Independent Private Value Auctions

The following notes are based on the treatment in Krishna (2009); see also Milgrom (2004). I focus on only the simplest auction environments.

Consider \( I \geq 2 \) bidders with values independently and identically distributed on \([0, 1]\) (the analysis for other intervals is similar). The distribution of a bidder’s value has cumulative distribution function \( F \) and strictly positive density \( f \). Assume that each bidder has an outside option of 0 and that if bidder \( I \) wins the object and pays \( t_i \) then she gets utility

\[ v_i - t_i, \]

where \( v_i \) is bidder \( I \)’s value and \( t_i \) is her payment. Values are private: although the values of the other bidders will affect their actions and through that channel both the probability that bidder \( I \) wins and her payment, bidder \( I \)’s payoff does not depend directly on the values of the other bidders. At the time of bidding, only bidder \( I \) knows \( v_i \).

Consider anonymous direct auctions. By anonymous, I mean that the auction treats bidders symmetrically. By direct I mean that a bidder’s action is to submit a bid, which is simply a real number. Consider only auctions in which the high bidder wins. In the examples below, I focus on three special cases.

1. First Price Sealed Bid Auctions. Each bidder \( I \) submits a bid \( b_i \in \mathbb{R} \) to a referee. The referee gives the object to the highest bidder, who pays her bid. All other bidders get a payoff of 0. A NE (in fact, the NE) of this auction is computed below.

2. Second Price Sealed Bid Auctions. Each bidder \( I \) submits a bid \( b_i \in \mathbb{R} \) to a referee. The referee gives the object to the highest bidder, who pays the bid of the second highest bidder. All other bidders get a payoff of 0. This is the VCG mechanism in an auction setting. There is a truth-telling NE: bidder \( I \) bids \( b_i = v_i \). In fact, truth telling is weakly dominant. But there are other NE. In particular, it is a NE for bidder 1 to bid 1 and for all other bidders to bid 0.

3. All Pay Auctions. Each bidder \( I \) submits a bid \( b_i \in \mathbb{R} \) to a referee. The referee gives the object to the highest bidder. EVERY bidders pays her bid. This auction is sometimes used to model political lobbying. A NE of this auction is computed below.

Assume that the NE is symmetric (every bidder uses the same bid function). Henceforth I drop the \( I \) subscript. Assume that in this NE, bids are strictly increasing in type. Finally, assume that the auction has the property that the interim
expected utility of a bidder with value \( v = 0 \), is 0. In any given NE of any given auction, these assumptions have to be verified.

Under these assumptions, the auction is ex-post efficient. By almost the same argument as in MyersoI-Satterthwaite, \( T(v) \), the interim expected payment by a bidder with value \( v \) is

\[
T(v) = Q(v)v - U(0) - \int_0^v Q(x) \, dx, \tag{1}
\]

where \( Q(v) \) is the interim probability of winning the object when of type \( v \) and \( U \) is the interim expected utility when of type 0. By assumption, \( U(0) = 0 \). Therefore, integrating by parts yields

\[
T(v) = \int_0^v Q'(x)x \, dx. \tag{2}
\]

Remark 1. One way to remember equation (2) is to note that the interim expected utility of a bidder when of type \( v \) but bidding as though of type \( \hat{v} \) is

\[
Q(\hat{v})v - T(\hat{v}).
\]

If both \( Q \) and \( T \) are differentiable (which we cannot, in fact, assume, because \( Q \) and \( T \) are endogenous), then the first order condition for an interior maximum is

\[
Q'(\hat{v})v - T'(\hat{v}) = 0.
\]

If it is in fact optimal for the bidder to bid like a type \( v \) when a type \( v \) then \( \hat{v} = v \), hence,

\[
T'(v) = Q'(v)v.
\]

Integration gives (2). □

Because bids functions are symmetric and increasing in type, \( Q(v) \) is the probability that \( v \) is the highest value, which is to say, the probability that all other values are less than \( v \), which is

\[
Q(v) = \left[F(v)\right]^{I-1}.
\]

\( Q(v) \) can also be interpreted as the probability that the highest value of the other bidders is less than \( v \). Call this highest value of the other bidders \( Y \). Then \( G(v) = \left[F(v)\right]^{I-1} \) is the cumulative distribution function for \( Y \). Let \( g(Y) = G'(Y) \) be the density for \( Y \). Then

\[
T(v) = \int_0^v g(Y)Y \, dY.
\]
Note that
\[
\mathbb{E}[Y|Y < v] = \int_0^v Y \frac{g(Y)}{\text{Prob}[Y < v]} \, dY = \frac{1}{\text{Prob}[Y < v]} \int_0^v Y g(Y) \, dY.
\]

Therefore
\[
T(v) = \text{Prob}[Y < v] \mathbb{E}[Y|Y < v].
\]

Again, this is the interim expected payment by one bidder. The overall ex-ante expected payment for \( I \) bidders, under the above assumptions, is
\[
I[\mathbb{E}[\text{Prob}[Y < v] \mathbb{E}[Y|Y < v]]].
\]

Equation (3) has a number of very powerful implications. First, since it gives the interim expected payment in a form that depends only on the value distribution, and not on any details of the auction, it implies that, in this setting, any two symmetric equilibria of any two auctions will generate the same expected revenue if in both equilibria (a) bids are strictly increasing in type and (b) the interim expected utility of a bidder of type \( v = 0 \) is 0. This is true even if the auctions appear to be very different. As may be evident by inspection, the right-hand side of (3) is the expected revenue from the truth-telling NE of the second price auction. What (3) is saying, then, is that a wide variety of auctions have NE with the same expected revenue as the truth-telling NE of the second price auction.

Moreover, the ex-ante expected revenue in equation (4) is the highest ex-ante expected revenue that can be achieved in any ex-post efficient NE that respects interim individual rationality, across all possible auctions in this class of environments. As in Myerson-Satterthwaite, (1) implies that the seller’s ex-ante expected revenue is entirely determined by \( Q \) and \( U(0) \). But \( Q \) is pinned down by ex-post efficiency. And interim individual rationality holds iff \( U(0) \geq 0 \). Since \( T(v) \) is decreasing, \( U(0) \), the seller’s expected revenue is maximized when, as here, \( U(0) \) is set to its minimum value, namely 0.

**Example 1.** Suppose that the \( v \) are uniformly distributed on \([0,1]\). Then \( G(v) = v^{I-1}, \, g(v) = (I - 1)v^{I-2} \) and

\[
T(v) = \int_0^v (I - 1)Y^{I-2} \, dY = (I - 1) \int_0^v Y^{I-1} \, dY = \frac{I - 1}{I} v^I.
\]

It follows that ex ante expected revenue to the seller is

\[
I[\mathbb{E}[T(v)]] = I \int_0^1 \frac{I - 1}{I} v^I \, dv = \frac{I - 1}{I + 1}.
\]
In particular, if $I = 2$ then ex ante expected revenue is $1/3$. As $I$ increases, expected revenue converges to 1. This should make intuitive sense, since with large $I$ there is a high probability that there are many bidders with values close to 1. □

Equation (3) can be used to compute NE in auctions. I illustrate this with examples.

**Example 2. First price sealed bid auction.** Since the high bidder wins and pays her own bid $\beta(v)$, it must be that,

$$T(v) = \text{Prob}[Y < v] \beta(v).$$

Hence from (3),

$$\beta(v) = \mathbb{E}[Y | Y < v].$$

Note that this is increasing in $v$ and it is always less than $v$: bidders shave their bids, trading off a lower probability of winning against lower payments when they win. One can verify that there is, indeed, a NE with this bid function and it should be obvious that in this NE, interim expected utility when $v = 0$ is 0.

In the particular case in which $v$ is uniformly distributed on $[0, 1]$,

$$\beta(v) = \frac{1}{v^{I-1}} \int_0^v (I - 1)Y^{I-2} Y dY$$

$$= \frac{I - 1}{v^{I-1}} \int_0^v Y^{I-1} dY$$

$$= \frac{I - 1}{v^{I-1}} \left( \frac{1}{I} v^I \right)$$

$$= \frac{I - 1}{I} v.$$

If $I = 2$ then $\beta(v) = v/2$. As $I$ increases, the bid converges to $v$: competition from other bidders becomes keener and therefore shaving becomes less advantageous, although the bid is always less than $v$.

One can confirm by direct calculation that the ex-ante expected revenue to the seller is as calculated in Example 1. □

**Example 3. All pay auction.** Since each bidder always pays her bid regardless of whether she wins,

$$T(v) = \beta(v).$$

Hence from (3),

$$\beta(v) = \text{Prob}[Y < v] \mathbb{E}[Y | Y < v].$$

Again, note that this is increasing in $v$. Again, one can verify that there is, indeed, a NE with this bid function and it should be obvious that in this NE, interim expected utility when $v = 0$ is 0.
In the particular case in which \( v \) is uniformly distributed on \([0, 1]\),

\[
\beta(v) = \int_0^v (I - 1)Y^{I-2}Y \, dY \\
= (I - 1) \int_0^v Y^{I-1} \, dY \\
= (I - 1) \left( \frac{1}{I} v^I \right) \\
= \frac{I - 1}{I} v^I.
\]

If \( I = 2 \) then \( \beta(v) = v^2/2 \). As \( I \) increases, the bid function becomes increasingly convex, with bids approximately 0 unless \( v \) is very close to 1. Intuitively, a bidder (although risk neutral) is willing to risk a non-trivial bid only when the probability of winning is sufficiently high.

Again, one can confirm by direct calculation that the ex-ante expected revenue to the seller is as calculated in Example 1. □

As a final remark, this analysis becomes substantially more complicated if any of the assumptions are relaxed. It can, for example, be difficult if not impossible to solve for the equilibrium of a first price auction if the values are not identically distributed.

References
