implies equicontinuity.

Consider the sup metric on $C([0,1])$ given by, for any $f, g \in C([0,1])$

$$d_{\text{sup}}(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|.$$

This metric is well defined since $[0,1]$ is compact, $f - g$ is continuous, and absolute value is continuous. It is easy to verify (by a now standard argument) that this metric is indeed a metric (in particular, satisfies the triangle inequality). Convergence under $d_{\text{sup}}$ is uniform convergence: $f_t \to f^*$ iff for any $\varepsilon > 0$ there is a $T$ such that for all $t > T$ and all $x \in [0,1]$, $|f_t(x) - f^*(x)| < \varepsilon$. Henceforth, fix $d_{\text{sup}}$ as the metric for $C([0,1])$. $C([0,1])$ is complete; the proof is almost identical to the proof that $(\ell^\infty, d_{\text{sup}})$ is complete.
Theorem 1. If $F \subseteq C([0,1])$ is equicontinuous then so is $F$.

Proof. Fix $\varepsilon > 0$ and fix any $\hat{\varepsilon} \in (0,\varepsilon)$. Since $F$ is equicontinuous, there is a $\delta > 0$ such that for any $g \in F$ and $x, \hat{x} \in [0,1]$, if $|x - \hat{x}| < \delta$ then $|g(x) - g(\hat{x})| < \hat{\varepsilon}$.

Consider, then, any $f \in \overline{F}$. There is a sequence $(f_t)$ in $F$ such that $f_t \rightarrow f$. For any $x, \hat{x} \in [0,1]$, if $|x - \hat{x}| < \delta$ then, by the triangle inequality,

$$|f(x) - f(\hat{x})| \leq |f(x) - f_t(x)| + |f_t(x) - f_t(\hat{x})| + |f(\hat{x}) - f_t(\hat{x})| < \hat{\varepsilon} + |f(x) - f_t(x)| + |f(\hat{x}) - f_t(\hat{x})|.$$ 

Hence, since $f_t \rightarrow f$ and $\hat{\varepsilon} < \varepsilon$,

$$|f(x) - f(\hat{x})| < \varepsilon.$$ 


The basic Arzelà-Ascoli Theorem, formalized below, says that if a set $F \subseteq C([0,1])$ is closed, bounded, and equicontinuous, then it is compact. Recall that the Heine-Borel Theorem states that, in $\mathbb{R}^N$, a set that is closed and bounded is compact. In contrast, in infinite-dimensional normed vector spaces, including $C([0,1])$, closed and bounded sets need not be compact and closed balls are never compact. The basic Arzelà-Ascoli Theorem can be viewed as fixing the problems of Heine-Borel in $C([0,1])$ by adding “equicontinuous” as an extra condition.

Theorem 2 (Basic Arzelà-Ascoli). If $F \subseteq C([0,1])$ is closed, bounded, and equicontinuous then it is compact.

Proof. Let $F \subseteq C([0,1])$ be equicontinuous and bounded. By boundedness, there is an $M > 0$ such that for all $f \in F$ and all $x \in [0,1]$, $|f(x)| < M$.

Let $(f_t)$ be a sequence in $F$. I need to show that there is a subsequence $f_{t_k}$ and a function $f^* \in F$ such that $f_{t_k} \rightarrow f^*$.

The proof has two main steps. The first uses boundedness to argue that there exists a subsequence $f_{t_k}$ that converges pointwise at every rational number in $[0,1]$. The second step then uses the fact that $F$ is closed and equicontinuous, together with the fact that $[0,1]$ is compact, to argue that $(f_{t_k})$ is Cauchy.

- **Step 1.** Let $A$ be the set of rational elements of $[0,1]$ and let $(a_1, a_2, \ldots)$ be an enumeration of the rationals in $[0,1]$. Then $(f_t(a_1))$ is a sequence in the compact set $[-M,M] \subseteq \mathbb{R}$ and hence has a convergent subsequence; denote a generic index in this subsequence $t_{(1,1)}$. Set $t_1$, the first index in the subsequence $(f_{t_k})$ that I am constructing, equal to $t_{(1,1)}$.

And so on. At stage $k + 1$, $f_{t_{(k,1)}}(a_{k+1})$ is a sequence in the compact set $[-M,M] \subseteq \mathbb{R}$ and hence has a convergent subsequence; denote a generic index in this subsequence $t_{(i,k+1)}$. Set $t_{k+1} = t_{(k+1,k+1)}$. Note that $t_{k+1} > t_k$. 


By construction, for every \( j \), \((f_t^k(a_j))\) is a convergent subsequence of \((f_t(a_j))\); in particular, all but (at most) finitely many terms of \((f_t^k(a_j))\) belong to \((f_{t_{i(j)}}(a_j))\). For every \( j \), set \( f^*(a_j) = \lim_{k \to \infty} f_t^k(a_j) \).

- **Step 2.** I claim that \((f_t^k)\) is Cauchy. Since, by assumption, \( F \) is a closed subset of \( C([0,1]) \), which is complete, \( F \) is complete, and the proof follows. It remains to show that \((f_t^k)\) is Cauchy.

  Fix \( \varepsilon > 0 \) and any \( \hat{\varepsilon} \in (0,\varepsilon) \). By equicontinuity, there is a \( \delta > 0 \) such that for any \( x \in [0,1] \) and any \( a \in A \), if \( |x - a| < \delta \) then, for any \( k \),

  \[
  |f_t^k(x) - f_t^k(a)| < \frac{\hat{\varepsilon}}{3}.
  \]

  Since \([0,1] \) is compact, it can be covered by a finite number of open intervals with rational centers and radius \( \delta \). (Since \( A \) is dense in \([0,1] \), the set of all open intervals with rational centers covers \([0,1] \). Since \([0,1] \) is compact, a finite subset of these also covers \([0,1] \).) Let \( A_\delta \subseteq A \) denote this finite set of rational centers. For each \( a \in A_\delta \), since \( f_t^k(a) \to f^*(a) \), and hence \((f_t^k(a))\) is Cauchy, there is a \( K_a \) such that for all \( k, \ell > K_a \),

  \[
  |f_t^k(a) - f_t^\ell(a)| < \frac{\hat{\varepsilon}}{3}.
  \]

  Let \( K = \max_{a \in A_\delta} K_a \). Since \( A_\delta \) is finite, \( K \) is well defined (in particular, finite).

  Consider, then, any \( x \in [0,1] \). Choose \( a \in A_\delta \) such that \( x \) is in the \( \delta \) interval around \( a \). Then for any \( k, \ell > K \),

  \[
  |f_t^k(x) - f_t^\ell(x)| \leq |f_t^k(x) - f_t^k(a)| + |f_t^k(a) - f_t^\ell(a)| + |f_t^\ell(x) - f_t^\ell(a)| < \hat{\varepsilon}.
  \]

  Therefore, \( \sup_{x \in [0,1]} |f_t^k(x) - f_t^\ell(x)| \leq \hat{\varepsilon} < \varepsilon \), which proves the claim that \( f_t^k \) is Cauchy.

Finally, since \( F \) is a closed subset of a complete space, it is complete, and hence, since \((f_t^k)\) is Cauchy, there is an \( f^* \in F \) to which \((f_t^k)\) converges. ■

The basic Arzelà-Ascoli Theorem implies that if \( F \) is bounded and equicontinuous then its closure is compact. (The closure of \( F \) is equicontinuous, by Theorem 1, and it is bounded because, in any metric space, the closure of a bounded set is bounded; see the notes on Metric Spaces.) This implies the following corollary, which is frequently the form in which the basic Arzelà-Ascoli Theorem is stated.

**Theorem 3** (Basic Arzelà-Ascoli - Alternate Form). If \( F \subseteq C([0,1]) \) is bounded and equicontinuous then for any sequence \((f_t^i)\) in \( F \) there is a subsequence \((f_t^k)\) and an \( f^* \in C([0,1]) \) such that \( f_t^k \to f^* \).
The following examples illustrate the basic Arzelà-Ascoli Theorem and the role of equicontinuity.

**Example 1.** For \( t \in \{2, 3, \ldots\} \), define \( f_t : [0, 1] \to \mathbb{R} \) by

\[
f_t(x) = \begin{cases} 1 & \text{if } x \in [0, 1/t] \\ 0 & \text{if } x \in (1/t, 1]. \end{cases}
\]

This sequence converges pointwise to \( f^* \) defined by

\[
f^*(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \in (0, 1]. \end{cases}
\]

Convergence is not uniform, however. Indeed, \( d_{\sup}(f_t, f^*) = 1 \) for every \( t \), since for every \( t \) there are \( x > 0 \) for which \( f_t(x) = 1 \). There are no Cauchy subsequences of \( (f_t) \) (under \( d_{\sup} \)) and hence no convergent subsequences. Thus, setting \( F = \{f_t\} \), \( F \) is not compact. This example somewhat resembles the example in the \( \mathbb{R}^\omega \) notes showing that closed balls in \( (\ell_\infty, d_{\sup}) \) are not compact.

Note, however, that in this example the \( f_t \) are not continuous, hence are not members of \( C([0, 1]) \), and, therefore, \( F \) is not equicontinuous. \( \square \)

**Example 2.** Fix any \( \gamma \in (0, 1/4) \). I modify the functions in the previous example to make them continuous by inserting a segment of length \( \gamma \) and slope \(-1/\gamma\). Explicitly, for \( t \in \{2, 3, \ldots\} \), define \( g_t : [0, 1] \to \mathbb{R} \) by

\[
g_t(x) = \begin{cases} 1 & \text{if } x \in [0, 1/t] \\ -\frac{x}{\gamma} + \frac{1}{\gamma} + 1 & \text{if } x \in (1/t, 1/t + \gamma] \\ 0 & \text{if } x \in (1/t + \gamma, 1]. \end{cases}
\]

Let \( G = \{g_t\} \). Then \( G \) is equicontinuous; in particular, for any \( \varepsilon > 0 \), set \( \delta = \varepsilon \gamma \).

By Arzelà-Ascoli (alternate form), \( (g_t) \) has a subsequence that converges uniformly to a continuous function, and indeed the entire sequence converges uniformly to

\[
g^*(x) = \begin{cases} -\frac{x}{\gamma} + 1 & \text{if } x = [0, \gamma] \\ 0 & \text{if } x \in (\gamma, 1]. \end{cases}
\]

Note that \( \gamma \) can be made arbitrary small. So this example can be made close, in a sense, to that of Example 1. \( \square \)

**Example 3.** Finally, consider an example similar to Example 2 but now with \( \gamma_t \) that shrinks as \( t \) grows. In particular, fix \( \gamma \in (0, 1/4) \) and let \( \gamma_t = \gamma/t \). For \( t \in \{2, 3, \ldots\} \), define \( h_t : [0, 1] \to \mathbb{R} \) by

\[
h_t(x) = \begin{cases} 1 & \text{if } x \in [0, 1/t] \\ -\frac{x}{\gamma_t} + \frac{1}{\gamma} + 1 & \text{if } x \in (1/t, 1/t + \gamma/t] \\ 0 & \text{if } x \in (1/t + \gamma/t, 1]. \end{cases}
\]
Note, in particular, that the slope of the middle section is $-t/\gamma$, which is increasing in $t$.

Let $\mathcal{H} = \{h_t\}$. Then $\mathcal{H}$ is a subset of $C([0,1])$ but it is not equicontinuous. For any given $t$ and any $\varepsilon > 0$, uniform continuity for $f_t$ requires a $\delta_t$ of no more than $\varepsilon\gamma/t$, which depends on $t$. There is no single $\delta$ that will work for all $f_t$.

The sequence $(h_t)$ converges pointwise to $f^*$, the same $f^*$ as in Example 1. The sequence does not converge uniformly to $f^*$, however, and indeed it could not because $f^*$ is not continuous (remember that $C([0,1])$ is complete under $d_{\text{sup}}$). Under $d_{\text{sup}}$, $(h_t)$ has no Cauchy subsequences and hence no convergent subsequences. This helps illustrate the importance of equicontinuity in Arzelà-Ascoli. □

The ideas behind the basic Arzelà-Ascoli Theorem can be extended to more general environments in a number of ways, one example of which is the following. Given metric spaces $(X,d_X)$ and $(Y,d_Y)$, let $C(X,Y)$ denote the set of continuous functions from $X$ to $Y$. If $X$ is compact, define the sup metric on $C(X,Y)$ by, for any $f, g \in C(X,Y)$,

$$d_{\text{sup}}(f,g) = \sup_{x \in X} d_Y(f(x),g(x)).$$

This metric is well defined since $X$ is compact, $f$ and $g$ are continuous, and $d_Y$ is continuous with respect to the max metric on $Y \times Y$ (see the notes on Continuity). By essentially the same argument as for $(\ell^\infty,d_{\text{sup}})$, $C(X,Y)$ with the sup metric is complete. And, by essentially the same argument as for Theorem 1, the closure of an equicontinuous set of functions in $C(X,Y)$ is equicontinuous.

**Theorem 4** (Generalized Arzelà-Ascoli Theorem). Let $(X,d_X)$ and $(Y,d_Y)$ be compact metric spaces and let $F$ be a subset of $C(X,Y)$. If $F$ is closed and equicontinuous then it is compact.

**Proof.** Since $X$ is a compact metric space, it has a countable dense set $A$ (I leave showing this as an exercise). The proof is then almost the same as the proof of Theorem 2. In particular, the set $A$ here plays the same role in the proof of Theorem 4 as the rationals in $[0,1]$ did in the proof of Theorem 2. ■

The Arzelà-Ascoli Theorem can be generalized still further but I will not do so.